IOP Publishing

Electroweak Baryogenesis (Second Edition)
An introduction
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Chapter 10

Transport equations

We now have enough information to build a network of transport equations for a given model during a phase transition. In chapter 6 we derived how to relate the divergences of particle number current densities to functions of the particle's self-energies. In the previous chapter we derived the thermal masses and thermal widths which are inputs for the transport coefficients and CP violating sources. We also derived diffusion coefficients and bubble wall velocities as they will be useful in relating the divergence of the current density to derivatives in a single space–time variable of the number densities. In this section we finally put this all together in the MSSM. As we will discuss in chapter 14, electroweak baryogenesis in the minimal supersymmetric Standard Model (MSSM) is either dead or at least on life support. Even still, the MSSM contains scalar and fermionic sources as well as the knowledge of tricks used in many other models. Thus, the MSSM retains a status as being the most useful test model for pedagogical purposes.

10.1 The MSSM under supergauge equilibrium

A full calculation of charge transport during the electroweak transition requires solving a set of coupled differential equations with one equation per particle species (treating left doublets and right handed particles as separate species). We can immediately make a simplification by assuming only the top Yukawa has interactions fast enough amongst fermions to be numerically important. Second, we can assume that gauge and supergauge interactions are fast enough to allow us to consider the number densities of gauge singlets, summed with their superpartners as a single number density. Furthermore, there are some fast interactions that allow us to assume local equilibrium between the two Higgs doublets. Then, our network of transport equations only involve three number densities

doi:10.1088/978-0-7503-3571-3ch10

$$\begin{aligned} Q &= n_{t_L} + n_{b_L} + n_{\tilde{t}_L} + n_{\tilde{b}_L} \\ T &= n_{t_R} + n_{\tilde{t}_R} \\ H &= n_{H_u^+} + n_{H_u^0} - n_{H_d^-} - n_{H_d^0} n_{\tilde{H}_u^+} + n_{\tilde{H}_u^0} - n_{\tilde{H}_d^-} - n_{\tilde{H}_d^0}. \end{aligned} \tag{10.1}$$

We can relate chemical potentials to number densities using an assumption of local thermal equilibrium. Ignoring terms of $O(\mu^3/T^3)$, that is, performing a high temperature expansion, we can derive the relation

$$\mu_x = \frac{6}{T^2}\frac{n_x}{k_x} \tag{10.2}$$

with

$$k_x = k_x(0)\frac{c_{F,\,B}}{\pi^2}\int_{m/T}^{\infty} \mathrm{d}yy\frac{\mathrm{e}^y}{(\mathrm{e}^y \pm 1)^2}\sqrt{y^2 - m_x^2/T^2}, \tag{10.3}$$

where $c_{F(B)} = 6(3)$ and the sign in the denominator is $\pm$ for fermions and bosons, respectively. The factors $k_i(0)$ are 2 for Dirac fermions and complex scalars and 1 for chiral fermions. The k factors of our composite charge densities in equation (10.3) are the sum of the k factors for each component. Combining our number densities together in this way means we also combine the vacuum expectation value (VEV) insertion diagrams of tops and stops as well as Higgs and Higgsinos to each form a singlet transport coefficient:

$$\Gamma_M^{\pm} = \Gamma_M^{\pm}\tilde{t} + \Gamma_M^{\pm} t \Gamma_H^{\pm} = \Gamma_M^{\pm}\tilde{H}^{\pm} + \Gamma_M^{\pm}\tilde{H}^0. \tag{10.4}$$

Furthermore, all one-loop three-body rates involving t, Q and H and their superpartners, as cataloged in chapter 6, sum into a single three-body rate, Γ_Y, for the linear combinations of composite densities $\mu_t - \mu_H - \mu_Q$. The set of coupled differential equations is

$$\begin{aligned} \partial_\mu T^\mu = {} & \Gamma_M^+\left(\frac{T}{k_T} + \frac{Q}{k_Q}\right) - \Gamma_M^-\left(\frac{T}{k_T} - \frac{Q}{k_Q}\right) - \Gamma_Y\left(\frac{T}{k_T} - \frac{H}{k_H} - \frac{Q}{k_Q}\right) \\ & + \Gamma_{SS}\left(\frac{2Q}{k_Q} - \frac{T}{k_T} + \frac{9(Q+T)}{k_B}\right) + S_{\tilde{t}}^{C/P} \end{aligned} \tag{10.5}$$

$$\begin{aligned} \partial_\mu Q^\mu = {} & -\Gamma_M^+\left(\frac{T}{k_T} + \frac{Q}{k_Q}\right) - \Gamma_M^+\left(\frac{T}{k_T} - \frac{Q}{k_Q}\right) + \Gamma_Y\left(\frac{T}{k_T} - \frac{H}{k_H} - \frac{Q}{k_Q}\right) \\ & -2\Gamma_{SS}\left(\frac{2Q}{k_Q} - \frac{T}{k_T} + \frac{9(Q+T)}{k_B}\right) - S_{\tilde{t}}^{C/P} \end{aligned} \tag{10.6}$$

$$\partial_\mu H^\mu = -\Gamma_H\frac{H}{k_H} + \Gamma_Y\left(\frac{T}{k_T} - \frac{Q}{k_Q} - \frac{H}{k_H}\right) + S_{\tilde{H}}^{C/P}, \tag{10.7}$$

where the strong sphaleron rate is taken to be $\Gamma_{SS} \approx 16\alpha_S^4 T$ [1].

10.2 Solution using fast rates, diffusion approximation, and ultrathin wall approximations

Now we have a network of coupled transport equations we will outline the simplest approach to solving them. Unfortunately, the approach produces a number for the baryon asymmetry that disagrees with more complete methods by up to two orders of magnitude. Nonetheless, it is useful for pedagogical purposes, but we will not go to lengths to justify the approximation we use as they can be viewed as a stepping stone to more complete methods.

From the solution to the network of coupled transport, equations manifest particle–anti-particle asymmetries that diffuse into the symmetric phase ahead of the bubble wall, biasing electroweak sphalerons. The expanding bubble wall will capture some of the net baryon asymmetry. The first step in finding an approximate solution to the network of transport equations is to convert the left-hand sides of the network of transport equations into derivatives of number densities in a single space–time variable. This is achieved by first using the diffusion approximation that we introduced earlier

$$\partial_\mu J^\mu = v_W \dot{n} - D_n \nabla^2 n, \tag{10.8}$$

then making an assumption that the bubble wall is symmetric and we can ignore the curvature of the bubble wall, so that we can reduce the problem to a one-dimensional one. Shifting to the bubble wall rest frame using the variable $z \equiv | v_w t - x |$, we can write the set of coupled transport equations as a set of differential equations in z, whose left-hand sides look like $v_w n_x' - D_x n_x''$. Next we assume both Γ_Y and Γ_{SS} are fast enough such that the linear combination of number densities that these rates are coefficients for are in local equilibrium. That is,

$$\begin{aligned} \frac{T}{k_T} - \frac{H}{k_H} - \frac{Q}{k_Q} &= 0 \\ \frac{2Q}{k_Q} - \frac{T}{k_T} + \frac{9(Q+T)}{k_B} &= 0. \end{aligned} \tag{10.9}$$

The second equation can be used to solve T in terms of Q and both can be solved in terms of H using the first equation. The result is,

$$\begin{aligned} Q &= \frac{(k_B - 9k_T)k_Q}{(9k_T + 9k_Q + k_B)k_H} H \\ T &= \frac{(9k_T + 2k_B)k_T}{(9k_T + 9k_Q + k_B)k_H}. \end{aligned} \tag{10.10}$$

We have reduced our set of transport equations down to a single one. The remaining linearly independent transport equation is any that does not have any fast rate on the right-hand side. One such combination is 2 × (10.5) + (10.6) + (10.7). In this case the transport equation is

$$v_w H'(z) - \bar{D} H''(z) + \bar{\Gamma} H(z) = S(z), \tag{10.11}$$

where

$$
\begin{aligned}
\bar{D} &= \frac{(9k_Qk_T + k_Bk_Q + 4k_Tk_B)D_q + k_H(9k_T + 9k_Q + k_B)D_h}{9k_Qk_T + k_Bk_Q + 4k_Tk_B + k_H(9k_Q + 9k_T + k_B)} \\
\bar{\Gamma} &= \frac{(9k_Q + 9k_T + k_B)(\Gamma_M^- + \Gamma_H) - ((3k_B + 9k_Q - 9k_T)\Gamma_M^+)}{9k_Qk_T + k_Bk_Q + 4k_Tk_B + k_H(9k_Q + 9k_T + k_B)} \\
S(z) &= \frac{k_H(9k_Q + 9k_T + k_B)}{9k_Tk_Q + k_Bk_Q + 4k_Tk_B + k_H(9k_Q + 9k_T + k_B)}\left(S_{\tilde{t}} + S_{\tilde{H}}\right).
\end{aligned} \tag{10.12}
$$

Next we make the ultrathin wall approximation where the Higgs profile is assumed to be close to a step function, meaning $S(z)$ and $\bar{\Gamma}$ are zero in the symmetric phase, i.e. when $z > 0$. Furthermore, the functions $\bar{\Gamma}$ become constant in the broken phase in this approximation. This linearizes the remaining transport equations and allows us to solve it in two regions using the method of variable coefficients. The solution in the symmetric phase, up to yet-to-be-determined integration constants that are specified by the boundary and matching conditions, is

$$H(z) = A e^{v_w z/\bar{D}} + C \tag{10.13}$$

and the solution in the broken phase is

$$
\begin{aligned}
H(z) = &\frac{e^{\kappa_+ z}}{\bar{D}(\kappa_+ - \kappa_-)}\left(\int_0^z dy e^{-\kappa_+} S(y) - \beta_+\right) \\
&- \frac{e^{\kappa_- z}}{\bar{D}(\kappa_+ - \kappa_-)}\left(\int_0^z dy e^{-\kappa_-} S(y) - \beta_-\right).
\end{aligned} \tag{10.14}
$$

Here the exponentials solve the homogeneous equations and the exponents have the form

$$\kappa_\pm = \frac{v_w \pm \sqrt{v_w^2 + 4\bar{\Gamma}\bar{D}}}{2\bar{D}} \tag{10.15}$$

We have two boundary conditions in that the densities must go to zero at $\pm\infty$ which sets C to zero and β_+ to

$$\beta_+ = \int_0^\infty dy e^{-\kappa_+ y} S(y). \tag{10.16}$$

The other two integration constants are determined by the fact that both $H(z)$ and $H'(z)$ are both continuous at the boundary, $z = 0$. The matching conditions are then

$$
\begin{aligned}
A &= \frac{\beta_- - \beta_+}{\bar{D}(\kappa_+ - \kappa_-)} \\
A(\kappa_+ + \kappa_-) &= \frac{\beta_-\kappa_- - \beta_+\kappa_+}{\bar{D}(\kappa_+ - \kappa_-)}.
\end{aligned} \tag{10.17}
$$

Solving the above equations gives

$$\begin{aligned}\beta_- &= \beta_+\frac{\kappa_-}{\kappa_+} = \frac{\kappa_-}{\kappa_+}\int_0^\infty \mathrm{d}y \mathrm{e}^{-\kappa_+ y} S(y)\\ A &= -\frac{\beta_+}{\kappa_+} = \frac{1}{\bar{D}\kappa_+}\int_0^\infty \mathrm{d}y \mathrm{e}^{-\kappa_+ y} S(y).\end{aligned} \tag{10.18}$$

10.3 Solution without fast rates

Here we will now resolve the transport equations without making the assumption that the rates are fast. If we otherwise use the same assumptions as before we once again have a set of linearized differential equations which gives us hope for a semi-analytic solution. Note the structure of the transport equations—if we take the combination $\partial_\mu(T + Q)^\mu$ we have a transport equation that depends only on two number densities. If we can create another linear combination that is a function of three number densities only, then the first equation can solve T in terms of Q, the second can solve for H in terms of Q, and the final will have the CP violating source. To create a linear combination of transport equations that is a function of number densities T, Q and H only, we note that the two CP violating sources are proportional to each other

$$S_{\tilde{t}}^{C/P} = \frac{1}{2a} S_{\tilde{H}}^{C/P} \tag{10.19}$$

The set of equations we want is therefore the linear combinations (10.5) + (10.6), (1 $+a$) × (10.5) +(1−a) × (10.6) + (10.7) and 2 × (10.5) + (10.6) + (10.7).

10.4 Deriving the analytic solution

The problem in deriving an analytic solution quickly becomes messy due to all the k factors and transport coefficients. It is a great simplification to rewrite all transport coefficients in the form $a_{Xj}{}^i$, where $j \in \{1,2,3\}$ is the equation number, $X \in \{Q,T,H\}$ is the field index, and $i \in \{0,1,2\}$ is the power of the derivative in z

$$a_{Q1}^i \partial^i Q + a_{T1}^i \partial^i T = 0 \tag{10.20}$$

$$a_{Q2}^i \partial^i Q + a_{T2}^i \partial^i T + a_{H2}^i \partial^i H = 0 \tag{10.21}$$

$$a_{Q3}^i \partial^i Q + a_{T3}^i \partial^i T + a_{H3}^i \partial^i H = \Delta(z). \tag{10.22}$$

Here of we have used Einstein's summation convention. To solve the first equation for T in terms of Q, use the method of variable coefficients treating Q as an inhomogeneous source. Doing this gives

$$T = \frac{1}{a_{T1}^2}\sum_\pm \frac{1}{\kappa_\mp - \kappa_\pm} \mathrm{e}^{\kappa_\pm z}\left[\int^z \mathrm{e}^{-\kappa_\pm y}\left(a_{Q1}^i \frac{\partial^i Q}{\partial y^i}\right)\mathrm{d}y - \beta_i\right]. \tag{10.23}$$

It was demonstrated in [2] that the integration constants are zero. We will ignore them as they clutter notation and little is learned more than the overall counting exercise—we have three number densities and we take their second derivatives so at most we can have six boundary conditions and six matching conditions which specify a total of twelve integration constants. If we substitute the above solution back into the subsequent transport equations we are left with an integro-differential equation. To overcome this we will make a series of variable changes. First,

$$h_\pm = \int^z e^{-\kappa_\pm y} Q \mathrm{d}y. \tag{10.24}$$

This eliminates the exponent and integral in equation (10.23), but we now have two functions related via the identity

$$h_+' = e^{(\kappa_- - \kappa_+)z} h_-'. \tag{10.25}$$

Let us make another change of variables to remove the exponential outside the integral

$$j_\pm = e^{\kappa_\pm z} h_\pm. \tag{10.26}$$

Finally, we need to write everything in terms of a single variable. This is achieved via the substitution

$$k = e^{\kappa_\mp z} \int^z e^{-\kappa_\mp y} j_\pm \mathrm{d}y, \tag{10.27}$$

which allows us to relate $j_\pm$ to k with the equation

$$j_\pm = k' - \kappa_\mp k. \tag{10.28}$$

We can then relate both T and Q to derivatives in k, which leaves us with differential equations only. It is convenient to rescale k in a way such that

$$T = -a^i_{Q1} \partial^i k \tag{10.29}$$

$$Q = a^i_{T1} \partial^i k. \tag{10.30}$$

We have achieved our initial aim of rewriting the solutions to the differential equation without awkward derivatives or integrals. One can verify by direct substitution that the above indeed satisfies the first transport equation. Next we substitute these equations for T and Q in terms of k into equation (10.21), which gives a differential equation in terms of k and H only,

$$0 = a^i_{Q2} \partial^i Q + a^i_{T2} \partial^i T + a^i_{H2} \partial^i H \tag{10.31}$$

$$= \left(a^i_{Q2} a^j_{T1} - a^i_{T2} a^j_{Q1}\right) \partial^{i+j} k + a^i_{H2} \partial^i H. \tag{10.32}$$

Using the same method as before one finds that the solution for H is given by

$$H = \frac{1}{a^2_{H2}} \sum_\pm \frac{e^{\kappa_\pm z}}{\kappa_\mp - \kappa_\pm} \int^z e^{-\kappa_\pm y} \left(a^i_{Q2} a^j_{T1} - a^i_{T2} a^j_{Q1} \frac{\partial^{i+j} k}{\partial y^i} \right) \mathrm{d}y. \tag{10.33}$$

Following the same recipe as before we can write H and k in terms of a variable l

$$H = -\sum_{n=0}^{4} \delta_{i+j-n}(a_{Q2}^{i} a_{T1}^{j} - a_{T2}^{i} a_{Q1}^{j})\partial^{n} l \tag{10.34}$$

$$k = a_{H2}^{i} \partial^{i} l. \tag{10.35}$$

In the above we added a Kronecker delta so that the structure of the equations was made more clear. We can then substitute our solution for k into our solutions for T and Q to give all three number densities in terms of l

$$\begin{aligned} H &= -\sum_{n=0}^{4} \delta_{i+j-n}(a_{Q2}^{i} a_{T1}^{j} - a_{T2}^{i} a_{Q1}^{j})\partial^{n} l \\ T &= -\sum_{n=0}^{4} \delta_{i+j-n} a_{Q1}^{i} a_{H2}^{j} \partial^{n} l \\ Q &= \sum_{n=0}^{4} \delta_{i+j-n} a_{T1}^{i} a_{H2}^{j} \partial^{n} l. \end{aligned} \tag{10.36}$$

We now can write equation (10.22) in terms of a single variable, l, and a source

$$\begin{aligned} \Delta(z) &= \sum_{n=0}^{6} \delta_{i+j+k-n}\left(a_{T1}^{i} a_{H2}^{j} a_{Q3}^{k} - a_{Q1}^{i} a_{H2}^{j} a_{T3}^{k}\right)\left(-a_{T1}^{i} a_{Q2}^{j} a_{H3}^{k} + a_{Q1}^{i} a_{T2}^{j} a_{H3}^{k}\right)\partial^{n} l = \\ & \sum_{n=0}^{6} \delta_{i+j+k-n} \epsilon^{abc} a_{Ta}^{i} a_{Hb}^{j} a_{Qc}^{k} \partial^{n} l \equiv \sum_{n=0}^{6} a_{l}^{n} \partial^{n} l. \end{aligned} \tag{10.37}$$

The step that introduced the permutation symbol makes use of the fact that $a_{H1}^{i} = 0$. Once again, using the method of variable coefficients the solution to the above equation is

$$l = \sum_{i=1}^{6} x_i \mathrm{e}^{\alpha_i z}\left(\int^{z} \mathrm{e}^{-\alpha_i y} \Delta(y) \mathrm{d}y - \beta_i\right) \tag{10.38}$$

in the EWSB phase and

$$l = \sum_{i=1}^{6} y_i \mathrm{e}^{\gamma_i z} \tag{10.39}$$

in the symmetric phase. In the above we have defined α_i and γ_i as the roots of the polynomials

$$\begin{aligned} \sum_{n=0}^{6} a_{l}^{n} \alpha^{n} &= 0 \\ \sum_{n=0}^{6} a_{l}^{n} \gamma^{n} &= 0, \end{aligned} \tag{10.40}$$

which are the characteristic polynomials in the broken and symmetric phases, respectively. Furthermore the constants x_i are derived from the equation

$$\vec{x} = M^{-1}\vec{d}. \tag{10.41}$$

In the above the matrix $M_{ij} \equiv \alpha_i^{j-1}$ and j is both an exponent and an index ranging from 1 to 6. Also $\vec{d} \equiv [0, \ldots, 1/a_l^6]^{\mathrm{T}}$ The integration constants are denoted by β_i and y_i in the broken and symmetric phases, respectively. They are determined by matching and boundary conditions. The boundary conditions are that all number densities must be null at $\pm\infty$, whereas the matching conditions are that all number densities and their derivatives must be continuous at the bubble wall. For the symmetric phase the boundary conditions are simple to enforce,

$$y_i = 0 \;\; \forall \; \gamma_i \leqslant 0. \tag{10.42}$$

For the broken phase we obtain a condition on the positive exponents

$$x_i\beta_i = x_i \int_0^\infty \mathrm{d}y \mathrm{e}^{-\alpha_i y}\Delta(y) \equiv I_i \;\; \forall \; \alpha_i \geqslant 0. \tag{10.43}$$

Finally, the matching conditions. First, note that we cannot naively match all the derivatives of l at the bubble wall even though doing so indeed gives the correct number of conditions. Rather, we match T, Q, and H at the bubble wall along with their first derivatives. Suppose $\gamma_i > 0$ for $i \in \{4,5,6\}$ and $\alpha_i > 0$ for $i \in \{1,2,3\}$. Let us use the convenient notation that $A_X(\alpha_i) \equiv A_X^{bi}$ and $A_X(\gamma_i) \equiv A_X^{si}$. The matching conditions result in the following conditions

$$\begin{aligned}
&\left(x_4\beta_4 \;\; x_5\beta_5 \;\; x_6\beta_6 \;\; x_1\beta_1 \;\; x_2\beta_2 \;\; x_3\beta_3 \;\; y_1 \;\; y_2 \;\; y_3\right)^T = \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_Q^{b4} & A_Q^{b5} & A_Q^{b6} & A_Q^{b1} & A_Q^{b2} & A_Q^{b3} & A_Q^{s1} & A_Q^{s2} & A_Q^{s3} \\
\alpha_4 A_Q^{b4} & \alpha_5 A_Q^{b5} & \alpha_6 A_Q^{b6} & \alpha_1 A_Q^{b1} & \alpha_2 A_Q^{b2} & \alpha_3 A_Q^{b3} & \gamma_1 A_Q^{s1} & \gamma_2 A_Q^{s2} & \gamma_3 A_Q^{s3} \\
A_T^{b4} & A_T^{b5} & A_T^{b6} & A_T^{b1} & A_T^{b2} & A_T^{b3} & A_T^{s1} & A_T^{s2} & A_T^{s3} \\
\alpha_4 A_T^{b4} & \alpha_5 A_T^{b5} & \alpha_6 A_T^{b6} & \alpha_1 A_T^{b1} & \alpha_2 A_T^{b2} & \alpha_3 A_T^{b3} & \gamma_1 A_T^{s1} & \gamma_2 A_T^{s2} & \gamma_3 A_T^{s3} \\
A_H^{b4} & A_H^{b5} & A_H^{b6} & A_H^{b1} & A_H^{b2} & A_H^{b3} & A_H^{s1} & A_H^{s2} & A_H^{s3} \\
\alpha_4 A_H^{b4} & \alpha_5 A_H^{b5} & \alpha_6 A_H^{b6} & \alpha_1 A_H^{b1} & \alpha_2 A_H^{b2} & \alpha_3 A_H^{b3} & \gamma_1 A_H^{s1} & \gamma_2 A_H^{s2} & \gamma_3 A_H^{s3}
\end{pmatrix}^{-1}
\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\equiv \begin{pmatrix}
\mathbf{1}_{3\times3} & 0 \\
A_X(\alpha) & \mathrm{A}_X(\gamma) \\
(\alpha A)_X(\alpha) & (\alpha A)_X(\gamma)
\end{pmatrix}^{-1}
\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ 0 \end{pmatrix}.
\end{aligned} \tag{10.44}$$

We can now write the analytic form of the solutions for H, T, and Q:

$$\begin{aligned} H &= \sum_{i=1}^{6} A_H(\alpha_i) x_i e^{\alpha_i z} \left(\int^{z} e^{-\alpha_i y} \Delta(y) dy - \beta_i \right) \\ T &= \sum_{i=1}^{6} A_T(\alpha_i) x_i e^{\alpha_i z} \left(\int^{z} e^{-\alpha_i y} \Delta(y) dy - \beta_i \right) \\ Q &= \sum_{i=1}^{6} A_Q(\alpha_i) x_i e^{\alpha_i z} \left(\int^{z} e^{-\alpha_i y} \Delta(y) dy - \beta_i \right), \end{aligned} \tag{10.45}$$

with known functions defined as

$$\begin{aligned} A_H &= -\sum_{n=0}^{4} \delta_{i+j-n} (a_{Q2}^i a_{T1}^j - a_{T2}^i a_{Q1}^j) \alpha^n \\ A_T &= -\sum_{n=0}^{4} \delta_{i+j-n} a_{Q1}^i a_{H2}^j \alpha^n \\ A_Q &= \sum_{n=0}^{4} \delta_{i+j-n} a_{T1}^i a_{H2}^j \alpha^n. \end{aligned} \tag{10.46}$$

So we have derived a fairly simple solution to the transport equations for the MSSM case. However, this calculation is easily generalized to multiple transport equations. Also, it is significantly simpler to assume the form of the solutions, parametrized in terms of α, $A_X(\alpha)$, β, and x_i, and substitute back into the transport equations to obtain a set of conditions which specify the above parameters. Specifically, α and $A_X(\alpha)$ solve the homogeneous version of the transport equations and x_i is defined when substituting the full solution into the inhomogeneous transport equations. Furthermore, it is straightforward to deal with the case of multiple transport equations—simply replace x_i with x_{ij} and the sources $\Delta(z)$ become replaced with $\Delta_j(z)$. One can then proceed as described above, either deriving the solution or substituting the known form to derive the parameters.

10.5 Beyond ultrathin walls

When we approximate the mass relaxation terms as constant in the broken phase and zero in the other, we are linearizing our differential equations. Therefore, we can also write perturbations to the solution which take into account the space–time dependence of the mass terms. Consider the second transport equation (10.22), which contains mass terms proportional to a. Let us assume for simplicity that a is small so that the only mass terms are in equation (10.22). We will define the following error functions

$$\begin{aligned} \Delta(z) &= \Theta(z)\Delta(z) + (1 - \Theta(z))\Delta(z) \equiv \Delta_0(z) + \epsilon(z) \\ a_{l3}^0(z) &\equiv a_{l3}^0(z)\Theta(z) + a_{l3}^0(z)\Theta(-z) \\ &= a_{l3}^0(z_{\max}) - [a_{l3}^0(z_{\max}) - a_{l3}^0(z)] + a_{l3}^0(z)\Theta(-z) \\ &= a_{l3}^0 + \delta a_{l3}^0(z) \\ l(z) &= l_0(z) + \delta_1 l(z) + \delta_2 l(z) + \cdots. \end{aligned} \tag{10.47}$$

Note that $l_0(z)$ solves the linearized transport equations so we can make a dramatic simplification similar to the simplification we made when solving bubble wall profiles. Also, just like in the case of the bubble wall profiles, these perturbations are finite, which allows for a perturbative series. Inserting the perturbations into the transport equations gives

$$\begin{aligned} a_{l3}^i \partial^i l_0 + a_{l3}^i \partial i(\delta_1 l + \delta_2 l + \cdots) + \ \delta a_{l3}^0(z)(l_0 + \delta_1 l + \delta_2 l + \cdots) \\ = \Delta(z) + \epsilon(z). \end{aligned} \tag{10.48}$$

We can use the exact same method as before to find the corrections to m

$$\begin{aligned} \delta_1 l &= \sum_{i=0}^{6} \mathrm{e}^{\alpha_i z} x_i \left(\int^z \mathrm{e}^{-\alpha_i y} [\epsilon - l_0 \delta a_{l3}^0(z)] - \delta_1 \beta_i \right) \\ \delta_2 l &= \sum_{i=0}^{6} \mathrm{e}^{\alpha_i z} x_i \left(\int^z \mathrm{e}^{-\alpha_i y} [-\delta_1 l \delta a_{l3}^0(z)] - \delta_2 \beta_i \right), \end{aligned} \tag{10.49}$$

etc.

References

[1] Moore G D 2000 Sphaleron rate in the symmetric electroweak phase *Phys. Rev.* D **62** 8

[2] White G A 2016 General analytic methods for solving coupled transport equations: from cosmology to beyond *Phys. Rev.* D **93** 4