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Fundamentals of Quantum Entanglement

F J Duarte

Appendix A

Revisiting the Einstein–Podosky–Rosen (EPR) paper

The EPR argument is examined directly under the premises of Heisenberg's uncertainty principle $\Delta x \Delta p \approx h$.

A.1 Introduction

The contents of the EPR paper entitled 'Can quantum mechanical description of physical reality be considered complete?' (Einstein *et al* 1935) were challenged by Niels Bohr also in a paper entitled 'Can quantum mechanical description of physical reality be considered complete?' (Bohr 1935). In that six-page paper, Bohr centered his argument on Heisenberg's uncertainty principle (Heisenberg 1927)

$$\Delta x \Delta p \approx h \tag{A.1}$$

and on the *complementarity principle*. Bohr's argument, although extensive, apparently failed to convince sectors within the physics community that continued to doubt the completeness of quantum mechanics for decades. Nevertheless, Bohr invoked the two crucial words that led to the one equation that can be applied to neutralize the EPR argument: *uncertainty principle*.

In this regard, it should be mentioned that Dirac already in the 1947 edition of his celebrated book included a remark of extraordinary significance: 'it is evident physically that a state for which all values of q are equally probable, or one for which all values of p are equally probable, cannot be attained in practice' (Dirac 1978).

A.2 EPR and the uncertainty principle

There is a key sentence in the first part of the EPR paper: '*when the momentum of a particle is known, its coordinate has no physical reality*' (Einstein *et al* 1935). A direct confrontation of this central concept with the uncertainty principle leads to an interesting result. The explicit argument that follows is based on concepts previously articulated by Duarte (2014).

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Heisenberg's uncertainty principle stated in its alternative fractional form (Feynman *et al* 1965) is

$$\Delta x \approx \frac{h}{\Delta p}. \tag{A.2}$$

Measurement of the momentum of a particle p can only be performed according to

$$p \pm \Delta p. \tag{A.3}$$

An *absolutely exact measurement of momentum* p with $\Delta p = 0$ is *physically impossible* (Duarte 2014). In this regard, it should be mentioned that uncertainties and errors in measurements have been known to exist since the dawn physics (Newton 1686, 1704). The EPR sentence '*when the momentum of a particle is known, its coordinate has no physical reality*' (Einstein *et al* 1935) implies an exact and perfect measurement of momentum p with $\Delta p = 0$, which is a physical impossibility.

A real non-idealized measurement of momentum leads to $p \pm \Delta p$ with a specific and real non-zero Δp. Once Δp is available, then Δx can be found according to

$$\Delta x \approx \frac{h}{\Delta p}.$$

In this regard, the '*all values*' spread in the coordinate, as feared by Einstein *et al* (1935), is *not allowed*. Removal of the 'all values' spread in the coordinate x immediately neutralizes the claim of '*no physical reality*'. Hence, the EPR conclusion that '*the quantum mechanical description of physical reality … is not complete*' can be dismissed.

A.3 Conclusion

Here, it has been shown that Heisenberg's uncertainty principle can be effectively applied in a direct and transparent manner to counter EPR's 'all values' argument that led those authors to the conclusion that the description of reality as given by wave functions, or probability amplitudes, 'is not complete'. In this regard, 'the uncertainty principle "protects" quantum mechanics' (Feynman *et al* 1965).

The dismissal of the EPR argument, or the *EPR paradox*, as referred to by many authors, has a profound meaning since it was the EPR argument that led to the formulation of hidden variable theories as presented by Bohm and colleagues (Bohm 1952, Bohm and Bub 1966) and the eventual derivation of Bell's theorem (Bell 1964). This theme is given further discussion in chapter 28.

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Appendix B

Revisiting the Pryce–Ward probability amplitude

The work of Pryce and Ward leading to the probability amplitude, $|\psi\rangle = (|x\rangle\,|y\rangle - |y\rangle\,|x\rangle)$, is examined from a historical perspective. Ward's parallel interests at the time, on quantum electrodynamics, are brought forward.

B.1 Introduction

Some readers may wonder why neither Maurice Pryce nor John Ward published a separate dedicated journal paper on the probability amplitude for quantum entanglement, $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$, thus leaving Ward's doctoral thesis as the only explicit contemporaneous record of this development. Concurrently, some readers may also wonder why they never championed, or exploited, the ownership of this most crucial equation, as most physicists would do today if confronted by similar circumstances. These are questions that apparently were never asked of John Ward and one aspect of his physics that he never discussed, at least not in the written record. In this chapter, an attempt is made to find an explanation for this apparent ineffable set of affairs. This discussion is based on measured speculation, personal knowledge of the man, and the published record.

B.2 Exciting times and extreme succinctness

According to John Ward, he was introduced to $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$ via discussions with Maurice Pryce. These discussions were generated by the interest on the perpendicularity of the polarization states of two quanta moving in opposite directions (Ward 2004). At the time, John Ward was working on his doctorate under the supervision of Maurice Pryce at Oxford (Pryce and Ward 1947). However, this was not the only focus of Ward's attention at the time. As his doctoral thesis already hinted in its title, 'Some Properties of the Elementary Particles' (Ward 1949), he was also already attracted by particle physics and by quantum electrodynamics in particular. Perhaps the only other human thinking about quantum optics at the time was Dirac himself.

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An additional preamble forces a departure from physics to introduce some aspects of John Ward's personality that might be relevant: he was a master of succinctness and always got to the point in the most direct possible way. It is as if he were a living demonstration of the *principle of least action*. He was a fairly distant man with few close friends, physicist Richard Dalitz among them. By today's standards he was extremely honest, modest, and hated corruption and the practitioners of corruption. The reader can find further details on Ward in his autobiography (Ward 2004) and in writings about him (Fraser 2008, Close 2011).

The following facts reinforce and add to the concepts already expressed:

1. His paper with Maurice Pryce entitled 'Angular correlation effects with annihilation radiation' (Pryce and Ward 1947) is slightly longer than half a page. Ward says that Pryce initially refused being a co-author and that he only accepted upon Ward's insistence (Ward 2004).
2. His doctoral thesis *Some Properties of the Elementary Particles* (Ward 1949) was a mere 47 pages long and dealt with two subject matters. The first section was entitled 'Polarization effects of annihilation radiation' and the second section was entitled 'Some higher order effects in covariant quantum electrodynamics'.
3. His landmark paper on renormalization theory entitled 'An identity in quantum electrodynamics' (Ward 1950a) was less than half a page long. The importance of this paper to quantum field theory can be summarized via the statements of experienced practitioners in the field: 'the Ward identity … ensures the *universality of the electromagnetic interaction*' (Greiner and Reinhardt 2009) and 'the proof that QED can be renormalized relied on Ward's Identities … Ward's Identities lie at the very foundations of renormalization theory' (Close 2011).
4. Another of his papers on renormalization theory entitled 'A convergent non-linear field theory' (Ward 1950b) was about a third of a page long.

   The evidence above suggests the following as factors that may have prevented publication of a sole and dedicated journal disclosure on $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$:

   (a) In the paper that he wrote with Maurice Pryce, what mattered at the time was the final scattering result useful to experimentalists interested in testing the pair theory. In this regard, Pryce and Ward most likely considered $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$ only as a necessary *intermediate step* to reach that final result. Adhering to succinctness most likely dissuaded them from disclosure.

   (b) Ward's attitude appears to express increased appreciation toward this discovery by the time he presented his thesis since he wrote, 'it is essential to derive correctly the state vector which properly describes the state of the two quanta, including their relative polarization' (Ward 1949). However, it is quite possible that Ward qualified his thesis a publication as good as any other, and thus considered the subject closed. As a matter of fact he would author and co-author only some 20 papers in his entire career.

(c) A dedicated disclosure on the subject would have to have been a joint paper with Pryce, but their paths began to diverge around 1949.

(d) In reference to items 2–4 above it is also quite obvious that the attention of the young physicist quickly shifted from quantum optics, a futuristic subject almost not existing at the time, to quantum electrodynamics and renormalization theory, which were the focus and attention of the physics community. Indeed, Ward would go on to co-author some of the papers that took center stage in the development of the Standard Model (Salam and Ward 1959, 1961 1964a, 1964b).

(e) In the 1970s when teaching quantum mechanics, via the *Feynman Lectures on Physics* (Feynman *et al* 1965), he would acknowledge with a shy smile and few words his rendezvous with equations of the form $|\psi\rangle = (|R\rangle - |L\rangle)$. No further details added.

B.3 Conclusion

The matter of $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$ did resurface between this author and Ward in early 2000. His attitude was that everybody knew the score and that eventually it would be recognized as the work of Pryce and Ward.

What did eventually transpire was that those who knew the score were physicists of his generation, such as Richard Dalitz, Willis Lamb, Maurice Pryce, and C-S Wu, but there was a new score being written by a new generation vastly unaware of the origin of $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$.

In this regard, it is fitting to mention a pertinent quote on Ward's physics: ' … he has drawn attention to fundamental truths, and has laid down basic principles that physicists have followed in subsequent decades, often without knowing it, and generally without quoting him' (Dunhill 1995).

In summary, succinctness, renormalization, the Standard Model, and uncommon honesty were probably factors in preventing a unique and dedicated disclosure of $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$ by John Ward. Championing his own work was not part of his ethos.

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Appendix C

Classical and quantum interference

The distinctions between classical interference and quantum interference are emphasized.

C.1 Introduction

In this appendix a brief description of classical N-slit interference and quantum N-slit interference is provided. Here it is shown that although interference can be described classically, this description is only an approximation of the subtle experimental interferometric phenomenon.

C.2 The classical interference equation

From Maxwell's electromagnetic theory, Born and Wolf (1999) derived the interference equation for two-slit interference, or Young's interference, as

$$I = I_1 + I_2 + 2(I_1 I_2)^{-1/2} \cos \delta \tag{C.1}$$

which has the same form as the equation given by Michelson (1927) except that Michelson writes it as

$$i = a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta \tag{C.2}$$

where a_1 and a_2 are designated as amplitudes. Using the notation of Born and Wolf (1999), for an array of N-slits the interference equation (C.1) can be extended to

$$I = \sum_{n=1}^{N} I_n + 2\sum_{n=1}^{N} I_n^{-\frac{1}{2}} \left(\sum_{m=n+1}^{N} I_m^{-\frac{1}{2}} \cos \delta \right) \tag{C.3}$$

where I_1, I_2, $I_3 \ldots I_n$ refer to the *intensities* present at each of the 1, 2, 3…n slits, which are assumed to be uniform and separated by uniform distances. In these equations, δ is the phase angle derived from the interaction of the light illuminating the slits and the geometry of the slits.

The following observations are applicable to these classical equations:

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1. These are intensity equations.
2. These are not probability equations.
3. These equations are not applicable to single-photon interference.

C.3 The *N*-slit interferometer

As already discussed in chapters 2 and 26, the *N*-slit interferometer is perfectly described by the *generalized interferometric probability* in one dimension:

$$\langle x|s\rangle\langle x|s\rangle^* = \left(\sum_{j=1}^{N}\langle x|j\rangle\langle j|s\rangle\right)\left(\sum_{j=1}^{N}\langle x|j\rangle\langle j|s\rangle\right)^* \tag{C.4}$$

$$\langle x|s\rangle\langle x|s\rangle^* = \sum_{j=1}^{N}\Psi(r_j)\sum_{m=1}^{N}\Psi(r_m)e^{i(\Omega_m-\Omega_j)} \tag{C.5}$$

$$\langle x|s\rangle\langle x|s\rangle^* = \sum_{j=1}^{N}\Psi(r_j)^2 + 2\sum_{j=1}^{N}\Psi(r_j)\left(\sum_{m=j+1}^{N}\Psi(r_m)\cos(\Omega_m-\Omega_j)\right). \tag{C.6}$$

These are three equivalent equations that apply to single-photon propagation or to the propagation of ensembles of indistinguishable photons. Equations (C.5) and (C.6) are obtained from (C.4) while using complex wave forms to represent the probability amplitudes (Duarte 1993, 2003) following Dirac's lead (Dirac 1978).

These equations can also be expressed in two and three dimensions as given in chapter 2 (Duarte 1995).

The following observations are applicable to these *N*-slit quantum probability equations:

1. These are probability equations.
2. These are not intensity equations.
3. These equations describe single-photon interference and interference of populations of indistinguishable photons.

C.4 The difference between classical and quantum interference

In classical interference

$$\textit{Maxwell equations} \rightarrow \sum_{n=1}^{N} I_n + 2\sum_{n=1}^{N} I_n^{-\frac{1}{2}}\left(\sum_{m=n+1}^{N} I_m^{-\frac{1}{2}}\cos\delta\right). \tag{C.7}$$

In quantum interference

$$\sum_{j=1}^{N}\langle x|j\rangle\langle j|s\rangle \rightarrow \sum_{j=1}^{N}\Psi(r_j)\sum_{m=1}^{N}\Psi(r_m)e^{i(\Omega_m-\Omega_j)}. \tag{C.8}$$

It should be noticed that using the semi-coherent version for the quantum probability given in equation (C.5), that is equation (2.17), and the definition for intensity given in equation (2.19), the classical equation for interference (equation (C.3)) can be derived. This is one example that illustrates classical physics as an approximation of quantum mechanics.

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Appendix D

Interferometers and their probability amplitudes

The probability amplitudes applicable to the Mach–Zehnder interferometer, the Michelson interferometer, the Sagnac interferometer, and the N-slit interferometer are given.

D.1 Introduction

In this appendix the probability amplitudes for the Mach–Zehnder, the Michelson, the Sagnac, and the N-slit interferometer are given via Dirac's notation (Dirac 1978). This treatment follows the notation given by Duarte (2003, 2014). Excellent classical discussions on these interferometers can be found in Michelson (1927) and Steel (1967).

D.2 Interferometers

The Mach–Zehnder, Michelson, and Sagnac interferometers are comprised of beam splitters and mirrors. Here, the beam splitters are assumed to exhibit perfect 50% reflectivity and 50% transmission while the mirrors are assumed to be 100% perfect reflectors.

For a *single beam splitter*, as described in figure D1, the probability amplitude is given by

$$\langle x|s\rangle = \langle x|j'\rangle\langle j'|s\rangle + \langle x|j\rangle\langle j|s\rangle \tag{D.1}$$

where j represents reflection at the beam splitter and j' stands for transmission. This equation ultimately leads to

$$|s\rangle = \frac{1}{\sqrt{2}}(|A\rangle \pm |B\rangle) \tag{D.2}$$

where

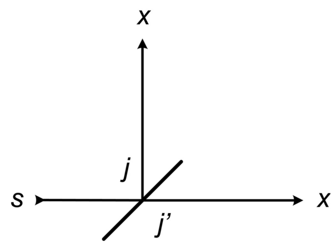

Figure D1. Schematics for a single ideal beam splitter. Interaction of the photon with the beam splitter in the reflection mode is labeled as j while interaction of the photon with the beam splitter in the transmission mode is assigned as j'. Both the transmitted and reflected photons are assumed to be incident on identical detectors x.

$$|A\rangle = |j'\rangle\langle j'|s\rangle \tag{D.3}$$

and

$$|B\rangle = |j\rangle\langle j|s\rangle \tag{D.4}$$

provided the two photons are detected by identical detectors x.

D.2.1 The Mach–Zehnder interferometer

Of particular interest to quantum computing is the Mach–Zehnder interferometer, which is comprised of an input beam splitter, an output beam splitter, and two mirrors M_1 and M_2, as illustrated in figure D2. It provides two interferometric outputs, one at x and the other at x'. The interference mechanics of the counter-propagating beams can be described via the following probability amplitude (Duarte 2003, 2014):

$$\langle x|s\rangle = \langle x|k'\rangle\langle k'|M_1\rangle\langle M_1|j\rangle\langle j|s\rangle + \langle x|k\rangle\langle k|M_2\rangle\langle M_2|j'\rangle\langle j'|s\rangle \tag{D.5}$$

where j and k refer to the beam splitters in the reflective mode while j' and k' refer to the beam splitters in the transmission mode. Assuming perfect reflectivity at mirrors M_1 and M_2, equation (D.5) is equivalent to (Duarte 2003, 2014)

$$\langle x|s\rangle = \langle x|k'\rangle\langle k'|j\rangle\langle j|s\rangle + \langle x|k\rangle\langle k|j'\rangle\langle j'|s\rangle. \tag{D.6}$$

Using the Dirac identity $|\phi\rangle = |j\rangle\langle j|\phi\rangle$, and abstracting, equation (D.6) reduces to

$$\langle x|s\rangle = \langle x|k'\rangle\langle k'|C\rangle + \langle x|k\rangle\langle k|D\rangle \tag{D.7}$$

where

$$|D\rangle = |j'\rangle\langle j'|s\rangle \tag{D.8}$$

and

$$|C\rangle = |j\rangle\langle j|s\rangle. \tag{D.9}$$

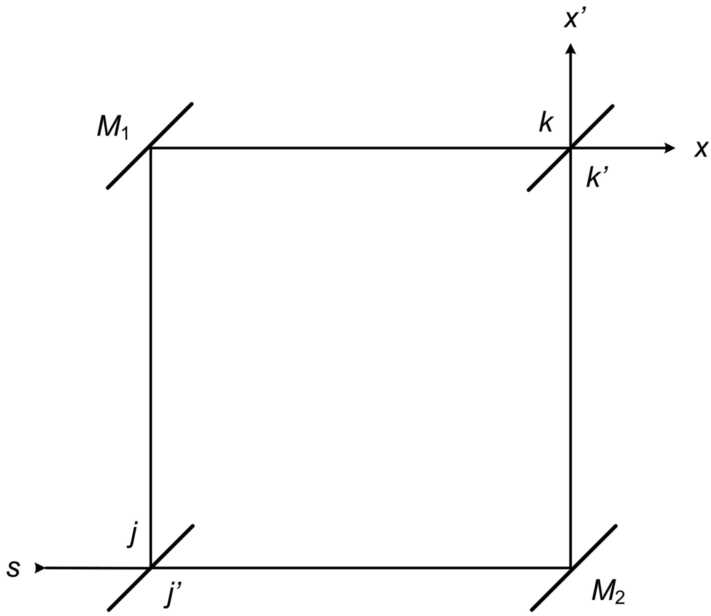

Figure D2. Schematics for the Mach–Zehnder interferometer. Mirrors M_1 and M_2 are assumed to be lossless and perfect. The beam splitters j and k are also assumed to be ideal, lossless, 50–50 partial reflectors. Interaction of the photon with the beam splitters in the reflection mode is labeled as j and k while interaction of the photon with the beam splitter in the transmission mode is assigned as j' and k' (see text).

Then, further abstracting the $\langle x$, and following normalization, equation (D.7) can be reduced to

$$|s\rangle = \frac{1}{\sqrt{2}}(|C\rangle + |D\rangle) \tag{D.10}$$

and once its linear combination is considered the overall probability amplitude becomes

$$|s\rangle = \frac{1}{\sqrt{2}}(|C\rangle \pm |D\rangle). \tag{D.11}$$

For the x' detector,

$$\langle x'|s\rangle = \langle x'|k\rangle\langle k|M_1\rangle\langle M_1|j\rangle\langle j|s\rangle + \langle x'|k'\rangle\langle k'|M_2\rangle\langle M_2|j'\rangle\langle j'|s\rangle \tag{D.12}$$

$$\langle x'|s\rangle = \langle x'|k\rangle\langle k|j\rangle\langle j|s\rangle + \langle x'|k'\rangle\langle k'|j'\rangle\langle j'|s\rangle \tag{D.13}$$

$$\langle x'|s\rangle = \langle x'|k\rangle\langle k|C\rangle + \langle x'|k'\rangle\langle k'|D\rangle \tag{D.14}$$

and ultimately to equation (D.11) again. It should be noticed that if the mirrors M_1 and M_2 are not abstracted, the final result is still given by equation (D.11).

Equation (D.11) gives the probability amplitudes that describe single-photon propagation, or the propagation of an ensemble of indistinguishable photons, in Mach–Zehnder interferometers. It should be noted that given the assumption of perfect mirrors only the beam splitters contribute to the final result. In essence, these equations describe single-photon propagation via two identical beam splitters.

The state $|C\rangle$ *is different* from $|A\rangle$, in equation (D.2), since it includes information about transmission via the first beam splitter, reflection at M_2, and reflection at the second beam splitter. The same observation is valid when comparing $|D\rangle$ to $|B\rangle$.

One final observation is that equation (D.7) can be directly derived from the generalized Dirac probability amplitude

$$\langle x|s\rangle = \sum_{j=1}^{N=2} \langle x|j\rangle\, \langle j|s\rangle \tag{D.15}$$

for $N = 2$, which is applicable to the double-slit interferometer (see chapter 17). However, physically speaking, a Mach–Zehnder interferometer is very different from a double-slit interferometer. In the double-slit interferometer, the single photon, or the population of indistinguishable photons, undergoes violent diffraction at the slits. This diffraction makes the physics between the two interferometers quite different. The only similarity between the two interferometers is that they are both two-path interferometers, that is, $N = 2$. However, while the Mach–Zehnder interferometer is a *two-beam* interferometer the double-slit, or two-slit, or Young, interferometer is a *parallel diffraction* interferometer.

D.2.2 The Michelson interferometer

The Michelson interferometer (Michelson 1927) is comprised of one beam splitter and two mirrors M_1 and M_2 in an L configuration, as depicted in figure D3. In reference to the schematics, the probability amplitude describing single-photon propagation from the source s to the detector x is given by

$$\langle x|s\rangle = \langle x|j\rangle\langle j|M_2\rangle\langle M_2|j'\rangle\langle j'|s\rangle + \langle x|j'\rangle\langle j'|M_1\rangle\langle M_1|j\rangle\langle j|s\rangle \tag{D.16}$$

where j represents reflection at the beam splitter and j' stands for transmission.

Assuming perfect reflectivity, equation (D.16) can be abstracted to

$$\langle x|s\rangle = \langle x|j\rangle\langle j|\,j'\rangle\langle j'|s\rangle + \langle x|j'\rangle\langle j'|\,j\rangle\langle j|s\rangle. \tag{D.17}$$

Further abstraction, using $|\phi\rangle = |\,j\rangle\langle j|\phi\rangle$, leads to

$$\langle x|s\rangle = \langle x|j\rangle\langle j|s\rangle + \langle x|\,j'\rangle\langle j'|s\rangle \tag{D.18}$$

which again leads to a probability amplitude of the form of equation (D.15).

As seen previously, this equation can be abstracted into

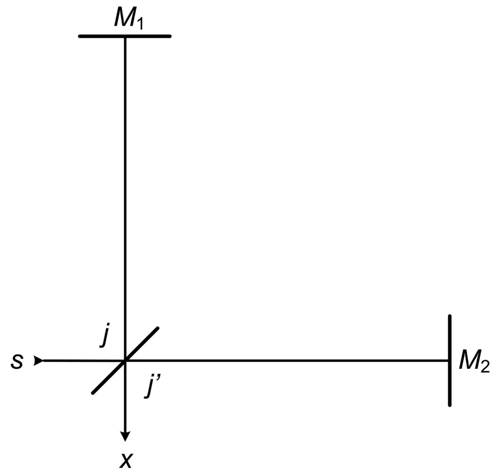

Figure D3. Schematics of the Michelson interferometer.

$$|s\rangle = \frac{1}{\sqrt{2}}(|E\rangle \pm |F\rangle) \tag{D.19}$$

where

$$|F\rangle = |j'\rangle\langle j'|s\rangle \tag{D.20}$$

and

$$|E\rangle = |j\rangle\langle j|s\rangle \tag{D.21}$$

which is not surprising since the mirrors M_1 and M_2 are being treated as idealized perfect mirrors leaving all the physics to the beam splitter. A variant of the Michelson interferometer uses retroreflectors (Steel 1967).

D.2.3 The Sagnac interferometer

The Sagnac interferometer is comprised of one beam splitter and three mirrors, as illustrated in figure D4. Using the same meaning for j and j' as previously, the corresponding probability amplitude can be expressed as

$$\begin{aligned}\langle x|s\rangle = &\langle x|j\rangle\langle j|M_3\rangle\langle M_3|M_2\rangle\langle M_2|M_1\rangle\langle M_1|j\rangle\langle j|s\rangle \\ &+ \langle x|j'\rangle\langle j'|M_1\rangle\langle M_1|M_2\rangle\langle M_2|M_3\rangle\langle M_3|j'\rangle\langle j'|s\rangle.\end{aligned} \tag{D.22}$$

Assuming perfect reflectivity at the mirrors,

$$\langle j|M_3\rangle\langle M_3|M_2\rangle\langle M_2|M_1\rangle\langle M_1|j\rangle = 1 \tag{D.23}$$

$$\langle j'|M_1\rangle\langle M_1|M_2\rangle\langle M_2|M_3\rangle\langle M_3|j'\rangle = 1 \tag{D.24}$$

and equation (D.22) reduces to

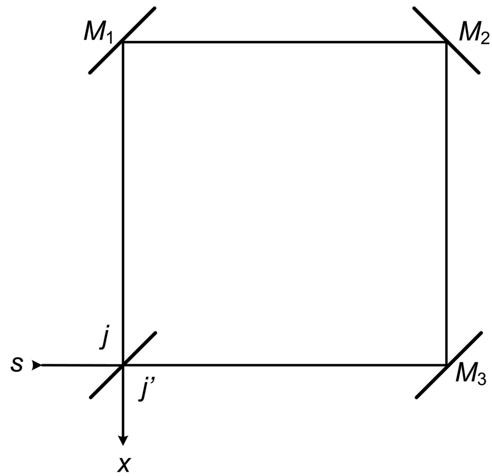

Figure D4. Schematics of the Sagnac interferometer with three mirrors M_1, M_2, and M_3.

$$\langle x|s\rangle = \langle x|j\rangle\langle j|s\rangle + \langle x|j'\rangle\langle j'|s\rangle \tag{D.25}$$

which can ultimately be expressed as

$$|s\rangle = \frac{1}{\sqrt{2}}(|G\rangle \pm |H\rangle) \tag{D.26}$$

where

$$|H\rangle = |j'\rangle\langle j'|s\rangle \tag{D.27}$$

and

$$|G\rangle = |j\rangle\langle j|s\rangle. \tag{D.28}$$

Again, this is due to simplifying assumptions made in equations (D.23) and (D.24).

Furthermore, it should be noted that the physics of equation (D.25) can be traced back to the probability amplitude given in equation (D.15).

The alternative triangular Sagnac interferometer, with only two mirrors (M_1 and M_2), illustrated in figure D5, leads to

$$\begin{aligned}\langle x|s\rangle = &\langle x|j\rangle\langle j|M_2\rangle\langle M_2|M_1\rangle\langle M_1|j\rangle\langle j|s\rangle \\ &+ \langle x|j'\rangle\langle j'|M_1\rangle\langle M_1|M_2\rangle\langle M_2|j'\rangle\langle j'|s\rangle\end{aligned} \tag{D.29}$$

with the same conclusions as with the Sagnac interferometer with three mirrors.

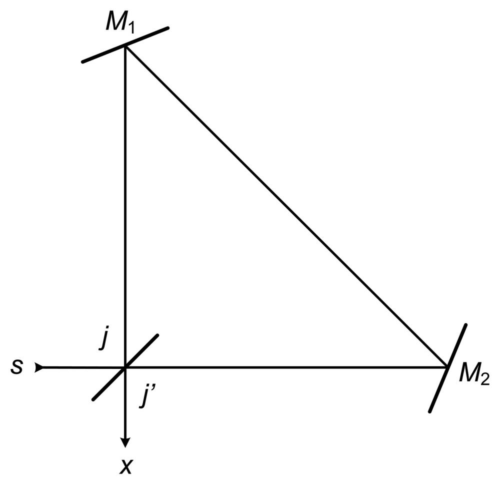

Figure D5. Schematics of the triangular Sagnac interferometer incorporating mirrors M_1 and M_2.

D.2.4 The *N*-slit interferometer

As already discussed in chapters 2 and 26, the N-slit interferometer is perfectly described by the Dirac–Feynman probability amplitude

$$\langle x|s\rangle = \sum_{j=1}^{N} \langle x|j\rangle\, \langle j|s\rangle \tag{D.30}$$

and leads the generalized probability in one dimension:

$$\langle x|s\rangle\langle x|s\rangle^* = \left(\sum_{j=1}^{N} \langle x|j\rangle\, \langle j|s\rangle\right)\left(\sum_{j=1}^{N} \langle x|j\rangle\, \langle j|s\rangle\right)^* \tag{D.31}$$

$$\langle x|s\rangle\langle x|s\rangle^* = \sum_{j=1}^{N} \Psi(r_j) \sum_{m=1}^{N} \Psi(r_m) e^{i(\Omega_m - \Omega_j)} \tag{D.32}$$

$$\langle x|s\rangle\langle x|s\rangle^* = \sum_{j=1}^{N} \Psi(r_j)^2 + 2\sum_{j=1}^{N} \Psi(r_j)\left(\sum_{m=j+1}^{N} \Psi(r_m)\cos(\Omega_m - \Omega_j)\right). \tag{D.33}$$

These three equivalent equations apply to single-photon propagation or to the propagation of ensembles of indistinguishable photons. For explicit long-hand versions of equation (D.33) for $N = 2, 3\ldots 5$, the reader should refer to Duarte (2014, 2015).

For instance, for $N = 7$(Duarte 2015),

$$\begin{aligned}|\langle x|s\rangle|^2 = &\Psi(r_1)^2 + \Psi(r_2)^2 + \Psi(r_3)^2 + \Psi(r_4)^2 + \Psi(r_5)^2 + \Psi(r_6)^2 + \Psi(r_7)^2\\ &+ 2(\Psi(r_1)\Psi(r_2)\cos(\Omega_2 - \Omega_1) + \Psi(r_1)\Psi(r_3)\cos(\Omega_3 - \Omega_1)\\ &+ \Psi(r_1)\Psi(r_4)\cos(\Omega_4 - \Omega_1) + \Psi(r_1)\Psi(r_5)\cos(\Omega_5 - \Omega_1)\\ &+ \Psi(r_1)\Psi(r_6)\cos(\Omega_6 - \Omega_1) + \Psi(r_1)\Psi(r_7)\cos(\Omega_7 - \Omega_1)\\ &+ \Psi(r_2)\Psi(r_3)\cos(\Omega_3 - \Omega_2) + \Psi(r_2)\Psi(r_4)\cos(\Omega_4 - \Omega_2)\\ &+ \Psi(r_2)\Psi(r_5)\cos(\Omega_5 - \Omega_2) + \Psi(r_2)\Psi(r_6)\cos(\Omega_6 - \Omega_2)\\ &+ \Psi(r_2)\Psi(r_7)\cos(\Omega_7 - \Omega_2) + \Psi(r_3)\Psi(r_4)\cos(\Omega_4 - \Omega_3)\\ &+ \Psi(r_3)\Psi(r_5)\cos(\Omega_5 - \Omega_3) + \Psi(r_3)\Psi(r_6)\cos(\Omega_6 - \Omega_3)\\ &+ \Psi(r_3)\Psi(r_7)\cos(\Omega_7 - \Omega_3) + \Psi(r_4)\Psi(r_5)\cos(\Omega_5 - \Omega_4)\\ &+ \Psi(r_4)\Psi(r_6)\cos(\Omega_6 - \Omega_4) + \Psi(r_4)\Psi(r_7)\cos(\Omega_7 - \Omega_4)\\ &+ \Psi(r_5)\Psi(r_6)\cos(\Omega_6 - \Omega_5) + \Psi(r_5)\Psi(r_7)\cos(\Omega_7 - \Omega_5)\\ &+ \Psi(r_6)\Psi(r_7)\cos(\Omega_7 - \Omega_6)).\end{aligned} \tag{D.34}$$

D.3 Beam splitter matrices

A straightforward non-polarizing beam-splitter is a partial mirror. The 2×2 transfer matrix describing the action of a partial reflector, or beam splitter, is simply the identity matrix (Siegman 1986; Duarte 2003)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{D.35}$$

If a photon polarized in the $|x\rangle$ state encounters a non-polarizing beam splitter, deployed at $\theta = \pi/4$, then there is a probability amplitude for straight passage and a probability amplitude for reflection onto a path orthogonal to initial direction of propagation. *No change in polarization is experienced.* The situation is different when dealing with the *Hadamard matrix* as a beam splitter. The 2×2 Hadamard matrix, which is described as a time symmetric beam splitter, can be expressed as (see chapter 24)

$$H = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{D.36}$$

which is equivalent to (see appendix F)

$$H = (|\psi\rangle_+ + |\psi\rangle^-) \tag{D.37}$$

The Hadamard H operating on the $|x\rangle$ and $|y\rangle$ states yields

$$H|x\rangle = 2^{-1/2}(|x\rangle + |y\rangle) \tag{D.38}$$

$$H|y\rangle = 2^{-1/2}(|x\rangle - |y\rangle) \tag{D.39}$$

It is immediately clear that equations (D.38) and (D.39) have the same form of equation (D.2).

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Appendix E

Polarization rotators

This is a brief introduction to the matrices used to describe polarization rotators such as wave plates, rhomboids, and broadband prismatic rotators.

E.1 Introduction

Polarization rotation devices are widely utilized in quantum optics. Here, an ultra-brief introduction is given with attention to the matrices governing the rotation. For an excellent theoretical review, the book of Robson (1974) is recommended along with Born and Wolf (1999). A more experimental perspective is given by Duarte (2014).

E.2 Wave plates

The generalized matrix for birefringent rotators is given by (Robson 1974):

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -e^{i\delta}\sin\theta & e^{i\delta}\cos\theta \end{pmatrix}. \tag{E.1}$$

For a quarter-wave plate $\delta = \pi/2$, the phase term is $e^{i\pi/2} = +i$, so that

$$R_{1/4} = \begin{pmatrix} \cos\theta & \sin\theta \\ -i\sin\theta & i\cos\theta \end{pmatrix}. \tag{E.2}$$

For a half-wave plate $\delta = \pi$ and $e^{i\pi} = -1$, so that

$$R_{1/2} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{E.3}$$

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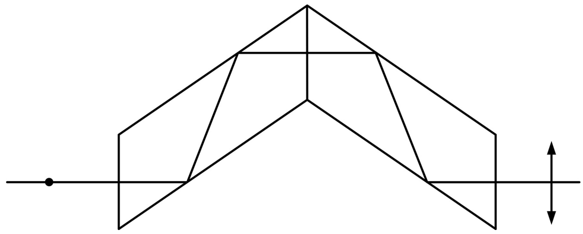

Figure E1. Schematics for a generic double Fresnel rhomb utilized for $\theta = \pi/2$ rotation of linearly polarized light.

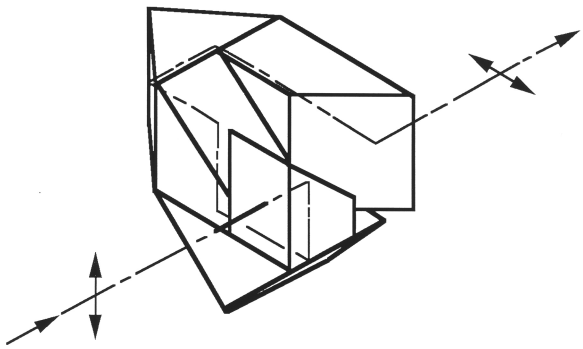

Figure E2. Schematics of the broadband collinear multiple-prism rotator utilized for $\theta = \pi/2$ rotation of linearly polarized light (from Duarte 1989).

Half-wave plates cause rotation of linearly polarized beams by $\theta = \pi/2$ so that the rotation matrix reduces to

$$R_{1/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{E.4}$$

Quarter-wave plates and half-wave plates are wavelength specific. However, their wavelength performance can be improved using achromatic designs incorporating multiple crystal materials.

E.3 Rhomboid and prismatic rotators

Other useful polarization rotators include the wavelength-specific double Fresnel rhomb, illustrated in figure E1, and the broadband collinear multiple-prism rotator (Duarte 1989), displayed in figure E2. Both these rotators turn linearly polarized light by $\theta = \pi/2$ so that their polarization matrix is

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The broadband collinear multiple-prism rotator has demonstrated high-fidelity transmission for $\theta = \pi/2$ rotation, at efficiencies approaching 95% (Duarte 1992).

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Fundamentals of Quantum Entanglement

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Appendix F

Vector products in quantum notation

Vector products are reviewed. Of particular interest are the vector products relevant to quantum probability amplitudes. These include the product utilized in density matrix calculations, the vector direct product, the vector outer product, and the Kronecker or tensor product. The equivalence in vector notation that represents $|x\rangle$ and $|y\rangle$ polarization states is also illustrated.

F.1 Introduction

In this appendix some aspects of vector algebra, vector products in particular, are described from a direct utilitarian perspective.

For a vector in three dimensions (x, y, z), the sum of two vectors $\mathbf{u} + \mathbf{w}$ is defined as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_1 + w_1 \\ u_2 + w_2 \\ u_3 + w_3 \end{pmatrix} \tag{F.1}$$

while subtraction $\mathbf{u} - \mathbf{w}$ is defined as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_1 - w_1 \\ u_2 - w_2 \\ u_3 - w_3 \end{pmatrix}. \tag{F.2}$$

Multiplication of a vector $\mathbf{u}$ with a scalar number a, yielding a new vector $a\mathbf{u}$, is defined as

$$a\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} au_1 \\ au_2 \\ au_3 \end{pmatrix}. \tag{F.3}$$

The length of a vector $\mathbf{u}$ is defined as $|\mathbf{u}|$:

doi:10.1088/2053-2563/ab2b33ch35

$$|\mathbf{u}|^2 = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}^2 = \left(u_1^2 + u_2^2 + u_3^2\right). \tag{F.4}$$

F.2 Vector products

Various vector products are useful in quantum optics. The vector quantum notation is used for the density matrix, the vector direct product, the tensor outer product, and the Kronecker product. The quantum vectors $|u\rangle$ and $|w\rangle$ are two-dimensional and thus compatible with 2×2 matrices.

F.2.1 Dot product

The dot product of two vectors $\mathbf{u} \cdot \mathbf{w}$ is a scalar defined as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = (u_1 w_1 + u_2 w_2 + u_3 w_3). \tag{F.5}$$

If the angle between the two vectors is defined as θ,

$$\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}||\mathbf{w}| \cos \theta. \tag{F.6}$$

F.2.2 Cross product

The cross product of two vectors leading to a new vector $\mathbf{u} \times \mathbf{w}$ is defined as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_2 w_3 - u_3 w_2 \\ u_3 w_1 - u_1 w_3 \\ u_1 w_2 - u_2 w_1 \end{pmatrix}. \tag{F.7}$$

F.2.3 Density matrix

The density matrix is defined as the product of two vectors, a *bra* vector and a *ket* vector (Dirac 1978):

$$\rho = |u\rangle\,\langle u| \tag{F.8}$$

$$\rho = |u\rangle\,\langle u| = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} u_1^* & u_2^* \end{pmatrix} = \begin{pmatrix} u_1 u_1^* & u_1 u_2^* \\ u_2 u_1^* & u_2 u_2^* \end{pmatrix}. \tag{F.9}$$

F.2.4 Vector direct product

Notice that this is different from the *dot product* (see, for example, Ayres 1965):

$$|u\rangle\,|w\rangle = |u\rangle \cdot |w\rangle^T \tag{F.10}$$

$$|u\rangle \cdot |w\rangle^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \end{pmatrix} = \begin{pmatrix} u_1 w_1 & u_1 w_2 \\ u_2 w_1 & u_2 w_2 \end{pmatrix}. \tag{F.11}$$

F.2.5 Vector outer product

The vector outer product is sometimes associated with the symbol $\otimes$. However, it is handled mechanically as the direct product (Ortega 1987)

$$|u\rangle\, |w\rangle^T = |u\rangle \otimes |w\rangle \tag{F.12}$$

$$|u\rangle \otimes |w\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \end{pmatrix} = \begin{pmatrix} u_1 w_1 & u_1 w_2 \\ u_2 w_1 & u_2 w_2 \end{pmatrix}. \tag{F.13}$$

The symbol $\otimes$ is also used for the Kronecker product, leading sometimes to confusion.

F.2.6 Kronecker product or tensor product

The Kronecker product, $\mathbf{U} \otimes \mathbf{W}$, is a form of matrix multiplication in which each element of the product matrix is comprised of each of the elements of the $\mathbf{U}$ matrix, u_{mn}, multiplying the whole $\mathbf{W}$ matrix, so that the first element is $u_{11}\,\mathbf{W}$, and the last element is $u_{mn}\,\mathbf{W}$ (Zehfuss 1858). This means, for example, that the Kronecker product of two 2×2 matrices yields a 4×4 matrix. For simple two-dimensional vectors this product can be expressed as

$$|u\rangle \otimes |w\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u_1 w_1 \\ u_1 w_2 \\ u_2 w_1 \\ u_2 w_2 \end{pmatrix}. \tag{F.14}$$

For

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{F.15}$$

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{F.16}$$

the following Kronecker products follow:

$$|1\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{F.17}$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{F.18}$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{F.19}$$

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{F.20}$$

Using the Kronecker product on the Pauli matrices yields the following 4×4 matrices:

$$\sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{F.21}$$

$$\sigma_x \otimes \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{F.22}$$

$$\sigma_y \otimes \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{F.23}$$

$$\sigma_y \otimes \sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{F.24}$$

$$\sigma_z \otimes \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{F.25}$$

$$\sigma_z \otimes \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. \tag{F.26}$$

F.3 Equivalence in vector notation for entangled polarizations

Here, for the sake of transparency, a clarification in the definition of vector notation is made explicit. In the notation utilized in this monograph, the polarization $|x\rangle$ and $|y\rangle$ states are represented by $|1\rangle$ and $|0\rangle$, and their corresponding vectors, as defined by Fowles (1968) and Robson (1974), are

$$|x\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{F.27}$$

and

$$|y\rangle = |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{F.28}$$

However, in the contemporaneous literature (see, for example, Nielsen and Chuang 2000) the convention

$$|x\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{F.29}$$

and

$$|y\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{F.30}$$

is used. The point to be made here is that both conventions are equivalent as long as consistency is maintained.

To illustrate the validity of the previous statement, first the definition expressed in equations (F.27) and (F. 28) is used in

$$|\psi\rangle_+ = 2^{-1/2}(|1\rangle\,|0\rangle + |0\rangle\,|1\rangle) \tag{F.31}$$

$$|\psi\rangle_- = 2^{-1/2}(|1\rangle\,|0\rangle - |0\rangle\,|1\rangle) \tag{F.32}$$

$$|\psi\rangle^+ = 2^{-1/2}(|1\rangle\,|1\rangle + |0\rangle\,|0\rangle) \tag{F.33}$$

$$|\psi\rangle^- = 2^{-1/2}(|1\rangle\,|1\rangle - |0\rangle\,|0\rangle) \tag{F.34}$$

to yield, using the vector direct product,

$$\begin{aligned}|\psi\rangle_+ &= 2^{-1/2}\left(\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right) = 2^{-1/2}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)\\ &= 2^{-1/2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{aligned} \tag{F.35}$$

$$\begin{aligned}|\psi\rangle_- &= 2^{-1/2}\left(\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}\right) = 2^{-1/2}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)\\ &= 2^{-1/2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\end{aligned} \tag{F.36}$$

$$\begin{aligned}|\psi\rangle^+ &= 2^{-1/2}\left(\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right) = 2^{-1/2}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)\\ &= 2^{-1/2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\end{aligned} \tag{F.37}$$

$$\begin{aligned}|\psi\rangle^- &= 2^{-1/2}\left(\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\right) = 2^{-1/2}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)\\ &= 2^{-1/2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{aligned} \tag{F.38}$$

which can be summarized in the following identities (Duarte *et al* 2019)

$$|\psi\rangle_+ = 2^{-1/2}\sigma_x \tag{F.39}$$

$$|\psi\rangle_- = 2^{-1/2}i\sigma_y \tag{F.40}$$

$$|\psi\rangle^+ = 2^{-1/2}I \tag{F.41}$$

$$|\psi\rangle^- = 2^{-1/2}\sigma_z. \tag{F.42}$$

Now, using instead the definitions of equations (F.29) and (F.30) and

$$|\psi\rangle_+ = 2^{-1/2}(|0\rangle\,|1\rangle + |1\rangle\,|0\rangle) \tag{F.43}$$

$$|\psi\rangle_- = 2^{-1/2}(|0\rangle\,|1\rangle - |1\rangle\,|0\rangle) \tag{F.44}$$

$$|\psi\rangle^+ = 2^{-1/2}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle) \tag{F.45}$$

$$|\psi\rangle^- = 2^{-1/2}(|0\rangle\,|0\rangle - |1\rangle\,|1\rangle) \tag{F.46}$$

the reader can verify that equations (F.35)–(F.38) are again reproduced, and so are the identities expressed in (F.39)–(F.42).

Furthermore, using the definition of equations (F.27) and (F.28)

$$\sigma_z\ |1\rangle = |1\rangle \tag{F.47}$$

$$\sigma_z\ |0\rangle = -|0\rangle \tag{F.48}$$

but, using the definitions of equations (F.29) and (F.30)

$$\sigma_z\ |0\rangle = |0\rangle \tag{F.49}$$

$$\sigma_z\ |1\rangle = -|1\rangle. \tag{F.50}$$

However, the explicit versions of equations (F.47) and (F.48) are

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{F.51}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{F.52}$$

which, as the reader can verify, are the same as the explicit versions of equations (F.49) and (F.50).

F.4 The Hadamard matrix and quantum entanglement

The nexus between the Hadamard matrix and the probabilities for quantum entanglement can be elucidated by considering equations (F.35) and (F.38) in their final form

$$|\ \psi\rangle_+ = 2^{-1/2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{F.53}$$

$$|\ \psi\rangle^- = 2^{-1/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{F.54}$$

which immediately lead to an expression for the Hadamard matrix in terms of the probability amplitudes for quantum entanglement (Duarte and Taylor 2019)

$$H = (|\psi\rangle_+ \ + |\psi\rangle^-) \tag{F.55}$$

which is equivalent to

$$H = 2^{-1/2}(\sigma_x + \sigma_z) \tag{F.56}$$

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Appendix G

Trigonometric identities

Trigonometric identities useful in the calculation of probability amplitudes and probabilities related to polarization are listed.

G.I Trigonometric identities

The following are timeless and useful trigonometric identities:

$$\sin^2\varphi + \cos^2\varphi = 1 \tag{G.1}$$

$$\sin(-\varphi) = -\sin\varphi \tag{G.2}$$

$$\cos(-\varphi) = \cos\varphi \tag{G.3}$$

$$\sin(\varphi + \theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta \tag{G.4}$$

$$\sin(\varphi - \theta) = \sin\varphi\cos\theta - \cos\varphi\sin\theta \tag{G.5}$$

$$\cos(\varphi + \theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta \tag{G.6}$$

$$\cos(\varphi - \theta) = \cos\varphi\cos\theta + \sin\varphi\sin\theta \tag{G.7}$$

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \tag{G.8}$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi \tag{G.9}$$

$$\cos 2\varphi = 1 - 2\sin^2\varphi \tag{G.10}$$

$$\cos 2\varphi = 2\cos^2\varphi - 1 \tag{G.11}$$

$$2\sin^2\varphi = 1 - \cos 2\varphi \tag{G.12}$$

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$$2\cos^2\varphi = 1 + \cos 2\varphi \tag{G.13}$$

$$\cos\varphi + \cos\theta = 2\cos(\varphi - \theta)\cos(\varphi + \theta) \tag{G.14}$$

$$\cos\varphi - \cos\theta = -2\sin(\varphi - \theta)\sin(\varphi + \theta) \tag{G.15}$$

$$-\cos\varphi + \cos\theta = 2\sin(\varphi - \theta)\sin(\varphi + \theta) \tag{G.16}$$

$$-\cos\varphi - \cos\theta = -2\cos(\varphi - \theta)\cos(\varphi + \theta) \tag{G.17}$$

$$\sin\varphi + \sin\theta = 2\cos(\varphi - \theta)\sin(\varphi + \theta) \tag{G.18}$$

$$\sin\varphi - \sin\theta = 2\sin(\varphi - \theta)\cos(\varphi + \theta) \tag{G.19}$$

$$-\sin\varphi + \sin\theta = -2\sin(\varphi - \theta)\cos(\varphi + \theta) \tag{G.20}$$

$$-\sin\varphi - \sin\theta = -2\cos(\varphi - \theta)\sin(\varphi + \theta) \tag{G.21}$$

$$\sin\varphi\sin\theta = \frac{1}{2}(\cos(\varphi - \theta) - \cos(\varphi + \theta)) \tag{G.22}$$

$$\cos\varphi\cos\theta = \frac{1}{2}(\cos(\varphi - \theta) + \cos(\varphi + \theta)) \tag{G.23}$$

$$\sin\varphi\cos\theta = \frac{1}{2}(\sin(\varphi + \theta) + \sin(\varphi - \theta)) \tag{G.24}$$

$$\cos\varphi\sin\theta = \frac{1}{2}(\sin(\varphi + \theta) - \sin(\varphi - \theta)). \tag{G.25}$$

Of particular interest are the identities related to the Pryce–Ward angle 2φ given in equations (G.8)–(G.13).

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Appendix H

More on quantum notation

Further aspects of quantum notation, as per *Dirac's identities*, are explored.

H.1 Introduction

Although it might be obvious to most readers, here the quantumness of equations such as $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$ is examined further.

H.2 Certainly not classical

Already in 1926, Born, Heisenberg, and Pascal had brought to the physics world their amazing discovery succinctly expressed as (Born *et al* 1926)

$$(PQ - QP) = \frac{h}{2\pi i} \tag{H.1}$$

$$(PQ - QP) = -i\hbar \tag{H.2}$$

which is known as the *commutation rule*. This equation, discovered by Heisenberg, Born, and Pascal, illustrates rather dramatically that in the quantum world expressions such as $(AB - BA)$ do not banish, that is,

$$(AB - BA) \neq 0. \tag{H.3}$$

This can be easily demonstrated via the expression

$$(\sigma_x\sigma_y - \sigma_y\sigma_x) \tag{H.4}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}. \tag{H.5}$$

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Moreover, the *Poisson bracket* is defined as (Dirac 1978)

$$(\hat{H}\hat{A} - \hat{A}\hat{H}) = [\hat{A}, \hat{H}]. \tag{H.6}$$

In this equation, $\hat{H}$ and $\hat{A}$ are operators, and $\hat{H}$ is known as the Hamiltonian (Feynman *et al* 1965).

H.3 Multiplication of probability amplitudes

In chapter III of his book, in a section entitled 'Developments in notation', Dirac introduces the commutative axiom of multiplication associated with *ket* vectors (Dirac 1978). In particular, he introduces the identity

$$|a\rangle|b\rangle = |b\rangle|a\rangle. \tag{H.7}$$

Dirac does so using the preamble, 'We assume that they have a product $|a\rangle|b\rangle$ for which the commutative and distributive axioms of multiplication hold'.

Here, a subtlety arises: Dirac's commutative axiom applies perfectly if $|a\rangle$ and $|b\rangle$ are probability amplitudes represented by complex wave functions. For instance,

$$|a\rangle|b\rangle = \psi_1 e^{-i(\phi_1+\theta_1)}\psi_2 e^{-i(\phi_2+\theta_2)} = \psi_1\psi_2 e^{-i(\phi_1+\theta_1+\phi_2+\theta_2)} \tag{H.8}$$

$$|b\rangle|a\rangle = \psi_2 e^{-i(\phi_2+\theta_2)}\psi_1 e^{-i(\phi_1+\theta_1)} = \psi_2\psi_1 e^{-i(\phi_2+\theta_2+\phi_1+\theta_1)} \tag{H.9}$$

thus clearly showing that $|a\rangle|b\rangle = |b\rangle|a\rangle$.

However, the same commutative axiom does not apply if the *kets* are associated with vectors. For instance, if

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{H.10}$$

$$|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{H.11}$$

using the direct vector product (see chapter 24, and appendix F) yields

$$|x\rangle|y\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{H.12}$$

and

$$|y\rangle|x\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{H.13}$$

clearly showing that $|x\rangle|y\rangle \neq |y\rangle|x\rangle$. If instead of the direct vector product the Kronecker $\otimes$ product is utilized,

$$|x\rangle|y\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \tag{H.14}$$

$$|y\rangle|x\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \tag{H.15}$$

thus confirming the previous result. The conversion from a 4×1 vector to a 2×2 vector is performed via the *vec* function (Neudecker 1969).

Assuming that $|a\rangle|b\rangle = |b\rangle|a\rangle$ applies, then some interesting effects in notation can arise. For instance, since

$$(|x, y\rangle - |y, x\rangle) = (|x\rangle|y\rangle - |y\rangle|x\rangle) \neq 0 \tag{H.16}$$

$$(|x\rangle|y\rangle - |x\rangle|y\rangle) \neq 0 \tag{H.17}$$

since the second term in parenthesis can be expressed as

$$|x\rangle|y\rangle = |y\rangle|x\rangle \tag{H.18}$$

This would establish a complete equivalence between equations (H.17) and (H.16).

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Appendix I

From quantum principles to classical optics

The Dirac–Feynman principle is utilized to derive the generalized interference equation that can then be applied to explain interference, diffraction, refraction, and reflection in a succinct, hierarchical, and unified manner. The uncertainty principle and the cavity linewidth equation are also explained in interferometric terms.

I.1 Introduction

In chapter 29, the idea articulated by Lamb (1987), essentially advocating that this is a quantum world, was endorsed via the introduction of the DFL doctrine. Dirac's name was incorporated since he was the first to describe macroscopic interference using quantum principles (Dirac 1978). Feynman's name was also incorporated since he used quantum path integrals to describe 'classical' beam divergence (Feynman and Hibbs 1965). Here it is shown that the generalized quantum interference equation can be applied to describe, in a unified, cohesive, and hierarchical approach, interference, diffraction, refraction, and reflection (Duarte 1997, 2003).

I.2 From quantum interference to generalized diffraction

The Dirac–Feynman probability amplitude

$$\langle x|s\rangle = \sum_{j=1}^{N} \langle x|j\rangle\langle j|s\rangle \tag{I.1}$$

leads, as explained in chapter 2, to the generalized probability in one dimension (Duarte 1991, 1993):

$$\langle x|s\rangle\langle x|s\rangle^* = \left(\sum_{j=1}^{N} \langle x|j\rangle\langle j|s\rangle\right)\left(\sum_{j=1}^{N} \langle x|j\rangle\langle j|s\rangle\right)^* \tag{I.2}$$

$$\langle x|s\rangle\langle x|s\rangle^* = \sum_{j=1}^{N}\Psi(r_j)\sum_{m=1}^{N}\Psi(r_m)e^{i(\Omega_m-\Omega_j)} \tag{I.3}$$

$$\langle x|s\rangle\langle x|s\rangle^* = \sum_{j=1}^{N}\Psi(r_j)^2 + 2\sum_{j=1}^{N}\Psi(r_j)\left(\sum_{m=j+1}^{N}\Psi(r_m)\cos(\Omega_m - \Omega_j)\right). \tag{I.4}$$

These three equivalent probability equations apply to single-photon propagation or to the propagation of ensembles of indistinguishable photons (Duarte 1993).

From the phase term of equation (I.4), it can be shown that (Duarte 1997, 2006)

$$d_m(\pm n_1 \sin\Theta_m \pm n_2 \sin\Phi_m)\frac{2\pi}{\lambda_v} = M\pi \tag{I.5}$$

where Θ_m and Φ_m are the angles of incidence and diffraction, respectively, n_1 and n_2 are the refractive indices prior- and post-diffraction, and $M = 0, 2, 4, 6, \ldots$.

For $n_1 = n_2$, $\lambda = \lambda_v$, and equation (I.5) reduces to the *generalized diffraction grating equation*

$$d_m(\pm \sin\Theta_m \pm \sin\Phi_m) = m\lambda \tag{I.6}$$

where $m = 0, 1, 2, 3, \ldots$ are the various *diffraction orders*.

I.3 From generalized diffraction to generalized refraction

For the condition $d_m \ll \lambda$, the diffraction grating equation can only be solved for (Duarte 1997)

$$d_m(\pm n_1 \sin\Theta_m \pm n_2 \sin\Phi_m)\frac{2\pi}{\lambda_v} = 0 \tag{I.7}$$

which leads directly to the *generalized refraction equation*

$$(\pm n_1 \sin\Theta_m \pm n_2 \sin\Phi_m) = 0. \tag{I.8}$$

For the case of incidence below the normal (−) and refraction above the normal (+), equation (I.8) becomes (Duarte 2006)

$$-n_1 \sin\Theta_m + n_2 \sin\Phi_m = 0 \tag{I.9}$$

and

$$n_1 \sin\Theta_m = n_2 \sin\Phi_m \tag{I.10}$$

is the well-known *equation of refraction*, also known as *Snell's law*.

For the case of incidence above the normal (+) and refraction above the normal (+) (Duarte 2006),

$$+n_1 \sin\Theta_m + n_2 \sin\Phi_m = 0 \tag{I.11}$$

and

$$n_1 \sin \Theta_m = -n_2 \sin \Phi_m \tag{I.12}$$

which is the refraction law for *negative refraction*. This subject is treated in greater detail by Duarte (2015).

I.4 From generalized refraction to reflection

From the generalized equation of refraction, equation (I.8) (Duarte 1997, 2015), for $n_1 = n_2$,

$$(\pm \sin \Theta_m \pm \sin \Phi_m) = 0. \tag{I.13}$$

For incidence above the normal (+) and reflection below the normal (−),

$$+\sin \Theta_m - \sin \Phi_m = 0 \tag{I.14}$$

which means

$$\Theta_m = \Phi_m \tag{I.15}$$

where Θ_m is the angle of incidence, and Φ_m is the angle of reflection. This is the well-known *law of reflection*.

I.5 From quantum interference to Heisenberg's uncertainty principle

In this section an approximate geometrical derivation of Heisenberg's uncertainty principle, via the generalized probability equation for interference, is illustrated. As already explained, from the generalized interferometric equation

$$|\langle x|s\rangle|^2 = \sum_{j=1}^{N} \Psi(r_j)^2 + 2\sum_{j=1}^{N} \Psi(r_j)\left(\sum_{m=j+1}^{N} \Psi(r_m)\cos(\Omega_m - \Omega_j)\right)$$

emerges the generalized diffraction equation (I.6) from which, for positive diffraction, the usual equation of diffraction

$$d_m(\sin \Theta_m \pm \sin \Phi_m) = m\lambda \tag{I.16}$$

emerges. For $\Theta_m \approx \Phi_m(=\theta)$, the Littrow grating equation

$$2d \sin \theta = m\lambda \tag{I.17}$$

can be established.

For an expanded beam of light incident on a reflection diffraction grating, as depicted in figure I1, $\sin \theta = \Delta x/l$, where l is the length of the grating, and Δx is the path difference. For an infinitesimal change in wavelength, at two infinitesimally different wavelengths, from equation (I.17)

$$\lambda_1 = \frac{2d}{m}\left(\frac{\Delta x_1}{l}\right) \tag{I.18}$$

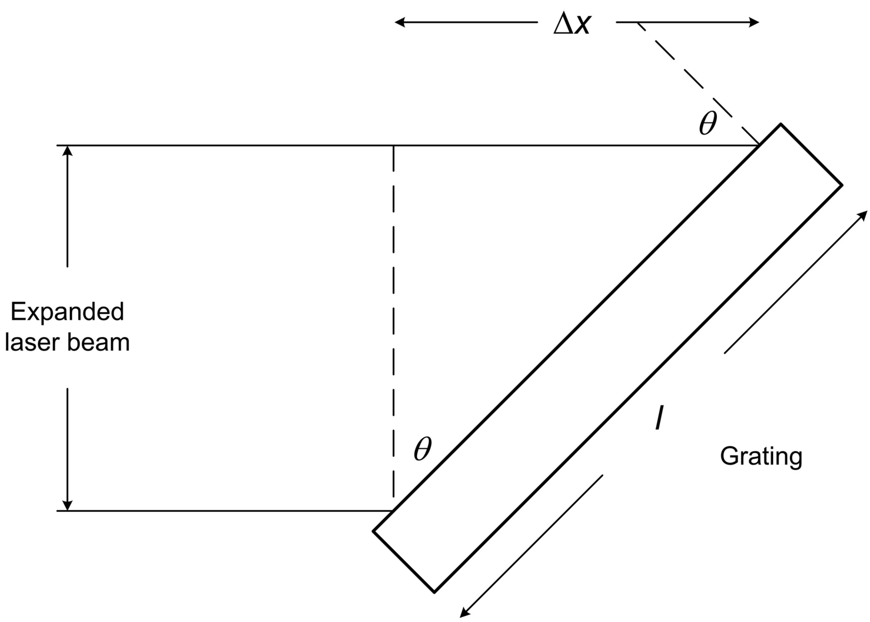

Figure I1. Reflection diffraction grating interacting with expanded laser beam.

$$\lambda_2 = \frac{2d}{m}\left(\frac{\Delta x_2}{l}\right). \tag{I.19}$$

Since equation (I.17) can also be expressed as

$$\frac{2d}{m} = \frac{l\lambda}{\Delta x} \tag{I.20}$$

$\Delta\lambda = (\lambda_1 - \lambda_2)$ yields

$$\Delta\lambda = \frac{l\lambda}{\Delta x}\left(\frac{\Delta x_1 - \Delta x_2}{l}\right). \tag{I.21}$$

To distinguish between a maxima and minima, the *difference* in path differences should be $(\Delta x_1 - \Delta x_2) \approx \lambda$. Thus, equation (I.21) reduces immediately to the diffraction identity

$$\Delta\lambda \approx \frac{\lambda^2}{\Delta x}. \tag{I.22}$$

The momentum expression $p = \hbar k$ for two slightly different wavelengths leads to

$$p_1 - p_2 = \frac{h(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2} \tag{I.23}$$

and by restating the assumption of two infinitesimally different wavelengths, this equation reduces to

$$\Delta p \approx h \frac{\Delta\lambda}{\lambda^2}. \tag{I.24}$$

Substitution of equation (I.22) into (I.24) immediately yields

$$\Delta p \Delta x \approx h \tag{I.25}$$

which is *Heisenberg's Uncertainty Principle* (Dirac 1978). Additional useful forms of the uncertainty principle are its frequency–spatial version

$$\Delta\nu \Delta x \approx c \tag{I.26}$$

and its frequency–time version

$$\Delta\nu \Delta t \approx 1. \tag{I.27}$$

I.6 The cavity linewidth equation

It has already been established that the generalized interferometric equation (I.4) leads to the generalized diffraction equation (I.6) from which, for positive diffraction, the usual equation of diffraction

$$d_m(\pm \sin \Theta_m \pm \sin \Phi_m) = m\lambda \tag{I.28}$$

is obtained. As seen previously, for $\Theta_m \approx \Phi_m(=\theta)$, the Littrow grating equation

$$2d \sin\theta = m\lambda$$

can be established.

Following Duarte (1992) and considering two slightly different wavelengths, an expression for $(\lambda_1 - \lambda_2) = \Delta\lambda$ can be written as

$$\Delta\lambda = \frac{2d}{m}(\sin\theta_1 - \sin\theta_2) \tag{I.29}$$

and for $\theta_1 \approx \theta_2(=\theta)$ as

$$\Delta\lambda \approx \frac{2d}{m}\Delta\theta\left(1 - \frac{3\theta^2}{3!} + \frac{5\theta^4}{5!} - \frac{7\theta^6}{7!} + \cdots\right). \tag{I.30}$$

Differentiation of the Littrow grating equation yields

$$\frac{\partial\theta}{\partial\lambda}\cos\theta = \frac{m}{2d} \tag{I.31}$$

and substitution of $(m/2d)$ into equation (I.30) leads to

$$\Delta\lambda \approx \Delta\theta\left(\frac{\partial\theta}{\partial\lambda}\right)^{-1}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}\ldots\right)(\cos\theta)^{-1}. \tag{I.32}$$

Since the expansion in θ approaches $\cos\theta$,

$$\Delta\lambda \approx \Delta\theta\left(\frac{\partial\theta}{\partial\lambda}\right)^{-1} \tag{I.33}$$

or

$$\Delta\lambda \approx \Delta\theta(\nabla_\lambda\theta)^{-1} \tag{I.34}$$

which is the well-known cavity linewidth equation (Duarte 1992).

I.7 Generalized multiple-prism dispersion

Since the cavity linewidth equation $\Delta\lambda \approx \Delta\theta(\nabla_\lambda\theta)^{-1}$ depends inversely on the overall cavity dispersion $(\nabla_\lambda\theta)$, it is important to have access to generalized equations of intracavity dispersion applicable to multiple-prism grating assemblies. Precise knowledge of the overall intracavity dispersion is essential for the design of optimized high-power pulsed tunable lasers. For instance, a high peak-power optimized multiple-prism solid-state dye laser oscillator can yield diffraction-limited beam divergence at a linewidth of $\Delta\nu \approx 350$ MHz for a near-Gaussian temporal pulse at $\Delta t \approx 3$ ns, so that $\Delta\nu\Delta t \approx 1.05$, leading to a performance near the frequency–time limit allowed by Heisenberg's uncertainty principle (Duarte 1999).

For a generalized multiple-prism array, as illustrated in figure I2, and the generalized diffraction equation (I.8) (Duarte and Piper 1982, Duarte 2006),

$$\phi_{1,m} + \phi_{2,m} = \varepsilon_m \pm \alpha_m \tag{I.35}$$

$$\psi_{1,m} + \psi_{2,m} = \alpha_m \tag{I.36}$$

$$\sin\phi_{1,m} = \pm n_m \sin\psi_{1,m} \tag{I.37}$$

$$\sin\phi_{2,m} = \pm n_m \sin\psi_{2,m}. \tag{I.38}$$

Here, $\phi_{1,m}$ and $\phi_{2,m}$ are the angles of incidence and emergence, and $\psi_{1,m}$ and $\psi_{2,m}$ are the corresponding angles of refraction at the mth prism. In these equations the positive sign + indicates positive refraction while the negative sign − refers to negative refraction.

Differentiating equations (I.37) and (I.38) and using the derivative identity $(d\psi_{1,m}/dn) = -(d\psi_{2,m}/dn)$ leads to the single-pass dispersion following the mth prism (Duarte and Piper, 1982, 1983, Duarte 2006):

$$\nabla_\lambda\phi_{2,m} = \pm\mathcal{H}_{2,m}\nabla_\lambda n_m \pm (k_{1,m}k_{2,m})^{-1}(\mathcal{H}_{1,m}\nabla_\lambda n_m(\pm)\nabla_\lambda\phi_{2,(m-1)}) \tag{I.39}$$

where $\nabla_\lambda = \partial/\partial\lambda$ and

$$k_{1,m} = \frac{\cos\psi_{1,m}}{\cos\phi_{1,m}} \tag{I.40}$$

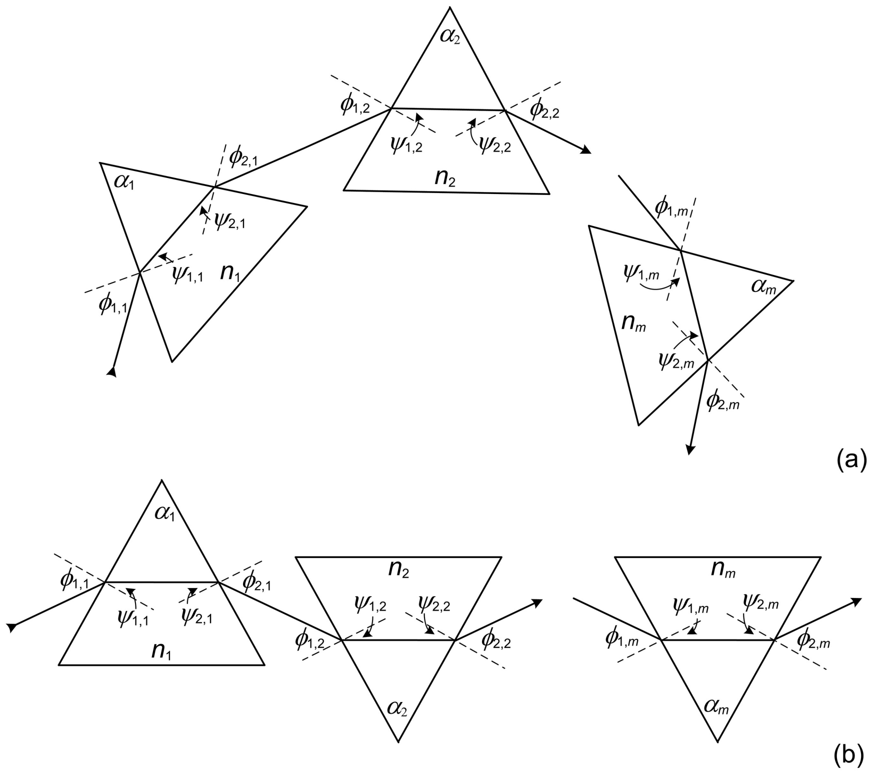

Figure I2. Multiple-prism arrays with the (a) additive configuration and (b) compensating configuration.

$$k_{2,m} = \frac{\cos \phi_{2,m}}{\cos \psi_{2,m}} \tag{I.41}$$

$$\mathcal{H}_{1,m} = \frac{\tan \phi_{1,m}}{n_m} \tag{I.42}$$

$$\mathcal{H}_{2,m} = \frac{\tan \phi_{2,m}}{n_m}. \tag{I.43}$$

$k_{1,m}$ and $k_{2,m}$ represent the beam expansion, at the mth prism, by the incidence and the emergence beams, respectively. In equation (I.39), (±) refers to deployment in either a positive (+) or compensating (−) configuration, while the simple ± indicates either positive or negative refraction (Duarte 2006).

Differentiation of equation (I.41) leads to the generalized single-pass dispersion equation for positive refraction

$$\nabla_\lambda \phi_{2,m} = \mathcal{H}_{2,m} \nabla_\lambda n_m + (k_{1,m} k_{2,m})^{-1} (\mathcal{H}_{1,m} \nabla_\lambda n_m \pm \nabla_\lambda \phi_{2,(m-1)}). \tag{I.44}$$

From this equation it can be shown that the generalized *double-pass* multiple-prism dispersion is given by (Duarte 1985, 1989)

$$\begin{aligned}\nabla_\lambda \Phi_P = 2M_1M_2 \sum_{m=1}^{r} (\pm 1)\mathcal{H}_{1,m}\left(\prod_{j=m}^{r} k_{1,j} \prod_{j=m}^{r} k_{2,j}\right)^{-1} \nabla_\lambda n_m \\ + 2\sum_{m=1}^{r} (\pm 1)\mathcal{H}_{2,m}\left(\prod_{j=1}^{m} k_{1,j} \prod_{j=1}^{m} k_{2,j}\right)\nabla_\lambda n_m\end{aligned} \tag{I.45}$$

$$M_1 = \prod_{m=1}^{r} k_{1,m} \tag{I.46}$$

$$M_2 = \prod_{m=1}^{r} k_{2,m}. \tag{I.47}$$

Furthermore, the overall multiple-prism grating multi return-pass laser linewidth is given by (Duarte and Piper 1984, Duarte 2001)

$$\Delta\lambda_R = \Delta\theta_R(RM\nabla_\lambda \Theta_G + R\nabla_\lambda \Phi_P)^{-1} \tag{I.48}$$

where R is the number of return intracavity passes elapsed from the leading edge of the laser excitation pulse to the onset of laser emission (Duarte 2001). This equation neatly shows the enormous effect on laser emission linewidth that intracavity multiple-prism factors in the $100 \leqslant M \leqslant 200$ range can have since the grating dispersion is multiplied by M (Duarte 2015).

I.7.1 Generalized multiple-prism dispersion for laser pulse compression

It has been established that the measured laser linewidth $\Delta\lambda \approx \Delta\theta(\nabla_\lambda\theta)^{-1}$ of an optimized multiple-prism grating laser oscillator can be very narrow and even approach the limit imposed by Heisenberg's uncertainty principle. For laser pulse compression, the reverse is desired. That is, since $\Delta\nu\Delta t \approx 1$, a very broadband laser emission can lead to a very narrow temporal pulse, and the least amount of intracavity dispersion is required. This requires detailed knowledge of the first, second, third, and even higher derivatives of the intracavity dispersion.

Using the identity (Duarte 1987),

$$\nabla_n \phi_{2,m} = \nabla_\lambda \phi_{2,m}(\nabla_\lambda n_m)^{-1} \tag{I.49}$$

where $\nabla_n = \partial/\partial n$, the generalized single-pass dispersion, that is, equation (I.44), becomes (Duarte 1987, 2009)

$$\nabla_n \phi_{2,m} = \mathcal{H}_{2,m} + (\mathcal{M})^{-1}(\mathcal{H}_{1,m} \pm \nabla_n \phi_{2,(m-1)}) \tag{I.50}$$

where

$$(\mathcal{M})^{-1} = k_{1,m}^{-1} k_{2,m}^{-1}. \tag{I.51}$$

The second derivative of $\phi_{2,m}$, that is, $\nabla_n^2\phi_{2,m}$, is given by (Duarte 1987, 2000)

$$\begin{aligned}\nabla_n^2\phi_{2,m} = \nabla_n\mathcal{H}_{2,m} \\ + (\nabla_n\mathcal{M}^{-1})(\mathcal{H}_{1,m} \pm \nabla_n\phi_{2,(m-1)}) \\ + (\mathcal{M}^{-1})\left(\nabla_n\mathcal{H}_{1,m} \pm \nabla_n^2\phi_{2,(m-1)}\right).\end{aligned} \tag{I.52}$$

The third derivative of $\phi_{2,m}$, $\nabla_n^3\phi_{2,m}$, is given by (Duarte 2009)

$$\begin{aligned}\nabla_n^3\phi_{2,m} = \nabla_n^2\mathcal{H}_{2,m} \\ + \left(\nabla_n^2\mathcal{M}^{-1}\right)(\mathcal{H}_{1,m} \pm \nabla_n\phi_{2,(m-1)}) \\ + 2(\nabla_n\mathcal{M}^{-1})\left(\nabla_n\mathcal{H}_{1,m} \pm \nabla_n^2\phi_{2,(m-1)}\right) \\ + (\mathcal{M}^{-1})\left(\nabla_n^2\mathcal{H}_{1,m} \pm \nabla_n^3\phi_{2,(m-1)}\right).\end{aligned} \tag{I.53}$$

The fourth derivative of $\phi_{2,m}$, $\nabla_n^4\phi_{2,m}$, is given by (Duarte 2009)

$$\begin{aligned}\nabla_n^4\phi_{2,m} = \nabla_n^3\mathcal{H}_{2,m} \\ + \left(\nabla_n^3\mathcal{M}^{-1}\right)(\mathcal{H}_{1,m} \pm \nabla_n\phi_{2,(m-1)}) \\ + 3\left(\nabla_n^2\mathcal{M}^{-1}\right)\left(\nabla_n\mathcal{H}_{1,m} \pm \nabla_n^2\phi_{2,(m-1)}\right) \\ + 3(\nabla_n\mathcal{M}^{-1})\left(\nabla_n^2\mathcal{H}_{1,m} \pm \nabla_n^3\phi_{2,(m-1)}\right) \\ + (\mathcal{M}^{-1})\left(\nabla_n^3\mathcal{H}_{1,m} \pm \nabla_n^4\phi_{2,(m-1)}\right)\end{aligned} \tag{I.54}$$

and so on.

Eventually, the seventh derivative of $\phi_{2,m}$, that is, $\nabla_n^7\phi_{2,m}$, is given by

$$\begin{aligned}\nabla_n^7\phi_{2,m} = \nabla_n^6\mathcal{H}_{2,m} \\ + \left(\nabla_n^6\mathcal{M}^{-1}\right)(\mathcal{H}_{1,m} \pm \nabla_n\phi_{2,(m-1)}) \\ + 6\left(\nabla_n^5\mathcal{M}^{-1}\right)\left(\nabla_n\mathcal{H}_{1,m} \pm \nabla_n^2\phi_{2,(m-1)}\right) \\ + 15\left(\nabla_n^4\mathcal{M}^{-1}\right)\left(\nabla_n^2\mathcal{H}_{1,m} \pm \nabla_n^3\phi_{2,(m-1)}\right) \\ + 20\left(\nabla_n^3\mathcal{M}^{-1}\right)\left(\nabla_n^3\mathcal{H}_{1,m} \pm \nabla_n^4\phi_{2,(m-1)}\right) \\ + 15\left(\nabla_n^2\mathcal{M}^{-1}\right)\left(\nabla_n^4\mathcal{H}_{1,m} \pm \nabla_n^5\phi_{2,(m-1)}\right) \\ + 6(\nabla_n\mathcal{M}^{-1})\left(\nabla_n^5\mathcal{H}_{1,m} \pm \nabla_n^6\phi_{2,(m-1)}\right) \\ + (\mathcal{M}^{-1})\left(\nabla_n^6\mathcal{H}_{1,m} \pm \nabla_n^7\phi_{2,(m-1)}\right).\end{aligned} \tag{I.55}$$

For this series of derivatives, the numerical factors can be predetermined from Pascal's triangle relative to N, where $(N + 1)$ is the order of the derivative (Duarte 2009, 2018).

Observing the series of derivatives, $\nabla_n^1\phi_{2,m}$, $\nabla_n^2\phi_{2,m}$, $\nabla_n^3\phi_{2,m}\ldots\nabla_n^7\phi_{2,m}$, a generalized equation for the higher derivatives is found (Duarte 2013):

$$\nabla_n^r\phi_{2,m} = \nabla_n^{r-1}\mathcal{H}_{2,m} + (\mathcal{M})^{-1}(\nabla_n + \zeta)^{r-1} \tag{I.56}$$

where

$$\zeta^s = \nabla_n^s\mathcal{H}_{1,m} \pm \nabla_n^{s+1}\phi_{2,(m-1)} \tag{I.57}$$

$$\zeta^0 = 1 = \mathcal{H}_{1,m} \pm \nabla_n\phi_{2,(m-1)}. \tag{I.58}$$

The lower derivatives have been used to design multiple-prism pulse compressors, including up to six prisms, for semiconductor lasers (Pang *et al* 1992). The lower derivatives have also been used by Osvay *et al* (2004, 2005) in practical femtosecond lasers to calculate intracavity dispersions and laser pulse durations for double-prism compressors, finding good agreement between theory and experiments.

I.8 Discussion

Here it was clearly and unambiguously demonstrated that from a purely quantum equation

$$\langle x|s\rangle\langle x|s\rangle^* = \left(\sum_{j=1}^{N}\langle x|j\rangle\langle j|s\rangle\right)\left(\sum_{j=1}^{N}\langle x|j\rangle\langle j|s\rangle\right)^*$$

and representing the probability amplitudes with complex wave equations, as encouraged by Dirac, the equations for generalized diffraction, generalized refraction, and reflection can be arrived at in a coherent and unified manner. The traditional way to present these equations, via classical optics, is in reverse and in the absence of cohesiveness.

It was also shown that Heisenberg's uncertainty principle can be derived from quantum interferometric principles, thus again demonstrating the enormous significance of interference at the foundations of quantum optics.

Other equations long thought to be entirely classical in nature and origin can be traced back to quantum interferometric principles. Such is the case for the cavity linewidth equation and multiple-prism grating dispersion, a subject that has been of interest since the times of Newton (1704). These are further examples that reinforce the DFL doctrine: the foundation of optics is quantum, and as Willis Lamb implied … it is time we 'learn to enjoy it' (Lamb 1987).

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Fundamentals of Quantum Entanglement

F J Duarte

Appendix J

Introduction to Hamilton's quaternions

Here an introduction to Hamilton's quaternions, as utilized in the development of the probability amplitudes for $n = N = 3, 6$, is given.

J.1 Introduction

Hamilton's quaternions were introduced by the mathematician of the same name around the mid-1800s (Hamilton 1866). A more recent review of this subject is given by Koecher and Remmert (1991).

J.2 Basic quaternion identities

Quaternions extend beyond the realm of complex numbers and obey the main relation

$$i^2 = j^2 = k^2 = ijk = -1 \tag{J.1}$$

and the basis elements i, j, and k obey the commutative law when multiplied by 1:

$$i \times 1 = 1 \times i = i \tag{J.2}$$

$$j \times 1 = 1 \times j = j \tag{J.3}$$

$$k \times 1 = 1 \times k = k. \tag{J.4}$$

The self-consistency of the main relation given in equation (J.1) also implies that

$$ij = k \tag{J.5}$$

$$ji = -k \tag{J.6}$$

$$jk = i \tag{J.7}$$

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$$kj = -i \tag{J.8}$$

$$ki = j \tag{J.9}$$

$$ik = -j. \tag{J.10}$$

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