Relativistic Quantum Field Theory, Volume 2

Path integral formalism

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This book is dedicated to my wife, Dr Veronica Antocheviz Dexheimer Strickland, and our amazing daughter Emily.

Contents

Preface

In introductory quantum mechanics one learns how to quantize a system given a fixed number of non-relativistic bosonic or fermionic particles. One uses a formalism in which the Hamiltonian is promoted from being a number to an operator resulting in the Schrödinger equation. The Hamiltonian operator itself is decomposed into kinetic and potential energy contributions and the potential energy form is typically taken as an external input. A question then arises: how does one arrive at the potential in the first place based on first principles? For example, how do we know (beyond experiment) that the Coulomb potential is the appropriate potential for charged particles? Are there quantum corrections to this potential? In addition, since potentials are not well-defined relativistically (instantaneous interactions), how can we generalize quantum mechanics to the relativistic case and satisfy causality?

In classical physics, fields are introduced in order to construct physical laws that are *causal* and *local*. Classical forces described by, for example Coulomb's law or Newton's universal gravitation, require action at a distance. As a result, the force felt by a body changes instantaneously if any other body's position (or charge, etc) changes, no matter how far away the other object is. In the classical field theories of Maxwell (electromagnetic field) and Einstein (gravitational field), the interactions between objects are mediated by a field that acts locally and causality is restored. So it seems that, since they are causal and local, we should figure out how to quantize classical field theories.

Another inconsistency in quantum mechanics was our somewhat haphazard treatment of wave-particle duality. For example, both electrons and photons act simultaneously like waves and particles and physically they share many common features. They both undergo wave-like diffraction from obstacles, but can also act like discrete particles (photoelectric effect, Compton scattering, etc). Despite this, in classical theory, electrons are simply postulated to exist as matter, while photons are interpreted as ripples in the electromagnetic field. Is it possible that electrons are themselves ripples in an 'electron field'? As we will learn over the course of these three volumes, the answer is a definitive yes. In general, the field is the fundamental object and particles are derived concepts that appear only after quantization of the field (e.g. the Higgs field gives birth to the Higgs boson).

In quantum mechanics, we take classical number-valued quantities and promote them to operators acting in a Hilbert space. As we will see, at least in 'canonical quantization', the rules for quantizing a field are only slightly different. The basic degrees of freedom in quantum field theory (QFT) are operator-valued functions of space and time and, since space and time are continuous, we are dealing with an infinite number of degrees of freedom, so we will need to (re-)learn how to deal with systems with a large number of degrees of freedom (many-body theory). Once we are done, we will be able to properly define QFTs that can be used in a variety of different contexts, for example high energy theory, condensed matter, cosmology, quantum gravity, etc.

Beyond this, Dirac taught us that a consistent theory of relativistic electrons requires the existence of anti-electrons (aka positrons). As a consequence, it is possible to create particle–antiparticle pairs once the energy available exceeds twice the electron rest mass $E > 2mc^2$ in a process called *pair production*. And, of course, the reverse can happen, which is called *pair annihilation*. And one can easily see both of these types of events using modern particle detectors. The conclusion we must draw from this is that particles are not indestructible objects; they can be created and destroyed and may only live for a short amount of time. They are merely excitations surfing on a quantum field.

But the story is more fantastical then this. If pair production has an energy threshold, then one could argue that as long as the energies available do not exceed this threshold ($E > 2mc^2$), then non-relativistic theories would be self-consistent; however, at this point the Heisenberg uncertainty principle comes into play. Let us say that we wanted to measure the position of a particle with a given spatial resolution L. The Heisenberg uncertainty principle tells us that the uncertainty in the momentum is $\Delta p \gtrsim 1/L$. In a relativistic setting, energy and momentum are connected, therefore, we also have an uncertainty in the energy $\Delta E \gtrsim 1/L$. However, when the energy plus its uncertainty exceeds $E + \Delta E > 2mc^2$, then it is possible to create purely quantum-mechanical particle–antiparticle pairs. Equating the two, we obtain a threshold distance $L_0 = 1/(2m) = \lambda_{\text{Compton}}/(4\pi)$ with $\lambda_{\text{Compton}} = 2\pi/m$. From this exercise we learn that the spontaneous production of particle–antiparticle pairs is important when a particle of mass m is localized in space to a distance, which is on order of less than its Compton wavelength.

A similar argument holds if one considers localizing particles in time. The energy uncertainty increases as the time interval is made shorter and one eventually turns on the possibility of pair production. The consequence of this is that as one considers shorter and shorter time (or space) intervals, one sees more and more particles! Why is this relevant? Since microscopic particles interacting with one another can resolve the other particles' microscropic dynamics (they behave like observers since they exchange quanta in order to interact), they 'see' their partners as being surrounded by an ensemble of particles and antiparticles that flit in and out of existence. In QFT, these ephemeral particles are called *virtual particles*. The 'cloud' of virtual particles that surround particles can modify its observable properties and must be taken into account to consistently understand them.

In figure 0.1, I show a typical Feynman graph for the splitting of an electron (solid line with the arrows) into virtual photons (sinusoidal lines) and virtual electron–positron pairs (closed loops). In this figure, time progresses from the left to the right. Starting from the incoming electron line we see the radiation of a virtual photon, which then splits into a virtual electron–positron pair, and so forth. In QFT, during this time, all possible configurations of the various intermediate particles are sampled, but some configurations are more probable than others. We will learn how to quantify this over the course of these three volumes.

A very similar phenomenon to pair production exists in condensed matter systems. When one considers the conduction of electrons in metals, for example, one finds that there is a valence band of electrons that are locally bound to atoms

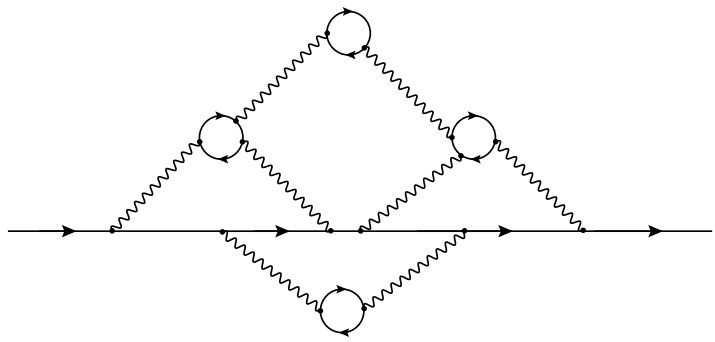

Figure 0.1. A typical Feynman graph for the splitting of a electron (solid line with the arrows) into virtual photons (sinusoidal lines) and virtual electron–positron pairs (closed loops).

and a conduction band in which electrons are able to move around. Because of the spin-statistics theorem, fermions obey the Pauli exclusion principle and 'stack up' to an energy called the Fermi energy. It is possible for a photon (or an energetic phonon) to excite one of the electrons that have an energy below the Fermi energy (in the Fermi sea) to above the Fermi sea. In metals, this process requires very little energy, however, in semiconductors the material is *gapped*. This introduces a lower limit on the amount of energy required to excite an electron into the conduction band which is called the gap. The processes of electron–hole pair production and annihilation can be described using Feynman graphs similar to the relativistic case and the mathematical machinery used to describe many body states in materials is very similar to what is encountered in relativistic QFT.

This three-volume series began as lecture notes for a two-semester introductory course in QFT and quantum chromodynamics (QCD) and, as such, is written rather informally. The text is intended to be used in an introductory graduate-level course in QFT or an advanced undergraduate course. Volume 1 introduces classical fields and the method of canonical quantization in order to have a bridge to the language and formalisms used by students focusing on condensed matter physics. Volume 2 builds upon what was learned in volume 1, but starts anew using the modern path integral formalism and focuses on applications to quantum electrodynamics and chromodynamics. Volume 3 continues with discussions of applications to particle physics phenomenology, the weak interaction, the Higgs mechanism, and finite temperature field theory.

Michael Strickland
30 June 2019

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Author biography

Michael Strickland

Dr Michael Strickland is a professor of physics at Kent State University. His primary interest is the physics of the quark-gluon plasma (QGP) and high-temperature QFT. The QGP is predicted by QCD to have existed until approximately 10^{-5} seconds after the Big Bang. The QGP is currently being studied terrestrially by experimentalists at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory and the Large Hadron Collider (LHC) at CERN. Dr Michael Strickland has published research papers on various topics related to the QGP, QFT, relativistic hydrodynamics, and many other topics. In addition, he has co-written a classic text on the physics of neural networks.

Units and conventions

- I will use natural units in which $\hbar = c = 1$. In these units, masses, energies, momenta, and temperatures have the same 'energy-like' units. I will typically use eV, MeV, or GeV as the standard unit for these energy-like quantities. For example, the mass of an electron is $m_e = 0.511\,\text{MeV}$, where this is understood implicitly to denote $m_e = 0.511\,\text{MeV}/c^2$ but, since $c = 1$, we typically suppress the division by c^2 when writing the mass.
- Another consequence of using natural units is that the vacuum electric permeability and magnetic permeabilities are both equal to one, $\varepsilon_0 = \mu_0 = 1$, which greatly simplifies the presentation of the formulas of electromagnetism. A similar simplification happens in thermodynamic formulas since $k_B = 1$.
- In natural units, spatial and temporal separations have units of inverse energy units (e.g. GeV^{-1}). In some places in the notes we will use a bracket notation to indicate the dimension of a quantity, for example [m] = 1 tells us that masses have units of energy to the power one and lengths [L] = −1 indicating that length scales scale like energy to the power −1.
- If the argument of a function, for example exp, should be dimensionless, then the argument should be equivalent to an energy times a length scale ($E \times L$), since lengths have units of inverse energies in natural units.
- In some cases you might need to switch from energy scales to explicit distance scales or vice versa. For this purpose, the following combination is very useful

$$\hbar c = 0.197\,326\,938\,\text{GeV fm},$$

  where fm stands for femtometers (1 fm = 10^{-15} m). For purposes of estimation, one can use the approximate value $\hbar c \approx 0.2\,\text{GeV fm}$. For example, if I were to tell you that a length (or time) scale is $L = 0.2\,\text{fm}$, you can divide by $\hbar c$ to find the equivalent in inverse GeV, for example $L = (0.2\,\text{fm})/(\hbar c) \simeq 1\,\text{GeV}^{-1}$. Similarly, we can work in reverse to convert energy (mass) scales to inverse distance scales. For example, from the electron mass $m_e = 0.511\,\text{MeV}$ we can determine the Compton wavelength $\lambda_{\text{Compton}} = 2\pi/m$ and then convert from GeV^{-1} to meters, that is $\lambda_{\text{Compton},e} \simeq 12\,296\,\text{GeV}^{-1} \times \hbar c \simeq 2.426 \times 10^{-12}\,\text{m}$.
- Finally, in the three volumes in this series the Minkowski space metric is assumed to be mostly negative which means that $(g^{\mu\nu})_{\text{Minkowski}} \equiv \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

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Michael Strickland

Chapter 1

Path integral formulation of quantum mechanics

In the standard path of 'second quantization', one derives classical equations of motion, formulates the appropriate Poisson brackets, and then simply makes the somewhat ad hoc assumption that one can replace Poisson brackets by commutators or anti-commutators for bosons and fermions, respectively. In the process, we transmute classical c-valued fields into operator-valued functions of space and time. One would like any alternative path to field quantization in which one does not have to invoke arbitrary mappings. Such a path is given to us by *path integral quantization*. It will provide us with a rigorous and systematic way to approach quantum field theory, which will reproduce known results coming from the path of canonical second-order quantization but it will also allow us to formulate things such as vertex functions and so on in a more intuitive manner. For more materials on this subject, I point the reader to refs. [1, 2].

1.1 The transition probability amplitude

In order to proceed, we first remind ourselves of the path integral formulation of non-relativistic quantum mechanics. We will do this for a one-dimensional system, but the derivation is straightforwardly extended to an arbitrary number of spatial dimensions. We start with a quantum wavefunction in the Schrödinger picture represented in the spatial basis

$$\psi(x,\, t) = \langle x|\psi(t)\rangle_S\,. \tag{1.1}$$

Next, we remind ourselves of a familiar relationship from introductory quantum mechanics which tells us how to evolve the Schrödinger-picture wave function given the wavefunction specified at a time t_i

$$|\psi(t)\rangle_S = e^{-iH(t-t_i)}|\psi(t)\rangle_S, \tag{1.2}$$

where, as usual, I have taken $\hbar = 1$ and H is understood to be an operator. Note that $|\psi(t_i)\rangle_S = |\psi(t_i)\rangle_H$ where the H indicates the Heisenberg picture.

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Defining

$$|x(t)\rangle \equiv e^{iH(t-t_i)}|x\rangle, \tag{1.3}$$

we see that

$$\langle x(t_f)|\psi(t_i)\rangle_S = \langle x|e^{-iH(t_f-t_i)}|\psi(t_i)\rangle = \langle x|\psi(t_f)\rangle_S = \psi(x_f,\, t_f)\,. \tag{1.4}$$

Using this and inserting a complete set of states

$$\begin{aligned}\psi(x_f,\, t_f) &= \langle x(t_f)|\psi(t_i)\rangle_S \\ &= \int dx_i \underbrace{\langle x(t_f)|x(t_i)\rangle}_{\equiv K(x_f,t_f;x_i,t_i)} \underbrace{\langle x(t_i)|\psi(t_i)\rangle_S}_{\psi(x_i,t_i)} \qquad (1.5)\\ &= \int dx_i\, K(x_f,\, t_f;\, x_i,\, t_i)\psi(x_i,\, t_i)\,. \qquad (1.6)\end{aligned}$$

The function K introduced above is sometimes called the *propagator* and tells us how to compute the wavefunction as a function of the final time (and position) given the wavefunction at the initial time. The wavefunction $\psi(x_f,\, t_f)$ gives the probability amplitude to find the particle at the point x_f at time t_f and $K(x_f,\, t_f;\, x_i,\, t_i)$ is the *transition probability amplitude*. The probability for the particle to transition from $(x_i,\, t_i)$ to $(x_f,\, t_f)$ is $P(x_f,\, t_f;\, x_i,\, t_i) = |K(x_f,\, t_f;\, x_i,\, t_i)|^2$.

Of course, we need not stop at inserting one complete set of states. Let us consider what happens when we insert another complete set of states

$$\begin{aligned}\psi(x_f,\, t_f) &= \int dx_i\, \langle x(t_f)|x(t_i)\rangle \psi(x_i,\, t_i) \qquad (1.7)\\ &= \int dx_i \int dx_1 \underbrace{\langle x(t_f)|x(t_1)\rangle}_{K(x_f,t_f;x_1,t_1)} \underbrace{\langle x(t_1)|x(t_i)\rangle}_{K(x_1,t_1;x_i,t_i)} \psi(x_i,\, t_i) \qquad (1.8)\\ &= \int dx_i \int dx_1 K(x_f,\, t_f;\, x_1,\, t_1)K(x_1,\, t_1;\, x_i,\, t_i)\psi(x_i,\, t_i)\,. \qquad (1.9)\end{aligned}$$

From this we learn that

$$K(x_f,\, t_f;\, x_i,\, t_i) = \int dx_1 K(x_f,\, t_f;\, x_1,\, t_1)K(x_1,\, t_1;\, x_i,\, t_i)\,. \tag{1.10}$$

1.2 Derivation of the quantum mechanical path integral

Equation (1.10) can be applied iteratively by inserting n intermediate points to obtain

$$K(x_f,\, t_f;\, x_i,\, t_i) = \int dx_1 \int dx_2 \cdots \int dx_n K_{f,n} K_{n,n-1} K_{n-1,n-2} \cdots K_{1,i}\,, \tag{1.11}$$

where I have introduced the shorthand notation $K_{k,l} \equiv K(x_k,\, t_k;\, x_l,\, t_l)$. To further compress this formula, we write $\int dx_1 \int dx_2 \cdots \int dx_n = \prod_{j=1}^{n} \int dx_j$ to obtain

$$K(x_f,\, t_f;\, x_i,\, t_i) = \prod_{j=1}^{n} \left(\int dx_j \right) K_{f,n} K_{n,n-1} K_{n-1,n-2} \cdots K_{1,i}\,. \tag{1.12}$$

In this formula, we imagine that a particle propagates from the initial position to the final position and we compute the total propagator by integrating over all possible intermediate positions at a finite number of intermediate times. This is illustrated for three different possible paths in figure 1.1.

In the next step, we are going to take $n \to \infty$. With this in mind, let us focus on a single element of the path

$$\begin{aligned} K_{j,j-1} &= \left\langle x_j \middle| \mathrm{e}^{-iHt_j} \mathrm{e}^{iHt_{j-1}} \middle| x_{j-1} \right\rangle \\ &= \left\langle x_j \middle| \mathrm{e}^{-iH\Delta t} \middle| x_{j-1} \right\rangle. \end{aligned} \tag{1.13}$$

In the limit $n \to \infty$, holding the initial and final times fixed, we have $\Delta t \to 0$, so we can Taylor expand the exponential to obtain

$$\begin{aligned} K_{j,j-1} &\simeq \left\langle x_j \middle| 1 - iH\Delta t \middle| x_{j-1} \right\rangle \\ &\simeq \delta(x_j - x_{j-1}) - i\Delta t \left\langle x_j \middle| H \middle| x_{j-1} \right\rangle, \end{aligned} \tag{1.14}$$

where we have discarded terms of $\mathcal{O}((\Delta t)^2)$ and higher. Note that, above, we have assumed that the Hamiltonian is time-independent, which is true for any closed energy-conserving system. To proceed, let us consider a Hamiltonian of the standard form

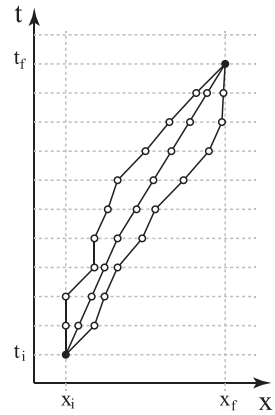

Figure 1.1. Three possible paths that could be followed by a particle that starts at spacetime point $(x_i,\, t_i)$ and ends and $(x_f,\, t_f)$.

$$H = T + V = \frac{p^2}{2m} + V(x), \tag{1.15}$$

and evaluate the expectation value appearing on the second line of equation (1.14). First, let us look at the contribution from the kinetic energy

$$\begin{aligned}\langle x_j|T|x_{j-1}\rangle &= \int \frac{dp'}{2\pi}\int \frac{dp}{2\pi}\ \underbrace{\langle x_j|p'\rangle}_{e^{ip'x_j}}\ \underbrace{\left\langle p'\left|\frac{p^2}{2m}\right|p\right\rangle}_{2\pi\delta(p-p')\frac{p^2}{2m}}\ \underbrace{\langle p|x_{j-1}\rangle}_{e^{-ipx_{j-1}}} \\ &= \int \frac{dp}{2\pi} e^{ip(x_j-x_{j-1})}\frac{p^2}{2m},\end{aligned} \tag{1.16}$$

where, in the first equality, I have inserted two complete sets of momentum states in order to make the evaluation of the kinetic energy operator trivial. Turning to the potential energy contribution, we obtain

$$\begin{aligned}\langle x_j|V|x_{j-1}\rangle &\simeq \langle x_j|\underbrace{V\left(\frac{x_j + x_{j-1}}{2}\right)}_{\equiv V(\bar{x})}|x_{j-1}\rangle \\ &\simeq V(\bar{x}_{j-1})\langle x_j|x_{j-1}\rangle \\ &\simeq V(\bar{x}_{j-1})\delta(x_j - x_{j-1}),\end{aligned} \tag{1.17}$$

where we have introduced the average position $\bar{x}_{j-1} \equiv (x_j + x_{j-1})/2$ in order to account for the fact that we could act either to the left or the right with the position operator. Strictly speaking, this is not necessary, since we will soon take the continuum limit, but this does remove some ambiguity in the evaluation for the time being.

Next, we express the delta function appearing on the last line of equation (1.17) as

$$\delta(x_j - x_{j-1}) = \int \frac{dp}{2\pi} e^{ip(x_j-x_{j-1})}, \tag{1.18}$$

to obtain

$$\langle x_j|V|x_{j-1}\rangle = \int \frac{dp}{2\pi} e^{ip(x_j-x_{j-1})}V(\bar{x}). \tag{1.19}$$

Using equation (1.18) for the delta function appearing in the second line of equation (1.14) and collecting the kinetic and potential energy terms, we can express equation (1.14) as

$$K_{j,j-1} \simeq \int \frac{dp}{2\pi} e^{ip(x_j-x_{j-1})}\left[1 - i\Delta t\left(\frac{p^2}{2m} + V(\bar{x}_j)\right) + \mathcal{O}((\Delta t)^2)\right]. \tag{1.20}$$

The term in the square brackets is simply the first-order expansion of the exponential of the *scalar* Hamiltonian. If one expands the propagator to higher orders in Δt, one

finds that, in the square brackets above, one generates an exponential, allowing us to write

$$K_{j,j-1} \simeq \int \frac{dp_{j-1}}{2\pi} \mathrm{e}^{ip_{j-1}(x_j - x_{j-1}) - i\Delta t H(p_{j-1},\, x_{j-1})}, \tag{1.21}$$

where I have relabeled the momentum integration variable to p_{j-1}.

Now that we know the expression for an infinitesimal element, we can multiply all of the propagators appearing in equation (1.12) to obtain

$$\begin{aligned} K(x_f,\, t_f;\, x_i,\, t_i) &= \int \prod_{j=1}^{n} dx_j\, K_{f,n} K_{n,n-1} \cdots K_{1,i} \\ &= \left(\prod_{j=1}^{n} \int dx_j \right) \left(\int \frac{dp_n}{2\pi} \int \frac{dp_{n-1}}{2\pi} \cdots \int \frac{dp_0}{2\pi} \right) \mathrm{e}^{ip_n(x_{n+1} - x_n) - i\Delta t H(p_n,\, x_n)} \\ &\quad \times \mathrm{e}^{ip_{n-1}(x_n - x_{n-1}) - i\Delta t H(p_{n-1},\, x_{n-1})} \cdots \mathrm{e}^{ip_0(x_1 - x_0) - i\Delta t H(p_0,\, x_0)} \\ &= \left(\prod_{j=1}^{n} \int dx_j \right) \left(\prod_{k=0}^{n} \int dp_k \right) \mathrm{e}^{i \sum_{\ell=0}^{n} [p_\ell (x_{\ell+1} - x_\ell) - \Delta t H(p_\ell,\, \bar{x}_\ell)]} \end{aligned} \tag{1.22}$$

where it is understood that $x_0 = x_i$ and $x_{n+1} = x_f$. Note that there is one more term in the product and sum over momentum since, for n intermediate points in x, we require $n + 1$ propagators to connect x_i to x_f. Focusing on the argument of the exponential above, we can write

$$\lim_{\Delta t \to 0} \sum_{\ell=0}^{n} \Delta t \left[p_\ell \left(\frac{x_{\ell+1} - x_\ell}{\Delta t} \right) - H(p_\ell,\, \bar{x}_\ell) \right] = \int_{t_i}^{t_f} dt [p\dot{x} - H(p(t),\, x(t))]. \tag{1.23}$$

Introducing the notations

$$\mathcal{D}x \equiv \lim_{n \to \infty} \prod_{j=1}^{n} \int dx_j, \tag{1.24}$$

and

$$\mathcal{D}p \equiv \lim_{n \to \infty} \prod_{j=0}^{n} \int dp_j, \tag{1.25}$$

we can express the continuum propagator compactly as

$$K(x_f,\, t_f;\, x_i,\, t_i) = \int \mathcal{D}x\, \mathcal{D}p \exp\left(i \int_{t_i}^{t_f} dt [p\dot{x} - H] \right). \tag{1.26}$$

Exercise 1.1 Show that if equation (1.20) is expanded to second order in Δt that one generates an expression, which is consistent with the Taylor series of the exponentiated Hamiltonian as stated below equation (1.20).

1.3 Path integral in terms of the Lagrangian

In the case that the Hamiltonian is of the form $H = (p^2/2m) + V(x)$ it is possible to perform the p_j integrations and reduce the path integral to integrations only over the coordinate x. To see this, we back up one step and look at the terms involving the momentum in one of the terms contributing to the sum in equation (1.23)

$$I_j \equiv \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(i\left[p_j(x_{j+1} - x_j) - \Delta t \frac{p_j^2}{2m} \right] \right). \tag{1.27}$$

This is a Gaussian integral that can be evaluated using

$$\int dx\, \mathrm{e}^{-ax^2+bx} = \mathrm{e}^{b^2/4a} \left(\frac{\pi}{a} \right)^{1/2}. \tag{1.28}$$

Identifying $a = i\Delta t/2m$ and $b = i(x_{j+1} - x_j)$ gives

$$I_j = \exp\left(i\Delta t \frac{m}{2} \left[\frac{x_{j+1} - x_j}{\Delta t} \right]^2 \right) \left(\frac{m}{2\pi i \Delta t} \right)^{1/2}. \tag{1.29}$$

Recognizing that

$$\frac{m}{2} \left[\frac{x_{j+1} - x_j}{\Delta t} \right]^2 = \frac{p_j^2}{2m}, \tag{1.30}$$

we obtain

$$K(x_f,\, t_f;\, x_i,\, t_i) = \left(\frac{m}{2\pi i \Delta t} \right)^{(n+1)/2} \left(\prod_{j=1}^{n} \int dx_j \right) \exp\left(i\Delta t \sum_{j=0}^{n} \underbrace{\left[\frac{p_j^2}{2m} - V(\bar{x}_j) \right]}_{T_j - V_j = L_j} \right), \tag{1.31}$$

where L_j is the Lagrangian evaluated at point j. Defining

$$\mathcal{N} \equiv \lim_{n\to\infty} \left(\frac{m}{2\pi i \Delta t} \right)^{(n+1)/2}, \tag{1.32}$$

and taking the continuum limit as before, we obtain

$$K(x_f,\, t_f;\, x_i,\, t_i) = \mathcal{N} \int \mathcal{D}x \exp\left(i \int_{t_i}^{t_f} dt\, L(x,\, \dot{x}) \right). \tag{1.33}$$

Finally, we note that the time-integral of the Lagrangian defines the action

$$S[x,\, \dot{x}] = \int_{t_i}^{t_f} dt\, L(x,\, \dot{x}), \tag{1.34}$$

where $S[x, \dot{x}]$ indicates that the action is a *functional* of x and $\dot{x}$. This allows us to write the propagator in the rather compact form

$$K(x_f, t_f; x_i, t_i) = \mathcal{N} \int \mathcal{D}x \ \ \mathrm{e}^{iS[x, \dot{x}]}. \tag{1.35}$$

This formula tells us that the propagator is computed from a sum over all paths weighted by the complex exponential of the action S. The action S is minimized on the classical trajectory for the particle and hence nearby paths will have nearly the same phase factor, leading to constructive interference between the terms contributing to the path integral. If, however, the path is 'far' from the classical path, then $\exp(iS)$ will rapidly oscillate, leading to a very small probably amplitude for that configuration. In natural units, 'far' is measured by when the action differs from the classical one by order one. In standard SI units, the argument of the exponential above would instead be $iS/\hbar$, allowing us to see that, in these units, paths whose action differs from the classical one by more than order $\hbar$ will suffer these oscillations. What this means in practice is that there is a quantum 'blurriness' of particle propagation which extends to paths whose action is within a window of a few multiples of $\hbar$ from the classical path. Paths that are completely different from the classical one are allowed, however, they are unlikely.

1.3.1 Summary

Let us summarize what we have learned thus far. We have found that, in the general case, the quantum mechanical propagator can be expressed in terms of an infinite dimensional path integral

$$K(q_f, t_f; q_i, t_i) = \int \mathcal{D}q\mathcal{D}p \exp\left(i \int_{t_i}^{t_f} dt[p\dot{q} - H]\right), \tag{1.36}$$

and that, in the case that $H = p^2/2m + V(q)$, one has[1]

$$K(q_f, t_f; q_i, t_i) = \mathcal{N} \int \mathcal{D}q \exp\left(i \int_{t_i}^{t_f} dt\, L(q, \dot{q})\right) = \mathcal{N} \int \mathcal{D}q\, \mathrm{e}^{iS[q, \dot{q}]}, \tag{1.37}$$

where we have switched notation to generalized coordinates and it is understood that, in both cases, the path integral is constrained at the end points to enforce the boundary condition $q(t_i) = q_i$ and $q(t_f) = q_f$.

1.3.2 Caveats and clarifications

- We glossed over something in the definition of the normalization factor in equation (1.32). By inspection, one can see that, in the limit $n \to \infty$ ($\Delta t \to 0$), this quantity *diverges*. The fact that it is infinite does not matter physically since this infinity can be absorbed by normalization of the quantum wavefunction. In practice this is done by normalizing to a known path integral

[1] See the caveats and clarifications below for a qualifying remark.

such as the path integral for a freely propagating particle. In fact, as we will see in the context of quantum field theory, this normalization constant always cancels in physical quantities, for example expectation values, etc. Formally speaking, one should have in mind the discrete form and only take the continuum limit at the end of the calculation, however, in practice it suffices to simply denote this normalization abstractly as one and proceed ignoring this subtlety.

- As noted during the derivation, obtaining equation (1.37) relied on the Hamiltonian being over the form $H = p^2/2m + V(x)$. This is not necessarily always the case, for example, if there are internal frictional forces proportional to the velocity. However, as it turns out, equation (1.37) is more general. It holds for a classical Lagrangian of the form

$$L(q, \dot{q}) = \frac{1}{2}\sum_{i,j}\dot{q}_i A_{ij}\dot{q}_j + \sum_i B_i(q)\dot{q}_i - V(q), \tag{1.38}$$

where A is a symmetric, non-singular matrix which does not depend on q. In this case, one can show that

$$\int \mathcal{D}p \ \exp\left(i\int_{t_i}^{t_f} dt[p\dot{q} - H]\right) \propto \exp\left(i\int_{t_i}^{t_f} dt\, L(q, \dot{q})\right)\prod_t (\det A)^{-1/2}, \tag{1.39}$$

where $\prod_t$ implies a product over all discrete time points. If A is independent of q, then the product at the end is a constant, which can be absorbed into the overall normalization constant. If, however, A does depend on q, then one can use

$$\prod_t (\det A)^{-1/2} = \exp\left(-\frac{1}{2}\sum_t \log(\det A)\right), \tag{1.40}$$

to obtain

$$K(q_f, t_f; q_i, t_i) = \mathcal{N}\int \mathcal{D}q \ \exp\left(i\int_{t_i}^{t_f} dt\left[L(q, \dot{q}) + \frac{i}{2}\delta(0)\log(\det A(q))\right]\right). \tag{1.41}$$

1.4 Computing simple path integrals

So far we found that, in the general case, the quantum mechanical propagator can be expressed in terms of an infinite dimensional path integral

$$K(q_f, t_f; q_i, t_i) = \int \mathcal{D}q\,\mathcal{D}p \exp\left(i\int_{t_i}^{t_f} dt[p\dot{q} - H]\right), \tag{1.42}$$

and that, in the case that the classical Lagrangian can be expressed in the form

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j}\dot{q}_i A_{ij}\dot{q}_j + \sum_i B_i(q)\dot{q}_i - V(q), \tag{1.43}$$

one has

$$K(q_f, t_f; q_i, t_i) = \mathcal{N}\int \mathcal{D}q \;\; \exp\left(i\int_{t_i}^{t_f} dt\, L(q,\dot{q})\right) = \mathcal{N}\int \mathcal{D}q\, \mathrm{e}^{iS[q,\dot{q}]}, \tag{1.44}$$

where we have switched notation to a generalized coordinate q and it is understood that, in both cases, the path integral is constrained at the end points to enforce the boundary condition $q(t_i) = q_i$ and $q(t_f) = q_f$.

In the next two sections, I would like to cover three additional topics in this context before proceeding to quantum field theory: (1) a practical example of the computation of the path integral for a free particle, a harmonic oscillator, and a general discussion of the quadratic approximation, (2) how to calculate time-ordered expectation values in the path integral formalism, and (3) how to describe asymptotic states. Before embarking on item (1), I will note that typically one does not have to actually evaluate a path integral explicitly to make use of the formalism. As you will see, the actual analytic evaluation can be a bit tricky, even for simple cases.

1.4.1 Free particle in the path integral formalism

The Hamiltonian for a free particle is $H = p^2/2m$ and the corresponding Lagrangian is $L = m\dot{q}^2/2$. In the discrete formulation, the propagator is given by

$$K(q_f, t_f; q_i, t_i) = \lim_{\Delta t\to 0}\frac{1}{A}\left(\prod_{j=1}^{n}\int\frac{dq_j}{A}\right)\exp\left(i\Delta t\sum_{j=0}^{n}\frac{m}{2}\left[\frac{q_{j+1}-q_j}{\Delta t}\right]^2\right), \tag{1.45}$$

where $A = \sqrt{2\pi i\Delta t/m}$, $\Delta t = (t_f - t_i)/(n+1)$, and it is understood that $q_0 = q_i$ and $q_{n+1} = q_f$. To evaluate this integral we can use the the 'Feynman slice' method, which makes use of the fact that an integral over a Gaussian gives another Gaussian and then we can proceed iteratively. To start, let us consider the integral over q_1 since it only appears in the first and second time slices

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} dq_1 \exp\left(\frac{im}{2\Delta t}\left[(q_1-q_0)^2 + (q_2-q_1)^2\right]\right) \\ &= \int_{-\infty}^{\infty} dq_1 \exp\left(\frac{im}{2\Delta t}\left[2\left(q_1 - \frac{1}{2}(q_0+q_2)\right)^2 + \frac{1}{2}(q_2-q_0)^2\right]\right) \\ &= \frac{A}{\sqrt{2}}\exp\left(\frac{im}{2\Delta t}\frac{1}{2}(q_2-q_0)^2\right). \end{aligned} \tag{1.46}$$

For the $n = 2$ term, there are two terms entering

$$
\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} dq_2 \exp\left(\frac{im}{2\Delta t}\left[\frac{1}{2}(q_2 - q_0)^2 + (q_3 - q_2)^2\right]\right) \\
&= \int_{-\infty}^{\infty} dq_2 \exp\left(\frac{im}{2\Delta t}\left[\frac{3}{2}\left(q_2 - \frac{1}{3}q_0 - \frac{2}{3}q_3\right)^2 + \frac{1}{3}(q_3 - q_0)^2\right]\right) \\
&= A\sqrt{\frac{2}{3}} \exp\left(\frac{im}{2\Delta t}\frac{1}{3}(q_3 - q_0)^2\right).
\end{aligned}
\tag{1.47}
$$

From here on, the pattern should be clear, with the nth integral given by

$$
\begin{aligned}
I_n &= \int_{-\infty}^{\infty} dq_n \exp\left(\frac{im}{2\Delta t}\left[\frac{1}{n}(q_n - q_0)^2 + (q_{n+1} - q_n)^2\right]\right) \\
&= \int_{-\infty}^{\infty} dq_n \exp\left(\frac{im}{2\Delta t}\left[\frac{n+1}{n}\left(q_n - \frac{1}{n+1}q_0 - \frac{n}{n+1}q_{n+1}\right)^2\right.\right. \\
&\quad \left.\left. + \frac{1}{n+1}(q_{n+1} - q_0)^2\right]\right) \\
&= A\sqrt{\frac{n}{n+1}} \exp\left(\frac{im}{2\Delta t}\frac{1}{n+1}(q_{n+1} - q_0)^2\right).
\end{aligned}
\tag{1.48}
$$

Putting the pieces together, we obtain

$$
\begin{aligned}
K(q_f, t_f; q_i, t_i) &= \lim_{\Delta t \to 0} \frac{1}{A}\sqrt{\frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{n}{n+1}} \exp\left(\frac{im}{2\Delta t}\frac{1}{n+1}(q_{n+1} - q_0)^2\right) \\
&= \lim_{\Delta t \to 0} \frac{1}{A\sqrt{n+1}} \exp\left(\frac{im}{2\Delta t}\frac{1}{n+1}(q_f - q_i)^2\right) \\
&= \sqrt{\frac{m}{2\pi i T}} \exp\left(\frac{im}{2}\frac{(q_f - q_i)^2}{T}\right),
\end{aligned}
\tag{1.49}
$$

where, in the last line, we used the fact that $(n+1)\Delta t = t_f - t_i \equiv T$.

1.4.2 Quantum harmonic oscillator in the path integral formalism

Next, we compute the path integral for a harmonic oscillator with a Hamiltonian of the form

$$
H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2, \tag{1.50}
$$

and an associated Lagrangian

$$
L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2. \tag{1.51}
$$

Since the path integral is dominated by paths close to the classical solution, we will change variables to

$$q(t) = \bar{q}(t) + \varepsilon(t), \tag{1.52}$$

where $\bar{q}(t)$ is a solution to the classical equation of motion

$$\frac{d^2\bar{q}}{dt^2} + \omega^2\bar{q} = 0. \tag{1.53}$$

The full solution is subject to the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$ and hence it follows that

$$\varepsilon(t_i) = \varepsilon(t_f) = 0. \tag{1.54}$$

To proceed, we expand the action in terms of this new set of variables to find

$$S[q, \dot{q}] = S[\bar{q}, \dot{\bar{q}}] + S[\varepsilon, \dot{\varepsilon}] + \int_{t_i}^{t_f} dt \frac{d}{dt}\left(m\varepsilon\frac{d\bar{q}}{dt}\right) - \int_{t_i}^{t_f} dt\, m\varepsilon\left(\frac{d^2\bar{q}}{dt^2} + \omega^2\bar{q}\right). \tag{1.55}$$

The third term above vanishes because of the boundary condition (1.54) and the fourth term vanishes because $\bar{q}$ is a solution to the classical equation of motion (1.53). The fact that these two terms vanish is generic and independent of our assumption that the system is a harmonic oscillator. The fact that the first two terms are both simple actions for a harmonic oscillator is specific to this case and does not happen in general.

As a result of this factorization, we can rewrite the path integral (1.44) in this case as

$$\begin{aligned} K(q_f, t_f; q_i, t_i) &= \mathcal{N} e^{iS[\bar{q}, \dot{\bar{q}}]} \int \mathcal{D}\varepsilon\, e^{iS[\varepsilon, \dot{\varepsilon}]} \\ &= \mathcal{N} e^{iS[\bar{q}, \dot{\bar{q}}]} \int \mathcal{D}\varepsilon \exp\left(i\int_{t_i}^{t_f} dt\left[\frac{m}{2}\dot{\varepsilon}^2 - \frac{m\omega^2}{2}\varepsilon^2\right]\right), \end{aligned} \tag{1.56}$$

where we have used the fact that the classical path is fixed by the equations of motion and therefore can be treated as constant with respect to the integration over paths. To begin, let us evaluate the classical action

$$\begin{aligned} S[\bar{q}, \dot{\bar{q}}] &= \frac{m}{2}\int_{t_i}^{t_f} dt(\dot{\bar{q}}^2 - \omega^2\bar{q}^2) \\ &= \frac{m}{2}\left\{[\bar{q}\dot{\bar{q}}]_{t_i}^{t_f} - \int_{t_i}^{t_f} dt(\bar{q}\ddot{\bar{q}} + \omega^2\bar{q}^2)\right\} \\ &= \frac{m}{2}[\bar{q}\dot{\bar{q}}]_{t_i}^{t_f}, \end{aligned} \tag{1.57}$$

where, in going from the first to second lines, we integrated the kinetic energy term by parts and, in going from the second to third lines, we used the fact that, for the harmonic oscillator, $\ddot{\bar{q}} = -\omega^2\bar{q}$. The last line above can be evaluated by inserting a

general solution of the form $\bar{q}(t) = A\sin(\omega t) + B\cos(\omega t)$ subject to the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$. The final result is

$$S(q_f, t_f, q_i, t_i) = \frac{m\omega}{2\sin(\omega T)}\left[\left(q_i^2 + q_f^2\right)\cos(\omega T) - 2q_i q_f\right] \tag{1.58}$$

where $T \equiv t_f - t_i$.

Exercise 1.2 Show that equation (1.55) is correct.

Exercise 1.3 Find the transition probability amplitude, K for a particle moving in one dimension and subject to a constant external force, which has a Lagrangian $L = \frac{1}{2}m\dot{q}^2 + Fq$.

Exercise 1.4 Show that equation (1.58) is correct.

1.4.3 Path integral for deviations from the classical solution

What remains is to evaluate the path integral over the deviations from the classical solution in equation (1.56). We will do so using the 'determinant method'. Our goal is compute

$$J(T) \equiv \int D\varepsilon \exp\left(i\int_{t_i}^{t_f} dt\,\frac{m}{2}[\dot{\varepsilon}^2 - \omega^2\varepsilon^2]\right), \tag{1.59}$$

subject to the boundary condition $\varepsilon_i = \varepsilon_f = 0$. Notationally, we have indicated that the quantity J only depends on T since the path integral that remains has trivial boundary conditions and, by time-translation invariance, can only depend on the difference of the initial and final times.

We begin integrating the first term in the argument of the integral by parts to obtain

$$i\int_{t_i}^{t_f} dt\,\frac{m}{2}[\dot{\varepsilon}^2 - \omega^2\varepsilon^2] = -i\int_{t_i}^{t_f} dt\,\frac{m}{2}\varepsilon\left[\frac{d^2}{dt^2} + \omega^2\right]\varepsilon = \frac{i}{2}\int_{t_i}^{t_f} dt\,\varepsilon\hat{O}\varepsilon, \tag{1.60}$$

where

$$\hat{O} \equiv -m\left[\frac{d^2}{dt^2} + \omega^2\right]. \tag{1.61}$$

On a formal level, we now simply have to evaluate the Gaussian integral with the result being proportional to $(\det\hat{O})^{-1/2}$, but since this is our first time around, let us see how this works in detail and also determine the proportionality constant.

Since the fluctuation vanishes at the endpoints, we can write the function as a linear combination of the form

$$\varepsilon(t) = \sum_n c_n \varphi_n(t), \tag{1.62}$$

where $\varphi_n(t)$ are the orthonormal eigenfunctions of $\hat{O}$

$$\varphi_n(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{n\pi(t - t_i)}{T}\right) \quad n \in \mathbb{Z}, \tag{1.63}$$

which satisfy

$$\hat{O}\varphi_n(t) = \lambda_n \varphi_n(t), \tag{1.64}$$

with

$$\lambda_n = m\left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right], \tag{1.65}$$

and are, as advertised above, orthonormal

$$\int_{t_i}^{t_f} dt\, \varphi_n(t)\varphi_m(t) = \delta_{nm}. \tag{1.66}$$

The functions $\varphi_n(t)$ form a complete basis, therefore, we can use the coefficients c_n as a discrete set of integration variables. Transforming to these variables, we have

$$\int \mathcal{D}\varepsilon = \mathcal{J} \prod_n \int \frac{dc_n}{\sqrt{2\pi i}}, \tag{1.67}$$

where $\mathcal{J}$ is related to the Jacobian of the transformation and we have introduced some explicit factors of $\sqrt{2\pi i}$ by hand for future convenience. Note that the Jacobian is independent of c_n and, hence, can be treated as a constant.

Using the basis states introduced above, we have

$$\int_{t_i}^{t_f} dt\, \varepsilon\hat{O}\varepsilon = \sum_{m,n} \int_{t_i}^{t_f} dt\, c_m c_n (\varphi_m \hat{O} \varphi_n) = \sum_n \lambda_n c_n^2. \tag{1.68}$$

As a result, we have

$$\begin{aligned} J(T) &= \int \mathcal{D}\varepsilon \ \exp\left(\frac{i}{2}\int_{t_i}^{t_f} dt \ \ \varepsilon\hat{O}\varepsilon\right) \\ &= \mathcal{J}\left(\prod_n \int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi i}}\right) \exp\left(\frac{i}{2}\sum_n \lambda_n c_n^2\right). \end{aligned} \tag{1.69}$$

Each of the integrals is now independent and given by a standard Gaussian integral, with the result being

$$J(T) = \mathcal{J}\left(\prod_n \lambda_n\right)^{-1/2} = \mathcal{J}(\det \hat{O})^{-1/2}. \tag{1.70}$$

The above expression gives us a concrete understanding of what the determinate of the operator $\hat{O}$ implies; it is given by the product of the eigenvalues of the operator.

At this point we would like to take the continuum limit, but this proves to be difficult and potentially ill-defined since both the Jacobian and the product of eigenvalues are potentially divergent. We can simplify our lives by normalizing to the free-particle case since the result above should reduce to it in the limit that $\omega \to 0$, which will allow us to efficiently determine the normalization. Before proceeding we note that, although the operator $\hat{O}$ depends on ω, both the eigenfunctions and the Jacobian do not depend on ω. As a result, we have

$$J_\omega(T) = \mathcal{J}(\det \hat{O}_\omega)^{-1/2} \tag{1.71}$$

and

$$J_0(T) = \mathcal{J}(\det \hat{O}_0)^{-1/2}\,. \tag{1.72}$$

Next, we consider the following ratio

$$\frac{J_\omega(T)}{J_0(T)} = \left(\frac{\det \hat{O}_0}{\det \hat{O}_\omega}\right)^{1/2} = \left(\prod_n \frac{\lambda_n(\omega = 0)}{\lambda_n(\omega)}\right)^{1/2}. \tag{1.73}$$

Computing the eigenvalue product, one finds

$$\prod_n \frac{\lambda_n(\omega = 0)}{\lambda_n(\omega)} = \prod_n \frac{\left(\frac{n\pi}{T}\right)^2}{\left(\frac{n\pi}{T}\right)^2 - \omega^2} = \prod_n \frac{1}{1 - \left(\frac{\omega T}{n\pi}\right)^2} \xrightarrow{n\to\infty} \frac{\omega T}{\sin(\omega T)}, \tag{1.74}$$

where, in the last step, we took the continuum limit ($n \to \infty$). Using equations (1.49) and (1.73) we then obtain

$$J_\omega(T) = J_0(T)\sqrt{\frac{\omega T}{\sin(\omega T)}} = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}}\,, \tag{1.75}$$

which gives us our final result for the propagator of a quantum harmonic oscillator

$$K(q_f, t_f; q_i, t_i) = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}}\; e^{i\frac{m\omega}{2\sin(\omega T)}\left[(q_i^2+q_f^2)\cos(\omega T) - 2q_i q_f\right]}. \tag{1.76}$$

Note also, that by matching to the free particle case, we have fixed $\mathcal{N} = 1$. One can verify explicitly that in the limit $\omega \to 0$ the result above reduces to equation (1.49).

Exercise 1.5 An alternative method to obtain equation (1.75) is to use the fact that $J(T)$ defined in equation (1.59) is a transition amplitude from point $q_i = 0$ to point $q_f = 0$ due to the trivial boundary conditions on the fluctuation integral, that is $J(T) = K(0, t_f; 0, t_i)$. Based on this insight, use equations (1.10) and (1.58) to

obtain equation (1.75) up to an overall constant. Then fix the remaining constant by requiring that the result reduces to that for a free particle in the limit $\omega \to 0$.

1.4.4 Connection to our usual understanding of the quantum harmonic oscillator

Since the result does not seem immediately useful, I will note that the propagator obtained above gives the full spacetime evolution of the wavefunction for an arbitrary initial condition via equation (1.6). As it turns out, we can also extract the energy eigenvalues directly from equation (1.76). To obtain this result, we note that for a system with a constant energy, it is possible to also obtain the propagator in terms of the eigenfunction basis, that is

$$\begin{aligned} K(q_f, t_f; q_i, t_i) &= \langle q_f, t_f | q_i, t_i \rangle = \langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle \\ &= \sum_{m,n} \langle q_f | \psi_n \rangle \langle \psi_n | e^{-iH(t_f - t_i)} | \psi_m \rangle \langle \psi_m | q_i \rangle \\ &= \sum_n \psi_n^*(q_f) \psi_n(q_i) e^{-iE_n(t_f - t_i)}. \end{aligned} \tag{1.77}$$

Defining the *partition function*

$$Z(T) \equiv \int_{-\infty}^{\infty} dq\, K(q, T; q, 0) = \sum_n \left(\int_{-\infty}^{\infty} dq\, |\psi_n(q)|^2 \right) e^{-iE_n T} = \sum_n e^{-iE_n T}. \tag{1.78}$$

This result is general for any system with a time-independent energy. Computing the partition function from equation (1.76) we obtain

$$\begin{aligned} Z(T) &= \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} \int_{-\infty}^{\infty} dq\, e^{i \frac{m\omega}{\sin(\omega T)} [\cos(\omega T) - 1] q^2} \\ &= \sqrt{\frac{1}{2i}} \left(\frac{1}{\cos(\omega T) - 1} \right)^{1/2} \\ &= \frac{1}{2i} \frac{1}{\sin(\omega T/2)} = \frac{e^{-i\omega T/2}}{1 - e^{-i\omega T}} \\ &= \sum_n e^{-i(2n+1)\omega T/2}. \end{aligned} \tag{1.79}$$

Comparing equations (1.78) and (1.79) we find

$$E_n = \omega \left(n + \frac{1}{2} \right), \tag{1.80}$$

which is the correct result. Note that one can also extract the probability amplitudes for all states by Taylor expanding $K(q, -iT; q, 0)$ around $T = \infty$.

Exercise 1.6 Above I demonstrated that the transition amplitude $K(q_f, t_f; q_i, t_i)$ for the harmonic oscillator contains the full information about the energy spectrum. It also contains *all of the wavefunctions*! To see this, start from equation (1.76) and take $T = t_f - t_i = -i\tau = i \log(\varepsilon)/\omega$, $q_i = q_f = q$, and then Taylor-expand around $\varepsilon = 0$. Following this procedure, show that the coefficient of the $\mathcal{O}(\sqrt{\varepsilon})$ term gives the ground state probability distribution. Then extract the $n = 3$ and $n = 7$ states. I recommend using Mathematica for this exercise, otherwise, you will spend a great deal of time doing tedious and mind-numbing work. Finally, can you prove why this works?

1.4.5 Generalization to an arbitrary potential and the WKB approximation

The preceding discussion relied heavily on the fact that the potential was quadratic and hence the action split into two separate contributions, the classical action and the fluctuation action. In the case of a general potential, $V(q)$, equation (1.55) becomes instead

$$\begin{aligned} S[q, \dot{q}] = S[\bar{q}, \dot{\bar{q}}] + S_{\text{eff}}[\varepsilon, \dot{\varepsilon}, \bar{q}] + \int_{t_i}^{t_f} dt \frac{d}{dt}\left(m\varepsilon\frac{d\bar{q}}{dt}\right) \\ - \int_{t_i}^{t_f} dt\varepsilon\left(m\frac{d^2\bar{q}}{dt^2} + \frac{dV}{dq}\bigg|_{q=\bar{q}}\right), \end{aligned} \tag{1.81}$$

where S_{eff} is the effective action determined by

$$L_{\text{eff}} = \frac{1}{2}m\dot{\varepsilon}^2 - \frac{1}{2}\frac{d^2V}{dq^2}\bigg|_{q=\bar{q}} \varepsilon^2 + \mathcal{O}(\varepsilon^3)\,. \tag{1.82}$$

As before, the third and fourth terms in equation (1.81) vanish due to boundary conditions and the classical equations of motion. One can now ask when is it appropriate to throw out terms of $\mathcal{O}(\varepsilon^3)$ and higher in equation (1.82). Restoring $\hbar$ by going back to SI units, we first note that in the limit $\hbar \to 0$ one should obtain the classical case. In the path integral, the action enters as $S/\hbar$. The fluctuation ε contributes at leading order quadratically to the path integral and, as a result, the integral is dominated by contributions $\varepsilon \sim \sqrt{\hbar}$. As a result, it makes sense to rescale the fluctuation field

$$\varepsilon \to \sqrt{\hbar}\,\tilde{\varepsilon}\,, \tag{1.83}$$

such that the action divided by $\hbar$ has the form

$$\frac{S[q, \dot{q}]}{\hbar} = \frac{S^{(0)}[\bar{q}, \dot{\bar{q}}]}{\hbar} + S^{(2)}[\tilde{\varepsilon}, \dot{\tilde{\varepsilon}}, \bar{q}] + \sum_{n=3}^{\infty} \hbar^{n/2} S^{(n)}[\tilde{\varepsilon}, \dot{\tilde{\varepsilon}}, \bar{q}], \tag{1.84}$$

where the superscript '(n)' indicates truncation at nth-order in the fluctuation. Typically (but not always), the odd terms in the sum over n above vanish by symmetry, leaving a fluctuation contribution of the form

$$K(q_f, t_f; q_i, t_i) = \mathcal{N} e^{iS^{(0)}[\bar{q}, \dot{\bar{q}}]/\hbar} \int \mathcal{D}\tilde{\varepsilon} \exp\left(i \int_{t_i}^{t_f} dt\, L^{(2)}_{\text{eff}}(\tilde{\varepsilon}, \dot{\tilde{\varepsilon}}, \bar{q}) \right) \times (1 + \mathcal{O}(\hbar)). \tag{1.85}$$

If we stop at this order and ignore the terms of $\mathcal{O}(\hbar)$ and higher, this corresponds to the semiclassical WKB approximation[2]. Note that in the small $\hbar$ limit the classical contribution $\exp(iS^{(0)}[\bar{q}, \dot{\bar{q}}]/\hbar)$ oscillates wildly when the action changes from its minimum, while the remaining multiplicative factors depend smoothly on $\hbar$.

1.5 Calculating time-ordered expectation values

Given a Schrödinger picture operator $\mathcal{O}_S$, the time-dependent Heisenberg-picture operator is

$$\hat{O}_H = e^{iHt} \hat{O}_S e^{-iHt}. \tag{1.86}$$

For example, the generalized position operator in the Heisenberg picture is

$$\hat{q}(t) = e^{iHt} \hat{q} e^{-iHt}, \tag{1.87}$$

where I have dropped the subscripts H and S in favor of simply labeling the operators as functions of t or not. The time-dependent position basis vectors in the Heisenberg picture are obtained from the time-independent basis vectors in the Schrödinger picture via

$$|q(t)\rangle = e^{iHt} \,|\, q\rangle, \tag{1.88}$$

and are eigenstates of $\hat{q}(t)$

$$\hat{q}(t) \,|q(t)\rangle = q(t) \,|\, q(t)\rangle. \tag{1.89}$$

Next, consider a function $F(q)$ which defines an operator on a Hilbert space

$$\hat{F}(\hat{q}) \equiv \int dq |q\rangle\langle q| F(q). \tag{1.90}$$

[2] For the quantum harmonic oscillator this is exact since all higher order terms vanish identically and the semiclassical approximation is exact.

Sandwiching $\hat{F}$ between our intial and final states gives

$$\begin{aligned}
\langle q_f,\, t_f|\hat{F}(\hat{q}(t))|q_i,\, t_i\rangle &= \langle q_f|\mathrm{e}^{-iHt_f}\hat{F}(\hat{q}(t))\mathrm{e}^{iHt_i}|q_i\rangle \\
&= \langle q_f|\mathrm{e}^{-iH(t_f-t)}\hat{F}(\hat{q})\mathrm{e}^{-iH(t-t_i)}|q_i\rangle \\
&= \int dq\, \langle q_f|\mathrm{e}^{-iH(t_f-t)}\hat{F}(\hat{q})|q\rangle\langle q|\mathrm{e}^{-iH(t-t_i)}|q_i\rangle \\
&= \int dq\, F(q(t))\, \langle q_f|\mathrm{e}^{-iH(t_f-t)}|q\rangle\langle q|\mathrm{e}^{-iH(t-t_i)}|q_i\rangle \\
&= \int dq\, F(q(t))\, K(q_f,\, t_f;\, q,\, t)\, K(q,\, t;\, q_i,\, t_i) \\
&= \mathcal{N}^2 \int dq\, F(q(t)) \left(\int_{q_{a,i}=q}^{q_{a,f}=q_f} \mathcal{D}q_a\, \mathrm{e}^{iS[q_a,\, \dot{q}_a]} \right) \\
&\quad \times \left(\int_{q_{b,i}=q_i}^{q_{b,f}=q} \mathcal{D}q_b\, \mathrm{e}^{iS[q_b,\, \dot{q}_b]} \right),
\end{aligned} \tag{1.91}$$

where, as usual $|q,\, t\rangle$ stands for $|q(t)\rangle$. We first recognize that we can absorb q into the path integration, that is

$$\begin{aligned}
\mathcal{N}^2 \int dq \int \mathcal{D}q_a \int \mathcal{D}q_b &= \frac{1}{A}\left(\int \frac{dq}{A} \right) \left(\prod_{i=1}^{n} \int \frac{dq_i}{A} \right) \left(\prod_{j=1}^{n} \int \frac{dq_j}{A} \right) \\
&= \frac{1}{A} \prod_{k=1}^{2n+1} \int \frac{dq_k}{A} = \mathcal{N} \int \mathcal{D}q_c
\end{aligned} \tag{1.92}$$

where $A = \sqrt{2\pi i \Delta t/m}$ with $\Delta t = (t_f - t_i)/(2n+2)$ and the final path integration over q_c now covers from $q_i \to q_f$. Secondly, we see that the arguments of the exponentials of the action appearing in the last line of equation (1.91) simply add and result in a single integral over the Lagrangian that spans $t_i \to t_f$ which defines $S[q_c,\, \dot{q}_c] = S[q_a,\, \dot{q}_a] + S[q_b,\, \dot{q}_b]$. As a result, we have

$$\langle q_f,\, t_f|\hat{F}(\hat{q}(t))|q_i,\, t_i\rangle = \mathcal{N} \int \mathcal{D}q\, F(q(t))\, \mathrm{e}^{iS[q,\, \dot{q}]}, \tag{1.93}$$

where we have relabeled $q_c \to q$ in the end.

Now let us consider two functions $F(q(t_1))$ and $G(q(t_2))$ and work in reverse, this time glossing over the details concerning the normalization factors until the end. We start with

$$\langle FG\rangle \equiv \mathcal{N} \int \mathcal{D}q\, F(q(t_1))\, G(q(t_2))\, \mathrm{e}^{iS[q,\, \dot{q}]}. \tag{1.94}$$

If $t_1 < t_2$, we can use the composition property demonstrated above to obtain

$$\langle FG\rangle \propto \int \mathcal{D}q_a \mathcal{D}q_b \mathcal{D}q_c\, dq_1\, dq_2\, F(q(t_1))\, G(q(t_2))\, \mathrm{e}^{i(S[q_a,\, \dot{q}_a]+S[q_b,\, \dot{q}_b]+S[q_c,\, \dot{q}_c])}, \tag{1.95}$$

where we have broken the integration in to three segments $t_i < t_1 < t_2 < t_f$ with segments a, b, and c covering $t_i \to t_1$, $t_1 \to t_2$, and $t_2 \to t_f$, respectively. With this time ordering we can write

$$\begin{aligned}\langle FG\rangle \propto \int dq_1\, dq_2\, \langle q_f,\, t_f|q_2,\, t_2\rangle\, G(q_2(t_2))\, \langle q_2,\, t_2|q_1,\, t_1\rangle \\ F(q_1(t_1))\, \langle q_1,\, t_1|q_i,\, t_i\rangle \\ = \langle q_f|\mathrm{e}^{-iH(t_f-t_2)}\hat{G}(\hat{q}_2)\mathrm{e}^{-iH(t_2-t_1)}\hat{F}(\hat{q}_1)\mathrm{e}^{-iH(t_1-t_i)}|q_i\rangle \\ = \langle q_f,\, t_f|\hat{G}(\hat{q}_2(t_2))\hat{F}(\hat{q}_1(t_1))|q_i,\, t_i\rangle .\end{aligned} \tag{1.96}$$

If, instead, we have $t_1 > t_2$ then a similar exercise gives

$$\langle FG\rangle \propto \langle q_f,\, t_f|\hat{F}(\hat{q}_1(t_1))\hat{G}(\hat{q}_2(t_2))|q_i,\, t_i\rangle . \tag{1.97}$$

Combining the two results, we obtain the following nice result

$$\langle q_f,\, t_f|\,\mathrm{T}\,\hat{F}(t_1)\hat{G}(t_2)\,|q_i,\, t_i\rangle = \mathcal{N}\int \mathcal{D}q\, F(q(t_1))\, G(q(t_2))\, \mathrm{e}^{iS[q,\,\dot{q}]}, \tag{1.98}$$

where T is the familiar time-ordering operator

$$T\,\hat{O}_1(t_1)\hat{O}_2(t_2) = \theta(t_2 - t_1)\hat{O}(t_2)\hat{O}(t_1) + \theta(t_1 - t_2)\hat{O}(t_1)\hat{O}(t_2), \tag{1.99}$$

and we have restored the proper normalization factor. Note that we have suppressed the explicit appearance of q_1 and q_2 on the left-hand side of equation (1.98) in order to make the formula more compact.

As we can see, this proves that the path integral of the scalar version of our operators automatically gives the time-ordered expectation value! This is quite handy. As one might guess, this pattern continues as one adds more operators, with the general result being

$$\begin{aligned}\langle q_f,\, t_f|\,T\,\hat{O}_1(t_1)\hat{O}_2(t_2)\cdots\hat{O}_n(t_n)\,|q_i,\, t_i\rangle \\ = \mathcal{N}\int \mathcal{D}q\, O_1(q(t_1))\, O_2(q(t_2))\cdots O_n(q(t_n))\, \mathrm{e}^{iS[q,\,\dot{q}]}.\end{aligned} \tag{1.100}$$

This is a quite nice result since we see that the path integral automatically takes care of the time ordering for us.

Exercise 1.7 Show that equation (1.97) is correct.

1.6 Adding sources

Next I would like to introduce the method of adding sources. In physics, we frequently consider a system that is subject to a source, which is an external perturbation, and study the response of the system. We can do this in the path

integral formalism by making the system interact with an arbitrary external forcing source

$$L(q,\dot{q}) \to L(q,\dot{q}) + J(t)q(t)\,, \tag{1.101}$$

and define a generalized propagator that is a functional of the source

$$K(q_f,\,t_f;\,q_i,\,t_i|J) = \mathcal{N}\int \mathcal{D}q \exp\left(i\int_{t_i}^{t_f} dt[L(q,\dot{q}) + J(t)q(t)]\right). \tag{1.102}$$

The functional K contains *all information* about the system, which is evidenced by the fact that functional derivatives of K generate all time-ordered n-point functions

$$\begin{aligned}\frac{1}{i^n}\frac{\delta^n K}{\delta J(t_1)\delta J(t_2)\cdots\delta J(t_n)}\bigg|_{J=0} &= \mathcal{N}\int \mathcal{D}q\, q(t_1)\,q(t_2)\cdots q(t_n) \\ &\qquad \exp\left(i\int_{t_i}^{t_f} dt\, L(q,\dot{q})\right) \\ &= \langle q_f,\,t_f|\,T\,\hat{q}(t_1)\hat{q}(t_2)\cdots\hat{q}(t_n)|q_i,\,t_i\rangle.\end{aligned} \tag{1.103}$$

For more complicated operators, one can build them up as power series, for example

$$\begin{aligned}\langle q_f,\,t_f|\exp(\hat{q}(t_1))|q_i,\,t_i\rangle &= \sum_{n=0}^{\infty}\frac{1}{i^n n!}\frac{\delta^n K}{\delta J^n(t_1)}\bigg|_{J=0} \\ &= \mathcal{N}\int \mathcal{D}q\ \ \exp(q(t))\exp\left(i\int_{t_i}^{t_f} dt\, L(q,\dot{q})\right).\end{aligned} \tag{1.104}$$

1.7 Asymptotic states and vacuum–vacuum transitions

Looking forward to the next chapter where we will apply this methodology to quantum field theory, we now want to consider a slightly different setup, namely that we have states with no excitations (the system is in its ground state) at $t_i = -\infty$ and $t_f = \infty$ and that we turn on a source J for a finite time in the interval t and t'. Before taking the limit we introduce finite (but large) times T and T' which satisfy $-T < t < t' < T'$ and then, in the end, we will take $T \to \infty$. We are interested in evaluating

$$\langle Q',T'|Q,T\rangle^J = \mathcal{N}\int \mathcal{D}q\ \ \exp\left(i\int_{T}^{T'} dt[L(q,\dot{q}) + J(t)q(t)]\right), \tag{1.105}$$

where the superscript J indicates that we are calculating this in the presence of a source. Inserting two intermediate spacetime points

$$\langle Q',T'|Q,T\rangle^J = \int dq \int dq'\,\langle Q',\,T'|q',\,t'\rangle\langle q',t'|q,t\rangle^J\langle q,\,t|Q,\,T\rangle\,, \tag{1.106}$$

where we have made it explicit that the source only acts in the interval $t \to t'$. Looking at the first expectation value in the integral we have

$$\begin{aligned}\langle Q',\, T'|q',\, t'\rangle &= \langle Q'|\mathrm{e}^{-iHT'}\mathrm{e}^{iHt'}|q'\rangle \\ &= \sum_n \langle Q'|\mathrm{e}^{-iHT'}|\psi_n\rangle\langle\psi_n|\mathrm{e}^{iHt'}|q'\rangle \\ &= \sum_n \psi_n^*(q')\psi_n(Q')\mathrm{e}^{iE_n(t'-T')},\end{aligned} \tag{1.107}$$

where ψ_n are a complete set of energy eigenstates. Using the same method, we find

$$\langle q,\, t|Q,\, T\rangle = \sum_n \psi_n^*(Q)\psi_n(q)\mathrm{e}^{-iE_n(t-T)}. \tag{1.108}$$

Motivated by the discussion above, we would like to somehow single out only the ground state from each of these sums when we take $T \to -\infty$ and $T' \to \infty$. To accomplish this, we make a rotation of the time integration $T' \to T'\mathrm{e}^{-i\delta}$ and $T \to T\mathrm{e}^{-i\delta}$ where $\delta \leqslant \pi/2$ is an arbitrary angle (which will we take to 0^+ in the limiting sense if we require physical, Minkowski space results)[3]. With this rotation, we have, for example

$$\begin{aligned}\lim_{T'\to\infty}\langle Q',\, T'|q',\, t'\rangle &= \lim_{T'\to\infty}\sum_n \psi_n^*(q')\psi_n(Q')\mathrm{e}^{iE_n(t'-T')} \\ &= \lim_{T'\to\infty}\sum_n \psi_n^*(q')\psi_n(Q')\mathrm{e}^{iE_n(t'-T'\cos\delta+iT'\sin\delta)} \\ &= \psi_0^*(q')\psi_0(Q')\mathrm{e}^{iE_0(t'-T')},\end{aligned} \tag{1.109}$$

and, likewise, $\lim_{T\to-\infty}\langle q,\, t|Q,\, T\rangle = \psi_0^*(Q)\psi_0(q)\exp(-iE_0(t-T))$. Combining these results, the limiting form of equation (1.106) becomes

$$\begin{aligned}&\lim_{\substack{T\to-\infty\mathrm{e}^{-i\delta}\\ T'\to\infty\mathrm{e}^{-i\delta}}}\langle Q',T'|Q,T\rangle^J \\ &\quad= \psi_0^*(Q)\psi_0(Q')\mathrm{e}^{-iE_0(T'-T)}\int dq\int dq'\,\psi_0^*(q')\langle q',t'|q,t\rangle^J\psi_0(q).\end{aligned} \tag{1.110}$$

The double integral on the right-hand side takes an initial ground state at time t, propagates it to t', and then projects it back onto the ground state, hence it is the expectation value of the vacuum to vacuum transition amplitude. Solving for the double integral, we obtain

$$\int dq\int dq'\,\psi_0^*(q')\langle q',t'|q,t\rangle^J\psi_n(q) = \lim_{\substack{T\to-\infty\mathrm{e}^{-i\delta}\\ T'\to\infty\mathrm{e}^{-i\delta}}}\frac{\langle Q',\, T'|Q,\, T\rangle^J}{\psi_0^*(Q)\psi_0(Q')\mathrm{e}^{-iE_0(T'-T)}}. \tag{1.111}$$

[3] This maps to a counterclockwise rotation of the end points of the time-integration contour by δ.

Note that the left-hand side does not depend on Q or Q' and hence neither can the right-hand side. As a result, we can choose any Q and Q' we like. Let us choose $Q = Q' = 0$. We will return to this point soon, but for now we will just take this as a prescription. This gives

$$\int dq \int dq' \psi_0^*(q')\langle q',t'|q,t\rangle^J \psi_0(q) = \lim_{\substack{T\to-\infty e^{-i\delta} \\ T'\to\infty e^{-i\delta}}} \frac{\langle 0,\, T'|0,\, T\rangle^J}{\psi_0^*(0)\psi_0(0)e^{-iE_0(T'-T)}}. \tag{1.112}$$

The quantity in the denominator of the right-hand side is just a numerical factor. Based on this, we define the *vacuum to vacuum transition amplitude* $Z[J]$

$$Z[J] \propto \langle 0,\, \infty|0,\, -\infty\rangle^J \propto \lim_{\substack{T\to-\infty e^{-i\delta} \\ T'\to\infty e^{-i\delta}}} \langle 0,\, T'|0,T\rangle^J . \tag{1.113}$$

In terms of the path integral, we have

$$Z[J] = \mathcal{N}\int \mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt[L(q,\,\dot{q})+J(t)q(t)+(1/2)i\epsilon q^2]} \propto \langle 0,\, \infty|0,\, -\infty\rangle^J. \tag{1.114}$$

To normalize $Z[J]$ we require that, in the absence of sources, the ground state to ground state transition amplitude returns identity as it should, that is $Z[0] = \langle 0|0\rangle = 1$. This gives

$$Z[J] = \frac{\int \mathcal{D}q\, e^{i\int dt\,[L(q,\,\dot{q})+J(t)q(t)]}}{\int \mathcal{D}q\, e^{i\int dt\, L(q,\,\dot{q})}}. \tag{1.115}$$

Above, we have suppressed the explicit time limits and additional imaginary convergence factor. These should be implicit in what follows. Using the results obtained in the previous section, it should be clear at this point that functional derivatives of $Z[J]$ generate time-ordered vacuum expectation values. In general, one has

$$G(t_1,\, t_2,\, \cdots,\, t_n) = \langle 0|T\,\hat{q}(t_1)\hat{q}(t_2)\cdots\hat{q}(t_n)|0\rangle = \frac{1}{i^n}\frac{\delta^n Z}{\delta J(t_1)\delta J(t_2)\cdots\delta J(t_n)}\bigg|_{J=0}. \tag{1.116}$$

As a result of this identity, $Z[J]$ is referred to as the *generating functional* for n-point functions (correlations functions).

1.8 Generating functional and Green's function for quadratic theories

One of the nice things about the generating functional is that, since the boundaries in time are at $\pm\infty$, the integrals involved become easier to deal with. Before dealing with the actual path integral, let us first consider a d-dimensional quadratic integral of the form

$$I \equiv \int d^d x \, \mathrm{e}^{i(1/2A_{ij}x_i x_j + j_i x_i)} = \left(\det \frac{A}{2\pi i}\right)^{-1/2} \mathrm{e}^{-iG_{ij}j_i j_j/2}, \tag{1.117}$$

where $G = A^{-1}$ is, as we shall see, the analogue of the Green's function. For this integral, the analogue of the generating funcitonal is

$$Z_0(J) \equiv \frac{\int d^d x \, e^{i(1/2A_{ij}x_i x_j + J_i x_i)}}{\int d^d x \, e^{i/2A_{ij}x_i x_j}} = e^{-iG_{ij}J_i J_j/2}, \tag{1.118}$$

where the subscript 0 above is to emphasize the assumption of a quadratic form for the integral. From this, it follows that the '2-point function'

$$\langle x_1 x_2 \rangle = \frac{\int d^d x \, x_1 x_2 \, \mathrm{e}^{(i/2)A_{ij}x_i x_j}}{\int d^d x \, \mathrm{e}^{(i/2)A_{ij}x_i x_j}}, \tag{1.119}$$

is given by

$$\langle x_k x_\ell \rangle \equiv \frac{1}{i^2} \frac{\partial^2 Z_0(J)}{\partial J_k \partial J_\ell} \bigg|_{J=0}. \tag{1.120}$$

Using equation (1.117) we see that

$$\langle x_k x_\ell \rangle = iG_{k\ell}. \tag{1.121}$$

Therefore, the '2-point function' is the 'Green's function' of the integral I. Higher n-point functions (aka moments or correlation functions) can be calculated similarly. For this theory, the odd n-point functions are zero by symmetry. For the four-point function, for example, one finds

$$\langle x_k x_\ell x_m x_n \rangle = \langle x_k x_\ell \rangle \langle x_m x_n \rangle + \langle x_k x_m \rangle \langle x_\ell x_n \rangle + \langle x_k x_n \rangle \langle x_m x_\ell \rangle. \tag{1.122}$$

For a general $2n$-point function, one obtains a sum over all possible $(2n-1)!!$ pairings, that is

$$\langle x_1 \cdots x_{2n} \rangle = \sum_{P(x_1, \cdots, x_{2n})} \langle x_{i_1} x_{i_2} \rangle \cdots \langle x_{i_{2n-1}} x_{i_{2n}} \rangle \tag{1.123}$$

where $P(x_1, \cdots, x_{2n})$ represents all possible pairings. This is related to *Wick's theorem*, which we learned about in volume 1 in the context of canonical quantization.

1.8.1 Quantum harmonic oscillator

Switching now to the generating functional for the quantum harmonic oscillator, using equation (1.115) one finds in this case

$$Z[J] = \frac{\int \mathcal{D}q \ \exp\left(i\int dt\left[\frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 + Jq\right]\right)}{\int \mathcal{D}q \ \exp\left(i\int dt\left[\frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2\right]\right)} = \exp\left(-\frac{i}{m}\int dt \int dt' \, J(t)G(t, t')J(t')\right), \tag{1.124}$$

where $Z[J]$ is now a functional of the time-dependent source, the integration limits are understood to be from $\pm\infty$, and now G is really a proper Green's function since it is the functional inverse of the operator $-\partial_t^2 - \omega^2$, that is

$$\left(\frac{d^2}{dt^2} + \omega^2\right)G(t, t') = -\delta(t - t') \, . \tag{1.125}$$

Above, we have integrated by parts to change

$$\int_{-\infty}^{\infty} dt \, \frac{m}{2}\left[\left(\frac{dq}{dt}\right)^2 - \omega^2 q^2\right] = -\int_{-\infty}^{\infty} dt \, \frac{m}{2} q\left[\frac{d^2}{dt^2} + \omega^2\right]q, \tag{1.126}$$

where we have used the fact that $q_i = q_f = 0$ at infinity.

Recall that with the infinite time interval we have to introduce the $i\varepsilon$ prescription implicitly. For the harmonic oscillator with a time-independent ω, one has

$$G(t, t') = \frac{1}{2i\omega}\mathrm{e}^{-i\omega|t-t'|}. \tag{1.127}$$

Having computed the generating functional (1.124), we can use it to calculate the n-point functions. For example, the two-point function

$$\langle 0|Tq(t)q(t')|0\rangle = \frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J(t)\delta J(t')}\bigg|_{J=0} = \frac{i}{m}G(t, t') \, , \tag{1.128}$$

from which we see that the time-ordered *vacuum expectation value* $\langle 0|Tq(t)q(t')|0\rangle$ is a Green's function. Similarly, one can show that for quadratic theories the higher n-point functions can be expressed in terms of products of two-point functions.

Exercise 1.8 Show that equation (1.117) is correct.

Exercise 1.9 Show that equation (1.122) is correct.

Exercise 1.10 Show that equation (1.124) is correct.

1.9 Euclidean path integral and the statistical mechanics partition function

Instead of computing the transition amplitude $\langle q_f,\ t_f\ |q_i,\ t_i\rangle = \langle q_f|\exp(-iH(t_f - t_i)|q_i\rangle$ as before, we could instead compute the *imaginary time* evolution

$$K_E \equiv \langle q_f|\mathrm{e}^{-\beta H}|q_i\rangle \quad \beta \in \mathbb{R}^+, \tag{1.129}$$

where E denotes 'Euclidean', which I will explain further below. This corresponds to an imaginary time interval $t_f - t_i = -i\beta$. One can show that using the same time slicing procedure used previously, that the imaginary-time amplitude can be written as a path integral of the form

$$K_E(q_f,\ q_i;\ \beta) \equiv \mathcal{N}\int \mathcal{D}q_E\,\mathrm{e}^{-S_E}, \tag{1.130}$$

where

$$S_E[q,\ \dot{q}] = \int_0^\beta d\tau\left[\frac{1}{2}m\dot{q}^2 + V(q)\right], \tag{1.131}$$

with $\dot{q} = dq/d\tau$.

The imaginary-time amplitude is an analytic continuation of the normal (Minkowski space) transition amplitude to imaginary time

$$K_E(q_f,\ q_i;\ \beta) = K(q_f,\ q_i;\ -i\beta)\,. \tag{1.132}$$

Equation (1.129) can be obtained directly from equation (1.37) by a change of variables $t = -i\tau$, $dt = -id\tau$, and $dx/dt = idx/d\tau$. With this change of variables, the Minkowski spacetime interval $ds^2 = dt^2 - d\mathbf{x}^2 \to -ds_E^2$ with $ds_E^2 = d\tau^2 + dx^2$. This explains why we call K_E the Euclidean path integral.

Exercise 1.11 Using discretized Euclidean spacetime, show that equation (1.130) is correct.

1.9.1 Connection to statistical mechanics

Using the Euclidean form and inserting a complete set of orthonormal energy eigenstates as we did in equation (1.77), we obtain

$$K_E(q_f,\ q_i;\ \beta) = \sum_{n,m}\psi_n^*(q_f)\psi_m(q_i)\mathrm{e}^{-\beta E_n}. \tag{1.133}$$

As before we can define the partition function

$$Z(\beta) \equiv \int dq\,K_E(q,\ q;\ \beta) = \sum_n \mathrm{e}^{-\beta E_n}. \tag{1.134}$$

The right-hand side above is the statistical mechanical partition function for a system of bosons with temperature $T = 1/\beta$ ($k_B = 1$ in natural units). Hence, we observe the equivalence

$$Z(\beta) = \int dq\, K_E(q, q; \beta) = \mathcal{N} \int_{q(0)=q(\beta)} \mathcal{D}q_E\, \mathrm{e}^{-S_E}. \tag{1.135}$$

As a consquence, the thermal statistical partition function for bosons is equal to a functional integral over a compact Euclidean time: $\tau \in [0, \beta]$ with periodic boundary conditions. We will return to this connection later in volume 3, when we discuss finite temperature field theory.

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