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Fundamentals of Quantum Entanglement

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Chapter 1

Introduction

The experimental origin of quantum mechanics is highlighted, and essential features of the photon are described. The philosophical path to quantum entanglement, as defined by the works of EPR → Bohm and Aharonov → Bell, and the physics path to quantum entanglement, as defined by the works of Dirac → Wheeler → Pryce and Ward, are introduced and discussed. The probability amplitude for quantum entanglement $|\psi\rangle = (|x_1, y_2\rangle - |y_1, x_2\rangle)$, discovered in 1947, is then introduced prior to a high-level overview of the field. This introduction then concludes with a chapter-by-chapter description of *Fundamentals of Quantum Entanglement* and its purpose.

1.1 Introduction

Very few subjects in physics evoke interest in the general population like that generated by quantum entanglement. This interest is intrinsic, fueled by general curiosity on everything quantum, and is also stimulated by writers trying to explain quantum entanglement effects while using words such as *strange*, *mysterious*, and even *weird*. The aim of this monograph is to illuminate, and elucidate, the physics and the fundamentals of quantum entanglement to scientists and engineers with a background in first-year physics and first-year mathematics.

1.2 A few words on quantum mechanics

Quantum mechanics was discovered around 1900 by Max Planck. It was the result of an experimental discovery in the macroscopic domain. Planck was attempting to explain the energy distribution of light sources as a function of frequency. In this endeavor, he introduced, in the absence of derivation, one of the most momentous equations in the history of physics (Planck 1901):

$$E = h\nu \tag{1.1}$$

where E is the energy in units of Joules (J), ν is the frequency of light in units of Hz, or s^{-1}, and $h = 6.626\ 069\ 57 \times 10^{-34}$ J s is Planck's constant. This equation links

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high frequencies, or wavelengths toward the blue end of the spectrum, with high energies. Meanwhile, low frequencies, or wavelengths toward the red end of the spectrum, are associated with low energies.

There are several equivalent ways to approach quantum mechanics, the most prevalent being:

1. Heisenberg's matrix mechanics (Heisenberg 1925, Born and Jordan 1925, Born *et al* 1926);
2. Schrödinger's equation (Schrödinger 1926);
3. Dirac's *bra–ket* notation (Dirac 1939);
4. Moyal's statistics (Moyal 1949);
5. Feynman's path integrals (Feynman and Hibbs 1965).

The presentation in this monograph is entirely based on Dirac's *bra–ket* notation. This notation is explained in chapter 2.

1.2.1 The photon from a quantum perspective

The concept of a single photon, or *quantum*, is extraordinarily important to the field of quantum entanglement. Equally important are pairs of photons or pairs of *quanta*. In this subsection, relevant thoughts and reflections on the photon by three of the brightest luminaries in the field of quantum physics are included in an effort to provide a manifold of useful concepts on the photon:

- 'The wave function gives information about the probability of *one* photon being in a particular place and not the probable number of photons in that place …. Each photon then interferes only with itself. Interference between two different photons never occurs' (Dirac 1978).
- 'Newton thought that light was made up of particles, but then it was discovered that it behaves like a wave … We say: "It is like *neither*"' (Feynman *et al* 1965).
- 'Photons cannot be localized in any meaningful manner, and they do not behave at all like particles, whether described by a wave function or not' (Lamb 1995).

In regard to Dirac's dictum on interference, it should be clarified that in quantum mechanics *two indistinguishable quanta are the same quantum*, or photon. Indistinguishable photons are photons of *the exact same frequency*.

The concept of *nonlocality* of the photon, brought up by Lamb, is of paramount importance to quantum optics and can be intuitive to experienced experimentalists: 'All the indistinguishable photons illuminate the array of N slits, or grating, simultaneously. If only one photon propagates, at any given time, then that individual photon illuminates the whole array of N slits simultaneously' (Duarte 2003). The property of nonlocality is highlighted because it is essential to quantum entanglement.

In more quantitative terms: a single photon moves, in vacuum, at the speed of light $c = 2.997\ 924\ 58\ \mathrm{m\ s^{-1}}$. It has a wavelength λ, and a frequency ν, so that

$$\lambda = \frac{c}{\nu}. \tag{1.2}$$

As already mentioned, a photon is related to a quantum energy $E = h\nu$, or

$$E = \hbar\omega \tag{1.3}$$

where $\hbar = h/2\pi$ and $\omega = 2\pi\nu$. A single photon exhibits a quantum momentum of (de Broglie 1924)

$$p = \hbar k \tag{1.4}$$

where $k = 2\pi/\lambda$ is known as the wave number.

As mentioned by Dirac (1978), a single photon is associated with *complex wave functions* of the form

$$\psi(x, t) = \psi_0 e^{-i(\omega t - kx)}, \tag{1.5}$$

and these complex wave functions can be used to provide a mathematical representation of the probability amplitudes.

Furthermore, it can be established from Heisenberg's uncertainty principle (Dirac 1978)

$$\Delta p \Delta x \approx h \tag{1.6}$$

that single photons can be extremely nonlocal and can exhibit *enormous coherence lengths* as described by

$$\Delta\nu\Delta x \approx c. \tag{1.7}$$

The extreme nonlocality comes from the fact that a single quantum can exhibit an extraordinarily narrow linewidth $\Delta\nu$. As will be seen further in this monograph, this condition of nonlocality is central to quantum entanglement. Finally, 'ensembles of indistinguishable photons exhibiting very narrow linewidth $\Delta\nu$ originating from nearly monochromatic sources, such as narrow-linewidth lasers, approximate the behavior of a single photon' (Duarte 2014).

1.3 Ward's observation

'($|x, y\rangle - |y, x\rangle$) … *was my first lesson in quantum mechanics, and in a very real sense my last, since all the rest is mere technique, which can be learnt from books*' (Ward 2004).

1.4 History of quantum entanglement

'*If I have seen further it is by standing on the sholders of giants,*' wrote Newton in 1675. This sentence is as relevant to physics now as it was then. It beautifully underscores the importance of history in physics. More specifically, it can be interpreted as a visionary call from the master to stimulate proper and meaningful referencing in the field of physics.

The vast majority of works published in the field of quantum entanglement present an incomplete version of history heavily biased toward the philosophical while neglecting the physics origin of quantum entanglement. Even though the philosophical perspective is extraordinarily enthralling, and has served quantum entanglement well by attracting the attention of the public at large, the less glamorous physics needs to be a participant in order to provide a complete and transparent exposition of the subject.

In this monograph, both the philosophical and the physics aspects of quantum entanglement are presented and discussed. This presentation is an extension of previous ideas and discussions on the subject (Dalitz and Duarte 2000, Duarte 2012, 2013a, 2013b, 2014, 2016).

1.4.1 The philosophical path

The philosophical path to quantum entanglement is well known to those familiar with the literature on the subject. It all started with a paper written by Einstein *et al* (1935) that became one of the most cited papers in physics. In this paper, which became known as EPR using the initials of the surnames of the three authors Einstein, Podolsky, and Rosen, questions were formulated on the completeness of quantum mechanics while introducing the notion of a 'possible' more complete formulation of the subject. At the same time, Einstein corresponded with Schrödinger, who also published on the subject, and introduced the word *entanglement* to describe the action at a distance introduced by the thought experiment considered by Einstein and colleagues (Schrödinger 1935, 1936). As it turns out, the EPR paper eventually generated an enormous citation following while the Schrödinger papers languished in the archives.

Key in deciding the chain of events that followed EPR was a paper by Bohm and Aharonov (1957) that discussed the EPR paper in the context of developments from the physics side of things, notably the work of Wu and Shaknov (1950), while also introducing a discussion on *hidden variables*. An additional key contribution was a famous paper by Bell (1964) that referenced Bohm and Aharonov while providing a transparent proof that quantum mechanics was incompatible with hidden variables. Incompatibility was determined by a probabilistic argument culminating in a probabilistic inequality. Violation of this inequality meant incompatibility of quantum mechanics with hidden variables. Here, it should be mentioned that hidden variable theories were though to provide a possible deterministic explanation of quantum experimental measurements.

In fairness to history, it should also be mentioned that the concept of incompatibility between quantum mechanics and hidden variables was introduced as early as 1932 by von Neumann; however, it was claimed that the proof leading to the von Neumann (1932) conclusion was found 'wanting' (Bell 1964).

Post-Bell came a paper by Clauser *et al* (1969) that made the use of Bell's theorem applicable to optical experiments and introduced modified Bell inequalities. Next came the use of those modified Bell inequalities by Aspect *et al* (1981, 1982a, 1982b) in optical experiments that demonstrated violation of Bell's inequalities. This chain of events is illustrated in figure 1.1.

Following the Aspect experiments, the citation trail of published papers in the field followed almost exclusively the philosophical path, which can be summarized as EPR → Bohm and Aharonov → Bell.

The label 'philosophical path' does not mean that it was all pure philosophy. There was also physics but the researchers were interested and motivated by a deeper meaning of quantum mechanics framed in a deterministic way of thinking compatible with classical notions. The fact that quantum mechanics denied that deeper understanding apparently made this philosophical avenue quite irresistible.

1.4.2 The physics path

The physics path is a pragmatic, measurement-driven, avenue to quantum entanglement. It began with a paper by Dirac on 'pair theory' (Dirac 1930) and some 16 years later was followed by a transparent and profound statement by John Wheeler that captures the essence of quantum entanglement: '*if one of these photons is linearly polarized in one plane, then the photon that goes off in the opposite direction with equal momentum is linearly polarized in the perpendicular plane*' (Wheeler 1946), our italics. Wheeler made his statement in reference to a positron–electron annihilation process that leads to the emission of two quanta in opposite directions, $e^+e^- \rightarrow \gamma_1\gamma_2$.

Wheeler's paper was followed by a publication that correctly predicted the quantum mechanical cross-section for γ ray scattering in experiments designed to test Wheeler's prediction (Pryce and Ward 1947). The quantum scattering equation

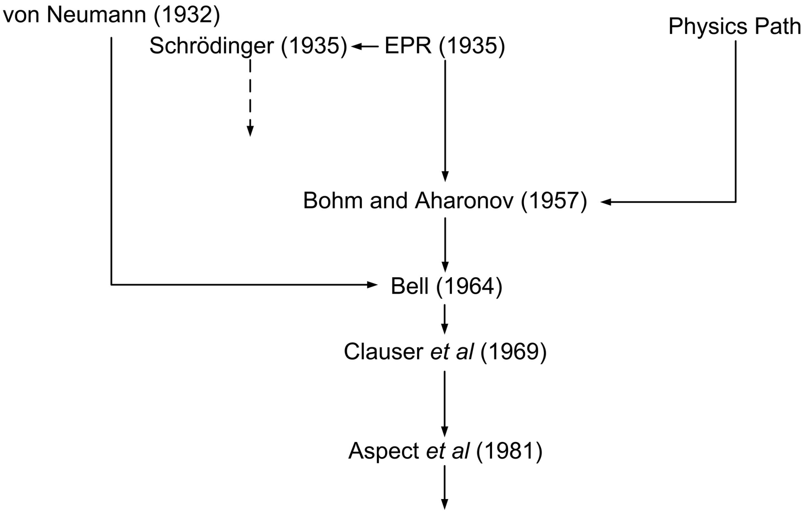

Figure 1.1. Philosophical path to quantum entanglement. Solid arrowed lines indicate a direct citation path in the literature. The main citation path runs vertically downward. For instance, von Neumann's contribution was cited by Bell but not necessarily by other authors in the direct vertical path. The papers by Schrödinger received intermittent citations for many years and only began to be cited regularly since the 1990s. The assigned years refer to literature emergence dates.

of Pryce and Ward was corroborated theoretically by Snyder *et al* (1948) and experimentally by Hanna (1948), Bleuler and Bradt (1948), and Wu and Shaknov (1950).

Following the publication of the ~½ page disclosure entitled ‘Angular correlation effects with annihilation radiation’ (Pryce and Ward 1947), Ward followed with a disclosure of the derivation of the quantum entanglement probability amplitude, $|\psi\rangle = (|x_1, y_2\rangle - |y_1, x_2\rangle)$, as part of his dual-topic doctoral thesis (Ward 1949). It should be noted that this probability amplitude is essential to the derivation of the final correct quantum scattering equation published by Pryce and Ward (1947).

It should also be stated that $|\psi\rangle = (|x_1, y_2\rangle - |y_1, x_2\rangle)$ includes and contains all the physics relevant to quantum entanglement experiments. All this was done in a complete vacuum of philosophical discussions and in the total absence of concern of, or preoccupation with, hidden variable theories.

In the physics path to quantum entanglement, illustrated in figure 1.2, it is apparent that after the Wu and Shaknov (1950) experiments, Wu and coworkers revisited the entanglement arena this time utilizing cross-field fertilization from the philosophical path, namely with knowledge of Bell’s theorem (Kasday *et al* 1975). As it turns out, practitioners in the philosophical path doubted the relevance of the Wu experiments from a local hidden variable perspective (Clauser *et al* 1969, Clauser and Horner 1974). Following this line of thought, Aspect *et al* (1981) wrote, ‘the experiments agree with QM predictions. However, because of the lack of efficient polarizers for 0.5 MeV photons, strong supplementary assumptions are

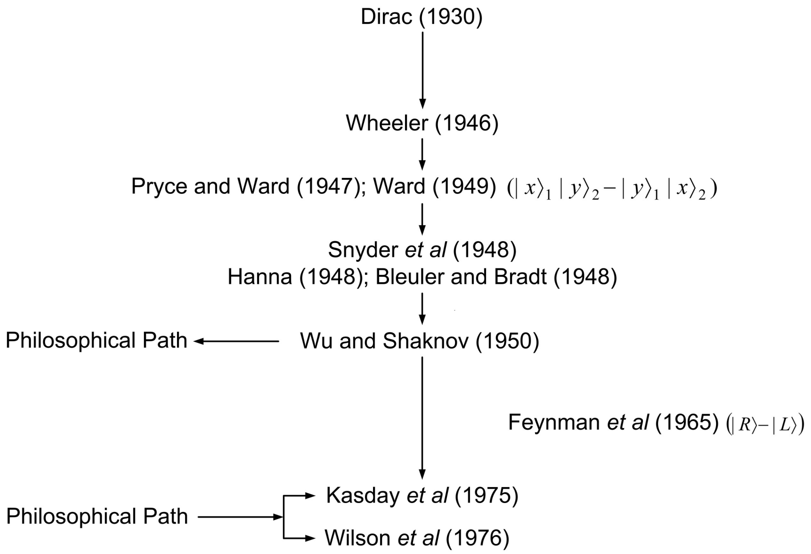

Figure 1.2. Physics path to quantum entanglement. Solid arrowed lines indicate a direct citation path in the literature. The main citation path runs vertically downward. The paper by Wu and Shaknov (1950) influenced the paper by Bohm and Aharonov (1957) in the philosophical avenue while the papers by Kasday *et al* (1975) and Wilson *et al* (1976) were influenced by Bell’s theorem. The assigned years refer to literature emergence dates.

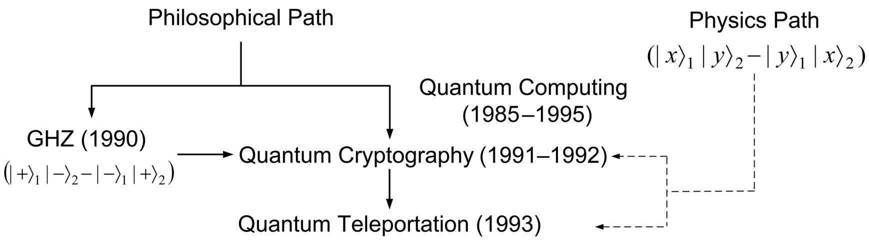

Figure 1.3. The literature from the philosophical path converges directly into quantum entanglement-based subfields such as quantum cryptography (and/or quantum communications) and quantum teleportation. The influence of the physics path is largely limited to the widespread use of $|\psi\rangle = (|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$, in the absence of acknowledgments. Quantum computing emerged in 1985 via a Feynman contribution and from the paper that introduced the term *qbit* in 1995 (see text). A noticeable fraction of authors in contemporaneous quantum computing research do refer to EPR and Bell's theorem.

necessary to interpret these results via Bell's theorem'. This was despite the initial favorable reception given by Bohm and Aharonov (1957) to the experiment by Wu and Shaknov (1950).

What should be kept in mind is that even though today all of the developments in the field of quantum entanglement revolve around the probability amplitude for quantum entanglement, in its several versions, there is almost no acknowledgment of its origin or the physics path that led to its discovery. This monograph is designed to provide a perspective on quantum entanglement from the philosophical and the physics perspectives by including all the relevant literature. This approach should help remove the cloud of mystery that surrounds quantum entanglement.

In conjunction with figures 1.1 and 1.2, the overall perspective of the field is given in figure 1.3.

1.5 The field of quantum entanglement

The emergence of the combined words *quantum entanglement*, in the open literature, appears to go back to the mid to late 1980s (Ghirardi *et al* 1987). This was a few years after the optical experiments on quantum entanglement by Aspect *et al* (1981, 1982a, 1982b).

Today the field of quantum entanglement is divided roughly into three subfields as outlined in figure 1.3: quantum cryptography (which includes quantum communications), quantum teleportation, and quantum computing. On paper, judging by citations, these subfields have been heavily influenced by the ideas and concepts derived from the philosophical path to quantum entanglement.

Also on paper, and judging from citations, the acknowledgment of the physics path has been miniscule. This almost non-existing recognition has persisted albeit the all-important probability amplitude for quantum entanglement, which was discovered in a vacuum of philosophical arguments:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2) \tag{1.8}$$

was reintroduced into the mainstream of quantum entanglement as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2) \tag{1.9}$$

by Greenberger *et al* (1990) in the same paper that introduced the GHZ probability amplitude.

In other words, although equation (1.8) began to be used heavily in quantum cryptography and quantum teleportation, virtually no mention has been made of its origin back in 1947. This is quite a remarkable situation.

In figure 1.3, quantum cryptography is dated as having been initiated in 1991 since that is the date applicable to the introduction of entangled photon pairs to the field (Ekert 1991). However, quantum cryptography as a concept goes back to the early–mid 1980s (see, for example, Bennett and Brassard 1984). Quantum teleportation was introduced by Bennett *et al* (1993).

Some of the first discussions on quantum computing are due to Feynman (1985, 1986), and reflecting on Feynman's independence, his approach to quantum computing was his own and thus it did not follow a pre-established literature path. The term *qbit*, short for quantum bit, was introduced by Schumacher (1995). Schumacher's approach, although it does not mention the EPR → Bohm and Aharonov → Bell philosophical path, does touch on issues of quantum interpretation and thus it might also qualify as a step in the philosophical path. A subsequent paper by Steane (1998) deals directly with EPR–Bell notions. The dates 1985–95 associated with quantum computing in figure 1.3 refer to the pre-philosophical association period of the field.

In quantum computing, alternative formulations of the basic probability amplitude for quantum entanglement are (see, for instance, Steane 1998)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \tag{1.10}$$

and

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle) \tag{1.11}$$

or

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle). \tag{1.12}$$

1.6 Fundamentals of Quantum Entanglement

Fundamentals of Quantum Entanglement is a focused monograph designed to deal almost directly with the origin and physics of the probability amplitude for quantum entanglement

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1|y\rangle_2 \pm |y\rangle_1|x\rangle_2) \tag{1.13}$$

in its minimal coverage, which is for two quanta ($n = 2$) and two propagation paths ($N = 2$), or $n = N = 2$. The approach taken to explain the physics is a generalized interferometric approach (Duarte 2013a, 2013b, 2014). The extended cases for $n = N = 2^1, 2^2, 2^4 \ldots 2^r$ and $n = N = 3, 6$ are also reviewed.

The selected style of presentation is a series of dedicated brief chapters. The aim is to convey in a focused style the essence of the subject at hand so as to offer the reader the opportunity to contemplate the subject matter before moving to another topic. In the first part of the book the presentation is chronological.

Next, the titles of the chapters, beyond this introduction, are listed with a brief explanation of their content.

Chapter 2: *Dirac's contribution*: provides a very brief non-technical description of Dirac's pair theory (Dirac 1930) followed by an introduction to Dirac's *bra–ket* notation (Dirac 1939). Both of these contributions went on to impact the field of quantum entanglement.

Chapter 3: *The EPR paper*: reviews the contribution of Einstein *et al* (1935) that was the genesis of the philosophical path to quantum entanglement.

Chapter 4: *The Schrödinger papers*: reviews the Schrödinger papers that first mentioned the words *entanglement* and *disentanglement* in a quantum mechanical framework (Schrödinger 1935, 1936).

Chapter 5: *The Wheeler paper*: reviews the paper that provided the first lucid and transparent description, in words, of the iconic quantum entanglement experiment (Wheeler 1946).

Chapter 6: *The probability amplitude for quantum entanglement*: introduces the Pryce–Ward probability amplitude, $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$, at the center of quantum entanglement physics and the experimental schematics of the original experiment (Pryce and Ward 1947, Ward 1949).

Chapter 7: *The quantum entanglement experiment*: provides a description of the concepts involved in the quantum entanglement experiment based on the probability amplitude $|\psi\rangle = (|x_1, y_2\rangle - |y_1, x_2\rangle)$.

Chapter 8: *The annihilation quantum entanglement experiments*: describes the first quantum entanglement experiments utilizing positron–electron annihilation, via $e^+e^- \rightarrow \gamma_1\gamma_2$, and detection utilizing gamma ray scattering configurations (Hanna 1948, Bleuler and Bradt 1948, Wu and Shaknov 1950).

Chapter 9: *The Bohm–Aharanov paper*: reviews the first paper that brought together the philosophical path, including aspects of local hidden variable theories, and the physics path. This paper became an additional point of origin, apart from EPR's work, for the mainstream literature on quantum entanglement (Bohm and Aharonov 1957).

Chapter 10: *Bell's theorem*: reviews the paper that provided a modern transparent proof that local hidden variable theories are incompatible with the predictions of quantum mechanics (Bell 1964).

Chapter 11: *Feynman's Hamiltonians*: outlines Feynman's contributions that led to equations of the form $|\psi\rangle = (|B\rangle - |A\rangle)$ but did not derive the probability amplitude for quantum entanglement (Feynman *et al* 1965).

Chapter 12: *The second Wu quantum entanglement experiment*: reviews a paper that recasts the quantum entanglement gamma ray scattering experiments in light of Bell's theorem (Kasday *et al* 1975).

Chapter 13: *The hidden variable theory experiments*: mentions efforts to extend Bell's theorem to make possible testing for local hidden variable theories in the optical domain (Clauser *et al.* 1969, Clauser and Horner 1974).

Chapter 14: *The optical quantum entanglement experiments*: describes the experiments of Aspect *et al* (1981, 1982a, 1982b) that were initiated in an effort to test for local hidden variable theories.

Chapter 15: *The quantum entanglement probability amplitude* 1947–91: reviews the various formats of $|\psi\rangle = (|x, y\rangle - |y, x\rangle)$ in the 1947–91 period.

Chapter 16: *The GHZ quantum entanglement probability amplitudes*: accounts for the reintroduction of $|\psi\rangle = (|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$ into the mainstream of quantum entanglement research and the introduction of the GHZ probability amplitudes for four particles and three particles (Greenberger *et al* 1990).

Chapter 17: *The interferometric derivation of the quantum entanglement probability amplitude for* $n = N = 2$: describes the derivation of $|\psi\rangle = (|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$ from a generalized N-slit interferometric perspective (Duarte 2013a, 2013b, 2014).

Chapter 18: *The interferometric derivation of the quantum entanglement probability amplitudes for* $n = N = 2^1, 2^2, 2^3 \ldots 2^r$: describes the systematic derivation of probability amplitudes for $n = N = 2^1, 2^2, 2^3 \ldots 2^r$ from the N-slit interferometric perspective (Duarte 2015, 2016, Duarte and Taylor 2017).

Chapter 19: *The interferometric derivation of the quantum entanglement probability amplitude for* $n = N = 3, 6$: describes the derivation of probability amplitudes for $n = N = 3, 6$ from the N-slit interferometric perspective (Duarte 2015, 2016).

Chapter 20: *What happens with the entanglement at* $n = 1$ and $N = 2$?: discusses reversibility from quantum entanglement probability amplitudes to interferometric probability amplitudes and vice versa. It also discusses the interferometric physics at $n = 1$ and $N = 2$.

Chapter 21: *The quantum entanglement probability amplitude and Bell's theorem*: describes how probabilities originating in the probability amplitude $|\psi\rangle = (|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$ are used in conjunction with Bell's inequalities.

Chapter 22: *Cryptography* via *quantum entanglement*: describes how secure cryptographic communications, in free space, can be accomplished via the physics of quantum entanglement.

Chapter 23: *Quantum entanglement and teleportation*: explain the quantum entanglement concepts applied to achieve teleportation of quantum states.

Chapter 24: *Quantum entanglement and quantum computing*: provides a succinct overview on the use of quantum entanglement concepts, via $|\psi\rangle = (|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$, to perform quantum logic operations. It also offers an additional avenue, via Pauli matrices, to derive $|\psi\rangle = (|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$.

Chapter 25: *Space-to-space and space-to-Earth communications* via *quantum entanglement*: revisits the concept of quantum cryptography with an specific emphasis on satellite communications.

Chapter 26: *Quantum interferometric communications as an alternative to quantum entanglement?*: explains the physics of secure space-to-space communications using quantum interference principles.

Chapter 27: *Quanta sources for quantum entanglement*: reviews the various sources of quanta pairs exhibiting orthogonal polarizations, and provides an outlook on possible future sources.

Chapter 28: *More on quantum entanglement*: provides a pragmatic perspective of quantum entanglement divergent from philosophical concerns.

Chapter 29: *On the interpretation of quantum mechanics*: this is a discussion on the interpretation of quantum mechanics from a pragmatic perspective along the lines of thought of Dirac, Feynman, Lamb, van Kampen, and Ward (Dirac 1987, Feynman *et al* 1965, Lamb 1987, 2001, van Kampen 1987, Ward 2004).

Appendix A: Revisiting the EPR paper.

Appendix B: Revisiting the Pryce–Ward probability amplitude.

Appendix C: Classical and quantum interference.

Appendix D: Interferometers and their probability amplitudes.

Appendix E: Polarization rotators.

Appendix F: Vector products in quantum notation.

Appendix G: Trigonometric identities.

Appendix H: More on probability amplitudes.

Appendix I: From quantum principles to classical optics.

Appendix J: Introduction to Hamilton’s quaternions.

1.7 Intent

Today, there are many young engineers working on many of the aspects, and applications, of quantum entanglement. Quantum mechanics works, the technology works, and this is a wonderful quantum world. However, many questions persist on the origin of quantum entanglement. Although various books and reviews expose the subject, they do so from the philosophical perspective. Moreover, they do so while introducing the probability amplitude $|\psi\rangle = (|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2)$ out of the blue, without explaining its origin or physics. This is not the case with *Fundamentals of Quantum Entanglement*. Here, the origin of this wondrous probability amplitude is explained via the beautiful interferometric principles laid down by Dirac and championed by Feynman. In the minds of many readers questions will probably remain; however, it is hoped that the tools and ideas presented here will help those driven by curiosity to continue the path of discovery.

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