Part 1

Foundation and Hands-on experiment of Fourier transform

IOP Publishing

Fourier Transform and Its Applications Using Microsoft EXCEL®
(Second Edition)

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Chapter 1

The principle of superposition and the Fourier series

There are numerous periodic motions—such as oscillations and waves—observed in nature. Differential equations describe their periodic motions in the time domain. They can also be analyzed in the frequency domain in order to acquire distributions of frequency components and their amplitudes because arbitrary oscillation patterns can be constructed with sinusoidal functions using the superposition principle. Such sets of sinusoidal functions are called the Fourier series, originating from a thermal conduction problem solved by Joseph Fourier.

In the first chapter of this book, we show how oscillations on a string and a membrane, and diffusion of heat and particles can be constructed using a set of possible solutions of their differential equations to introduce the Fourier series. We also describe how to construct a Fourier series of a given periodic function, and show examples of periodic functions in terms of their Fourier series.

1.1 The principle of superposition

Let us consider an oscillating string fixed at both ends to explain the important principle of periodic motion. A continuous train of sinusoidal waves is traveling back and forth to produce standing waves under an appropriate condition of tension on the string and its length. Figures 1.1–1.3 are snapshots of theses standing waves. Each of the standing waves corresponds to a normal mode of motion. Because the string is fixed at both ends, both ends must be nodes with motion. With the fixed boundary condition, adjacent nodes are one-half wavelength apart and the length of the string may be any integer number of one-half wavelengths. The frequency that gives the longest wavelength is called the first harmonic mode or the fundamental mode, and the others are integral multiples of the fundamental, called the higher harmonics.

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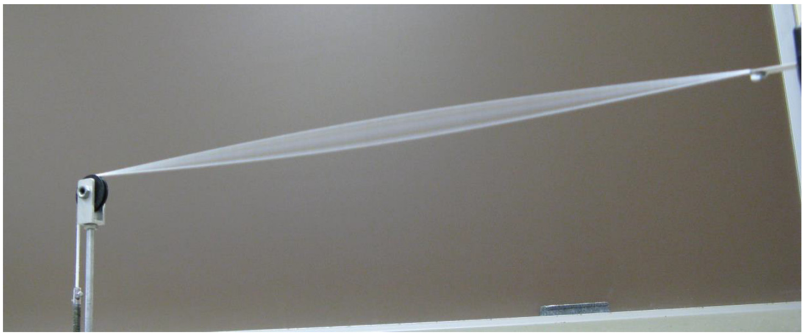

Figure 1.1. First harmonic (fundamental) mode.

Figure 1.2. Second harmonic mode.

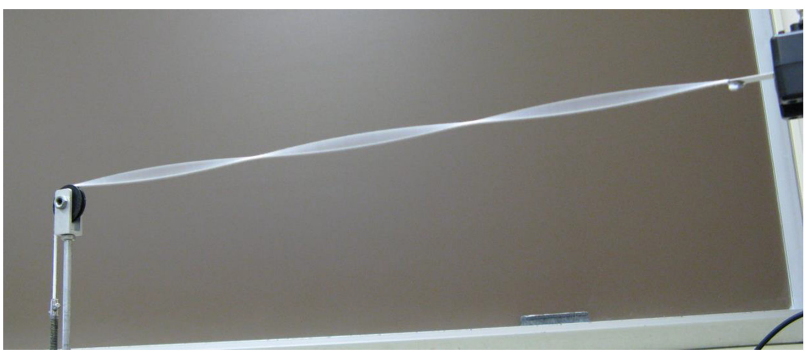

Figure 1.3. Third harmonic mode.

When a string is initially struck for a short time, the subsequent oscillating string will be described by a linear combination of normal modes. In other words, there are multiple motions of different frequencies on an arbitrarily oscillating string, and the displacement of the arbitrary point of the string is the algebraic sum of the wave displacements of propagation and reflection. This is called the principle of superposition. While it is a simple statement, it is an essential principle in periodic motion. Let us see how the superposition principle appears in mathematics.

1.2 One-dimensional standing wave

The normal modes are the solutions of the linear differential equation for the wave motion with a particular boundary condition

$$\frac{\partial^2 u(x,\, t)}{\partial t^2} = v^2\,\frac{\partial^2 u(x,\, t)}{\partial x^2} \tag{1.1}$$

where $u(x, t)$ is the displacement, $v = (F/\mu)^{1/2}$ is the wave speed, F is the tension in the string, and μ is line density of the string. By separating the variables, $u(x, t) = U(x)\Gamma(t)$, the equation for the x-coordinate, $U(x)$, becomes the Helmholtz equation whereas the one for time, $\Gamma(t)$, is an equation of harmonic oscillation:

$$\frac{d^2U(x)}{dx^2} + \lambda U\,(x) = 0 \text{ and } \frac{d^2\Gamma(x)}{dt^2}\lambda v^2\Gamma(t) = 0 \tag{1.2}$$

with the boundary condition of fixed ends, $U(0) = U(L) = 0$.

(1) Solutions of $U(x)$ are $\sin(n\pi x/L)$, where $\lambda_n = (n\pi/L)^2$ and $n = 1, 2, 3, \ldots$. By the superposition principle, a general solution or an arbitrary wave form with the same boundary condition is given by

$$U(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}\,x\right). \tag{1.3}$$

(2) The solution of $\Gamma(t)$ for a given λ_n is a harmonic oscillation and is given by

$$\Gamma_n(t) = C_n \sin\left[\left(\frac{n\pi}{L}\,v\right)t + \delta_n\right].$$

Therefore,

$$u(x,\, t) = \sum_{n=1} A_n \sin\left(\frac{n\pi}{L}\,x\right)\sin\left(\frac{n\pi v}{L}\,t + \delta_n\right) \tag{1.4}$$

where $A_n = a_n{\cdot}C_n$ and δ are determined by a given initial condition.

1.3 Fourier series

The previous section describes wave phenomena using linear differential equations where their solutions are given by sinusoidal periodic functions, and an arbitrary wave pattern on the string can be expressed by the superposition of sinusoidal harmonic modes. Likewise, we should be able express an arbitrary periodic function $f(\xi)$, where the variable ξ is a spatial coordinate or time, as a series of sinusoidal functions. This is called the Fourier series of $f(\xi)$. Because the sinusoidal functions are well-known and easy to apply, Fourier series are very useful for analyzing periodic motions such as wave phenomena.

1.3.1 Fourier theorem

In the following discussion, we use the periodicity of time. A periodic function $f(t)$ of period T $(-T/2 < t \leqslant +T/2)$ can be expressed by a Fourier series [1]:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \tag{1.5}$$

where $\omega_0 = 2\pi/T$ is the angular frequency of the fundamental mode, and the Fourier coefficients a_m and b_m are given by

$$a_m = \frac{2}{T}\int_{--T/2}^{T/2} f(t)\cos(m\omega_0 t)dt \;\text{ and }\; b_m = \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\sin(m\omega_0 t)dt. \tag{1.6}$$

The Fourier coefficients are calculated using the orthogonal property of sinusoidal functions.

Using the inner product, $\langle \ldots \mid \ldots \rangle$, defined by equation (1.17), we obtain

$$\langle \cos(m\omega_0 t) | \cos(n\omega_0 t)\rangle = \int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = 0 \; (m \neq n)$$

$$\langle \sin(m\omega_0 t) | \sin(n\omega_0 t)\rangle = \int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = 0 \; (m \neq n)$$

and

$$\langle \cos(m\omega_0 t) | \sin(n\omega_0 t)\rangle = \int_{-T/2}^{T/2} \cos(m\omega_0 t)\sin(n\omega_0 t)dt = 0 \; (\text{including } m = n).$$

Depending on the property of the original periodic function $f(t)$, a Fourier series may have only sine-terms or cosine-terms. If the function $f(t)$ is an *even* function in the interval $[-T/2, +T/2]$, the sine-terms must be excluded, and the Fourier series has only cosine-terms:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t). \tag{1.7}$$

Similarly, if the periodic function $f(t)$ is an *odd* function in the interval $[-T/2, +T/2]$, the Fourier series has only sine-terms:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t). \tag{1.8}$$

We can also obtain a Fourier series in a complex exponential form. By applying Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, the Fourier series (1.5) becomes

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{a_n}{2}[e^{in\omega_0 t} + e^{-in\omega_0 t}] + \sum_{n=1}^{\infty}\frac{b_n}{2i}[e^{in\omega_0 t} - e^{-in\omega_0 t}] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - ib_n)e^{in\omega_0 t} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n + ib_n)e^{-in\omega_0 t} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - ib_n)e^{in\omega_0 t} + \sum_{n=-1}^{-\infty}\frac{1}{2}(a_{-n} + ib_{-n})e^{-in\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}
\end{aligned}
\tag{1.9}
$$

$$
\text{where } c_n = \frac{1}{2}(a_n - ib_n) = \frac{2}{T}\int_{-T}^{T} f(t)\{\cos(n\omega_0 t) - i\,\sin(n\omega_0 t)\}dt = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega_0 t}dt \tag{1.10}
$$

for $n = 0,1,2,$ ….. For $n<0$, we can use the same equation (1.10) by defining $a_{-n} = a_n$ and $b_{-n} = b_n$. The complex Fourier coefficient represents the magnitude of the frequency component and the phase.

Instead of a periodic function in time, a Fourier series can be also applied to a periodic function of coordinates. The spatial periodicity can be observed through electromagnetic waves including optical interferences and diffractions, and wave packets of a particle. An arbitrary periodic function $f(x)$ in the interval $[-\pi, +\pi]$

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{1.11}
$$

where the coefficients $\{a_n; n = 0, 1, 2, 3, \ldots\}$ and $\{b_n; n = 1, 2, 3 \ldots\}$ are given by

$$
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{1.12}
$$

The Fourier series for the spatial interval $[-\pi, +\pi]$ can be changed to $[-L, +L]$ by using another variable $\xi = (L/\pi)x$ or $x = (\pi/L)\xi$

$$
f(\xi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\xi\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\xi\right) \tag{1.13}
$$

where

$$
a_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\left(\frac{n\pi}{L}\xi\right)d\xi \;\text{ and }\; b_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi. \tag{1.14}
$$

If we use complex functions, Fourier series (1.13) becomes

$$
f(\xi) = \sum_{n=-\infty}^{\infty} c_k e^{i\pi n\xi/L} \tag{1.15}
$$

$$
\text{where } c_n = \frac{1}{2L}\int_{-L}^{L} f(\xi)\exp\left(-i\frac{n\pi}{L}\xi\right)d\xi. \tag{1.16}
$$

Example 1: the Fourier series of a square wave. The first example of the Fourier series is a square wave train:

$$f(x) = -1 \text{ if } -\pi < x < 0; \text{ and } +1 \text{ if } 0 < x < \pi.$$

The graph of this square wave train for $x > 0$ is shown in figure 1.4. The Fourier series of the square wave train is given by:

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin[(2n-1)x] = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\right).$$

Figure 1.5 is the actual Fourier series of the first 10 terms—except the multiplication factor π/4—by iterative summation using EXCEL®. Its macro source program using the built-in Visual Basic for Applications (VBA) is listed in the appendix, A13.1. If readers are not familiar with EXCEL VBA macro, refer to A12.3 in the appendix.

If we calculate the sum of 100 terms, the Fourier series gets much closer to the square wave train with noticeable oscillations at the rising and falling edges (figure 1.6).

Remark: Gibbs phenomenon. The fine oscillations do not disappear, especially at the edges, even if the Fourier series takes many more terms. This overshoot behavior occurs at a jump discontinuity, and it is called the Gibbs phenomenon [2]. The size of the overshoot is tuned out to be proportional to the magnitude of the discontinuity. For the square wave train, the maximum peak value of the partial sum approaches approximately $\frac{2}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi \approx 1.1789$.

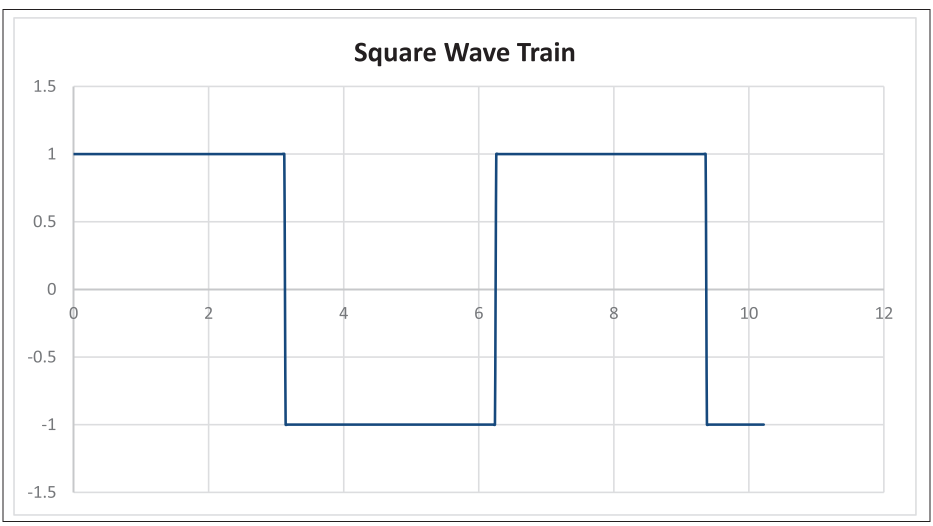

Figure 1.4. Square wave train.

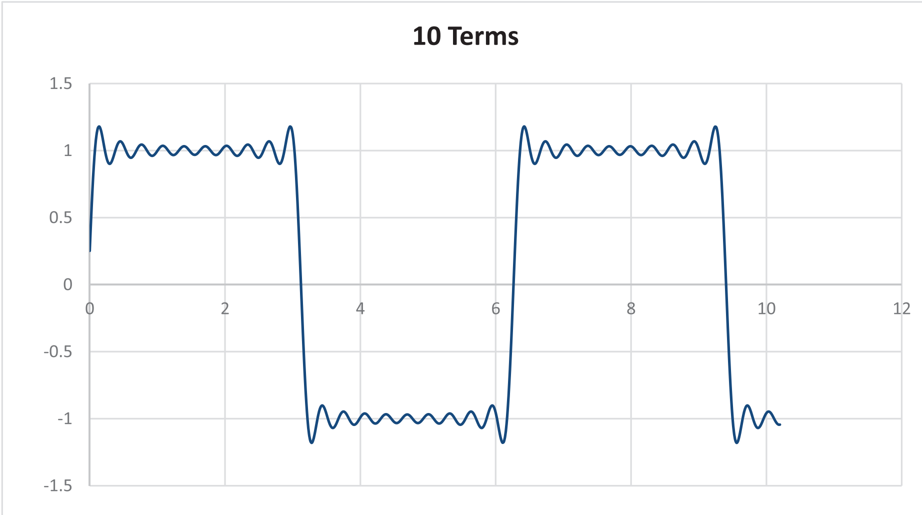

Figure 1.5. Fourier series of square wave train (10 terms).

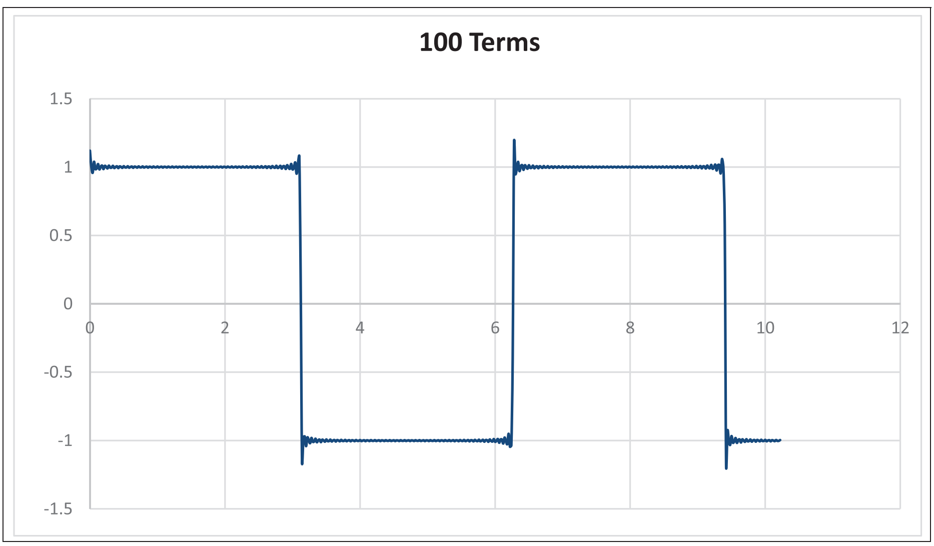

Figure 1.6. Fourier series of square wave train (100 terms).

Its calculation offers a good mathematical exercise and readers should try it. Refer to appendix A1 for a detailed calculation.

Example 2: the Fourier series of a sawtooth wave. The sawtooth wave is a repetition of the function $f(t) = x$ for $-\pi < x < +\pi$, and the period is 2π. Figure 1.7 shows this signal for $x \geqslant 0$. The Fourier series of the above sawtooth wave is

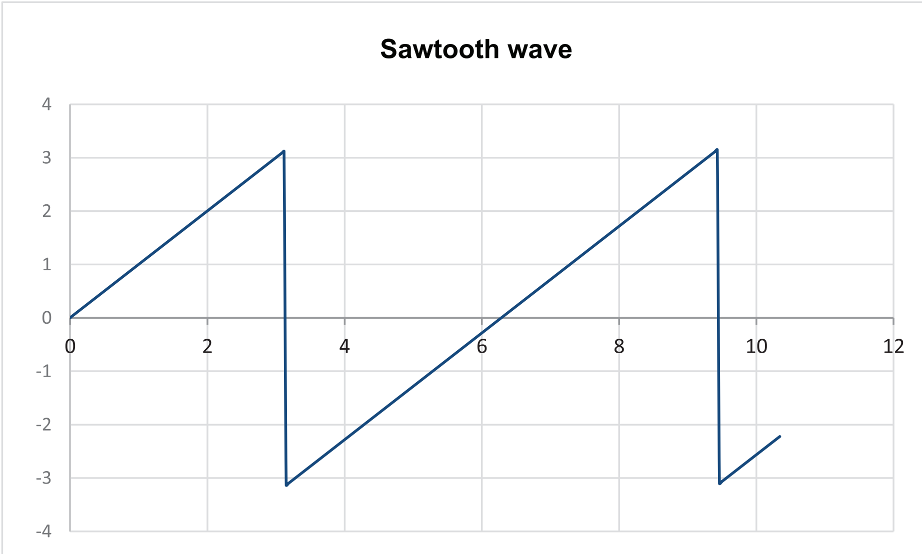

Figure 1.7. Sawtooth wave.

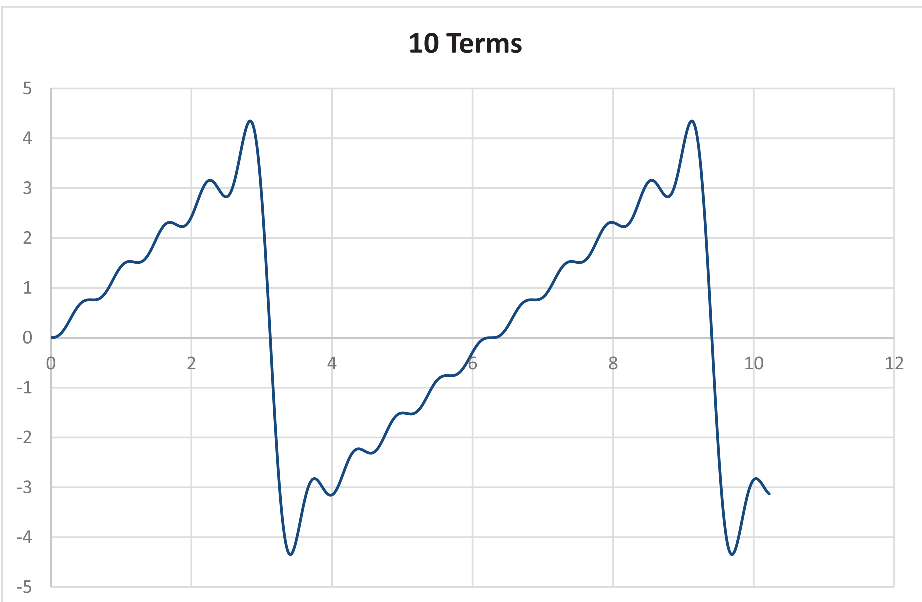

Figure 1.8. Fourier transform of sawtooth wave (10 terms).

$$f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + -\cdots\right].$$

The Fourier series up to 10 terms and 100 terms are shown in figures 1.8 and 1.9, respectively. The Gibbs phenomenon is also noticeable in this Fourier series.

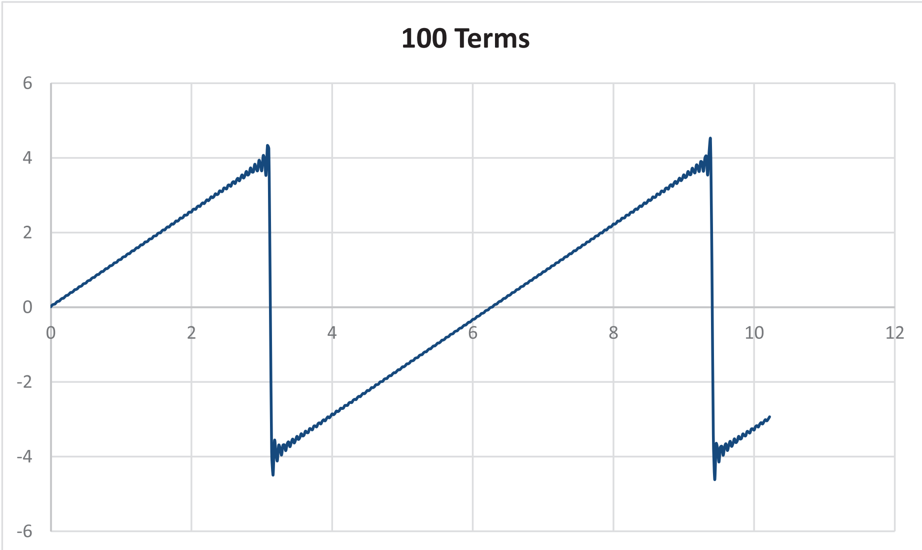

Figure 1.9. Fourier transform of sawtooth wave (100 terms).

1.4 Orthonormal basis

The concept of orthonormal basis is the foundation of the superposition principle and the theory of the Fourier series. Let us consider an n-dimensional Euclidian vector space for an intuitive discussion of the orthonormal basis. In this space, an arbitrary vector can be expressed as a linear combination of unit vectors:

$$\vec{v} = v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_n\vec{e_n} \tag{1.17}$$

where $\{v_j;\ j = 1, 2, \ldots., n\}$ are the components of the vector $\vec{v}$, and $\left\{\vec{e}_j; j = 1, 2, \ldots, n\right\}$ form a set of unit vectors associated with the given Cartesian coordinate frame. The component v_j is given by the inner product of the vector and the unit vector

$$v_i = \langle\vec{v}|\vec{e}\rangle = \sum_{j=1}^{n} v_j\langle\vec{e_j}|\vec{e_i}\rangle \tag{1.18}$$

because the unit vectors are orthonormal: $\langle\vec{e_j}|\ \vec{e_i}\rangle = \delta_{\text{ij}}$ where δ_{ij} is the Kronecker's delta, i.e., $\delta_{\text{ij}} = 1$ if $i = j$ and 0 otherwise. Furthermore, there are no other additional unit vectors required to express an arbitrary vector in the n-dimensional space. Thus, the unit vectors $\{\vec{e_i}\ ;\ i = 1, 2, \ldots, n\}$ form a complete orthonormal basis of the vector space.

As discussed in section 1.2, the standing wave equation has sinusoidal solutions as the normal modes in the given Cartesian coordinates. Because sinusoidal functions are orthogonal, for the string oscillation, the normal modes can be regarded as the unit vectors and an arbitrary string wave can be expressed in the form of the linear combination (superposition) of the normal modes, constituting the Fourier series. This is what equation (1.3) implies.

A time-dependence of temperature or particle diffusion in a rod computed from a heat or diffusion equation are other examples where we can apply Fourier series. Let us discuss an example below.

1.5 Heat and diffusion equations

Consider the following heat transfer or particle diffusion problem given by the heat equation:

$$\frac{\partial u(x,\, t)}{\partial t} = \kappa \frac{\partial^2 u(x,\, t)}{\partial x^2} \tag{1.19}$$

where $u(x,\, t)$ is the temperature distribution and κ is the thermal diffusivity. For particle diffusion, $u(x,\, t)$ is the particle density distribution and κ is the diffusion constant. The spatial part, $X(x)$, of the above equations also becomes the Helmholtz equation (1.2).

Let us find the solution of the heat equation with the following initial and the boundary conditions:

(i) initial condition: $u(x,\, 0) = f(x)$ for $0 < x < L$, and $u(0,0) = u(L,\, 0) = 0$; and
(ii) boundary condition: $u(0,\, t) = u(L,\, t) = 0$.

By setting $u(x,\, t) = \Gamma(t)X(x)$ to separate the variables x and t, we obtain $X'(x) + \lambda X(x) = 0$ and $\Gamma'(t) + \lambda\kappa\Gamma(t) = 0$ where λ is the separation constant.

(1) $d^2X(x)/dx^2 + \lambda X(x) = 0$: using the boundary condition, $U(0) = U(L) = 0$, we obtain
$X_n(x) = c_1 \sin(\frac{n\pi}{L}x)$ where c_1 is a constant, and $\lambda_n = \frac{n\pi}{L}$ where $n = 1, 2, 3, \ldots$
(2) $d\Gamma(t)/dt + \lambda\kappa\Gamma(t) = 0$: the solution is $\Gamma_n(t) = c_2 \exp[-D(\frac{n\pi}{L})^2 t]$ for each λ_n.

Thus, the temperature distribution, $u(x,\, t)$, is given by

$$u(x,\, t) = \sum_n \Gamma_n(t)X_n(x) = \sum_n A_n \exp\left[-\kappa\left(\frac{n\pi}{L}\right)^2 t\right]\sin\left(\frac{n\pi}{L}x\right) \tag{1.20}$$

where $A_n = c_1c_2$.

Now, we can include the initial condition, $u(x,\, 0) = f(x)$ by using the Fourier series

$$u(x,\, 0) = f(x) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right) \tag{1.21}$$

where the coefficient A_n is given by $A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L})dx$. Therefore, the complete solution becomes

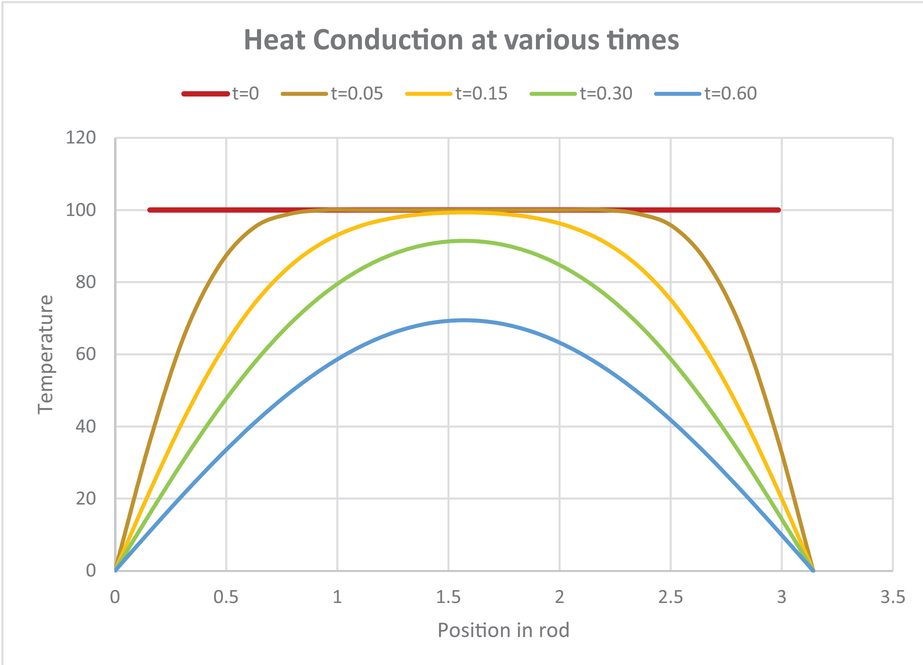

Figure 1.10. Temperature distribution in a rod.

$$u(x, t) = \frac{2}{L}\sum_{n}\left[\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)dx\right]\exp\left[-\kappa\left(\frac{n\pi}{L}\right)^2 t\right]\sin\left(\frac{n\pi}{L}x\right). \tag{1.22}$$

Figure 1.10 shows the time-dependence of the temperature distribution in a rod of length $L = \pi$, where both ends ($x = 0$ and π) are in contact with heat reservoirs of zero temperature. The rod is at a uniform temperature $f(x) = 100$ for $0<x<L$ at $t = 0$.

1.6 Two-dimensional standing wave and two-dimensional Fourier series

Two-dimensional standing wave: this is similar to the standing wave on a string discussed in section 1.2, the standing wave on a membrane is an example of a two-dimensional Fourier series. Let us discuss how to obtain two-dimensional standing waves. The wave equation on a membrane is

$$\frac{\partial^2 u(x, y, t)}{\partial t^2} = v^2\left[\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2}\right]. \tag{1.23}$$

The solution of equation (1.23) can be obtained by the variable separation. Let $u(x, y, t) = X(x)Y(y)\Gamma(t)$, then equation (1.23) becomes

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{v^2}\frac{\Gamma''(t)}{\Gamma(t)} \tag{1.24}$$

and we have

$$\frac{X''(x)}{X(x)} = -k_x^2 \text{ and } \frac{Y''(y)}{Y(y)} = -k_y^2. \tag{1.25}$$

Assume the following initial and boundary conditions for a rectangular membrane:

(i) initial condition: $u(x, y, 0) = f_1(x, y)$ and $(\partial u/\partial \mathrm{t})_{\mathrm{t}=0} = f_2(x, y)$; and
(ii) boundary condition: $u(0, y, t) = u(a, y, t) = 0$ and $u(x, 0, t) = u(x, b, t) = 0$.

The solutions of equation (1.24) including the boundary condition are

$$X_n(x) = A_n \sin\left(\frac{m\pi}{a}x\right) \text{ where } k_x = \frac{m\pi}{a} \text{ and } m = 1, 2, 3, \ldots; \tag{1.26}$$

$$Y_n(y) = B_n \sin\left(\frac{n\pi}{b}y\right) \text{ where } k_x = \frac{n\pi}{a} \text{ and } n = 1, 2, 3, \ldots;$$

and the time part is a harmonic oscillation given by

$$\frac{d^2\Gamma(t)}{dt^2} + \omega^2\Gamma(t) = 0 \text{ where } \omega^2 = v^2\left(k_x{}^2 + k_y{}^2\right). \tag{1.27}$$

Because possible k_x and k_y values are discrete, ω is also discrete. We obtain

$$\omega_{m,n}^2 = v^2\pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right], \text{ and thus } \Gamma_{m,n}(t) = C_{m,n}\sin(\omega_{m,n}t + \varepsilon_{m,n}). \tag{1.28}$$

The general solution is now given by

$$\begin{aligned} u_{m,n}(x, y, t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m(x)\, Y_n(y)\Gamma_{m,n}(t) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{m,n}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin(\omega_{m,n}t + \varepsilon_{m,n}) \end{aligned} \tag{1.29}$$

where $\{D_{m,n}\} = \{A_m B_n C_{m,n};\ m = 1,2,3, \ldots, \text{ and } n = 1,2,3,\ldots\}$ and $\varepsilon_{m,n}$ are determined by the initial condition. The spatial part is a two-dimensional (sine) Fourier series. Figure 1.11 shows the spatial part of $u_{3,2}$ with $a = b = 3.15$. Refer to appendix A13.3 for how to draw this 3D chart using EXCEL.

Note: a popular demonstration of standing waves on a Chladni plate has the boundary conditions of free ends and it is a forced oscillation because the plates are driven at the center vertically. Therefore, the resonant modes of a Chladni plate are different from equation (1.29). Refer to [3] for more information on Chladni plates.

Two-dimensional Fourier series: equation (1.29) is an example of a two-dimensional Fourier series. The two-dimensional Fourier series of an arbitrary periodic function $f(x,y)$ can be formulated in the following way. First, construct a Fourier series with respect to x whence coefficients are function of y. Second, construct

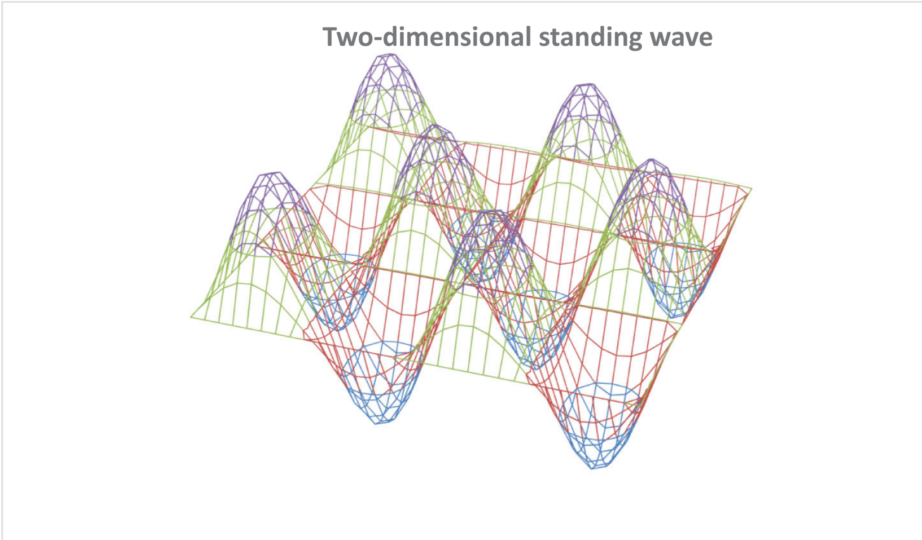

Figure 1.11. Two-dimensional standing wave on a membrane.

another Fourier series with respect to y from the coefficients. For example, from the one-dimensional complex Fourier series (1.13), we can formulate two-dimensional complex Fourier series:

$$f(x,\, y) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} c(k,\, \ell) \exp\left[i\pi \left(\frac{kx}{L_x} + \frac{\ell y}{L_y} \right) \right] \tag{1.30}$$

where $c(k,\, \ell) = \dfrac{1}{4L_xL_y} \displaystyle\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \, \exp\left[-i\pi \left(\frac{kx}{L_x} + \frac{\ell y}{L_y} \right) \right]$, $k,\, \ell = 0,\, \pm 1,\, \pm 2,\, \ldots$

Because we do not use the two-dimensional Fourier series, refer to [4] for a general discussion of multi-dimensional Fourier series as necessary.

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[1] Stewart J 2016 *Fourier Series—Stewart Calculus* (Boston, MA: Cengage Learning)

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