Appendix A

Unit conversion

Length

1 mile = 5280 feet = 1.61 km
1 inch = 25.4 mm = 2.54 cm
1 foot = 12 inches = 30.48 cm
1 meter = 1.094 yd =
1 yd = 3 feet = 36 inches

Mass

1 pound = 16 ounces
1 gallon = 4 quarts
1 quart = 2 pints
1 pound = 454 grams
1 liter = 1.06 quarts
1 ton = 908 kg

Time

1 year = 12 months = 365 days = 3.15×10^7 s
1 day = 24 h = 1440 min = 86 400 s

Capacity

1 fl oz = 29.57 ml
1 quart (qt.) = 2 pints (pt.) = 0.946 l
1 pt = 0.473 l
1 gallon (gal) = 4 qt = 3.785 l

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Techniques of Classical Mechanics
From Lagrangian to Newtonian mechanics
Samya Zain

Appendix B

Velocity and acceleration in various coordinates

B.1. Velocity and acceleration in polar coordinates

In polar coordinates, the position of a particle is found by knowing the radial distance (r) from a user defined origin (O) and the angle (ϕ) (figure B.1).

The position vector of a particle
The trajectory of a particle is found by evaluating r and ϕ as a function of t. The position vector of a particle is mathematically given as

$$\vec{r} = |\vec{r}|\hat{r}. \tag{B.1}$$

Velocity and acceleration of a particle
To find velocity and acceleration, let us begin with unit vectors $\hat{r}$ and $\hat{\phi}$ in terms of their Cartesian components along $\hat{i}$ and $\hat{j}$. The unit vectors expressed in Cartesian coordinates are

$$\begin{cases} \hat{r} &= \cos\phi\hat{i} + \sin\phi\hat{j} \\ \hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j}\,. \end{cases} \tag{B.2}$$

Differentiating we obtain

$$\begin{cases} \dfrac{d\hat{r}}{dr} &= 0 \\ \dfrac{d\hat{r}}{d\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j} \equiv \hat{\phi} \end{cases} \tag{B.3}$$

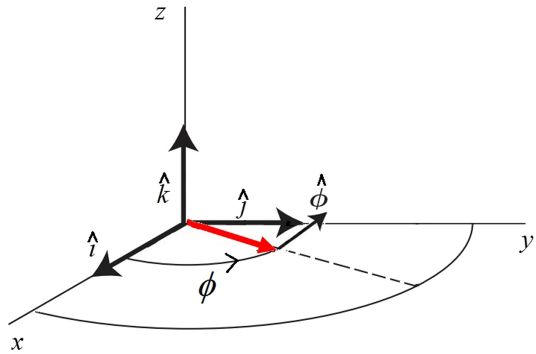

Figure B.1. Polar coordinates.

$$\begin{cases} \dfrac{d\hat{\phi}}{dr} = 0 \\ \dfrac{d\hat{\phi}}{d\phi} = -\cos\phi\hat{i} - \sin\phi\hat{j} \equiv -\hat{r}. \end{cases} \tag{B.4}$$

Velocity vector

We can now derive equation (B.1) with respect to time and write

$$\vec{v} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}. \tag{B.5}$$

Here, $\dot{r} \equiv v_r$ is called the *radial* velocity component and $r\dot{\phi} \equiv v_\phi$ is called the *transverse* or the *circumferential* velocity component. Note that, $\sqrt{v_r^2 + v_\phi^2} \equiv v$.

Acceleration vector

Differentiating equation (B.5) with respect to time, we obtain the acceleration as

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}}. \tag{B.6}$$

In this equation we can rewrite the second term as

$$\dot{r}\dot{\hat{r}} = \dot{r}\frac{d\hat{r}}{dt}\frac{d\phi}{d\phi} = \dot{r}\frac{d\hat{r}}{d\phi}\frac{d\phi}{dt} = \dot{r}\hat{\phi}\dot{\phi}$$

and the last term can be rewritten as

$$r\dot{\phi}\dot{\hat{\phi}} = r\dot{\phi}\frac{d\hat{\phi}}{dt}\frac{d\phi}{d\phi} = r\dot{\phi}\frac{d\hat{\phi}}{d\phi}\frac{d\phi}{dt} = r\dot{\phi}(-\hat{r})\dot{\phi} = -r\dot{\phi}^2\hat{r}.$$

Putting these values in equation (B.6) and rearranging we obtain

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}, \tag{B.7}$$

where $a_r \equiv (\ddot{r} - r\dot{\phi}^2)$ is called the *radial* acceleration component, and $a_\phi = (r\ddot{\phi} + 2\dot{r}\dot{\phi})$ is called the *transverse* or the *circumferential* acceleration component. Please note that $\sqrt{a_r^2 + a_\phi^2} \equiv a$.

B.2. Velocity and acceleration in cylindrical coordinates

In the case of 3D motion the position vector of a particle in cylindrical coordinators is given as (figure B.2):

$$\vec{r} = r\hat{r} + z\hat{z}. \tag{B.8}$$

We know from equation (B.2) the unit vectors $\hat{r}$ and $\hat{\phi}$, which are the same for cylindrical coordinates. Additionally we will have $\hat{z} = \hat{k}$ as the third coordinate. Their derivatives will be given as

$$\begin{cases} \dot{\hat{r}} & = & \dot{\phi}\hat{\phi} \\ \dot{\hat{\phi}} & = & -\dot{\phi}\hat{r} \\ \dot{\hat{k}} & = & 0. \end{cases} \tag{B.9}$$

The kinematic vectors can now be expressed relative to the unit vectors. Thus, the velocity vector will be given as

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{k}. \tag{B.10}$$

Taking the derivative w.r.t. time again, we obtain acceleration vector as

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} + \ddot{z}\hat{k}. \tag{B.11}$$

Note the similarities between equations (B.5) and (B.10) for the velocity vectors and between equations (B.7) and (B.11) for acceleration vectors.

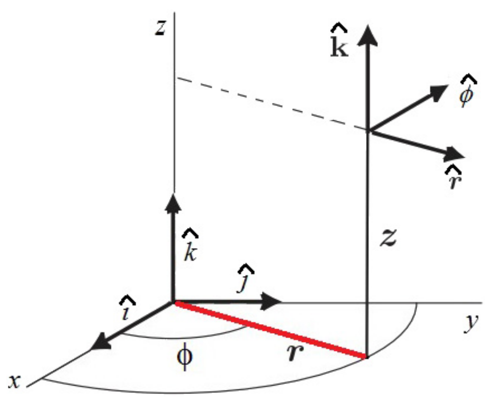

Figure B.2. Cylindrical coordinates.

B.3. Velocity and acceleration in spherical coordinates

In spherical coordinates, we use a distance (r) and two angles (θ, ϕ) to specify the position of a particle as shown in figure B.3.

The unit vectors written in Cartesian coordinates are

$$\begin{cases} \hat{r} &= \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \\ \hat{\theta} &= \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k} \\ \hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j}\,. \end{cases} \tag{B.12}$$

It can easily be seen that the derivatives of these unit vectors may be simplified using equation (B.12) as

$$\begin{cases} \dfrac{d\hat{r}}{dt} &= \dot{\phi}\sin\theta\hat{\phi} + \dot{\theta}\hat{\theta} \\ \dfrac{d\hat{\theta}}{dt} &= -\dot{\theta}\hat{r} + \dot{\phi}\cos\theta\hat{\phi} \\ \dfrac{d\hat{\phi}}{dt} &= -\dot{\phi}\sin\theta\hat{r} - \dot{\phi}\cos\theta\hat{\phi}. \end{cases} \tag{B.13}$$

The kinematic vectors can now be expressed relative to the unit vectors. Thus, the position vector will be given as

$$\vec{r} = r\hat{r}. \tag{B.14}$$

The velocity vector will be given as

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\sin\theta\hat{\phi} + r\dot{\theta}\hat{\theta}. \tag{B.15}$$

Taking derivative w.r.t. time again we obtain the acceleration vector as

$$\begin{aligned} \vec{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} \\ + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi}. \end{aligned} \tag{B.16}$$

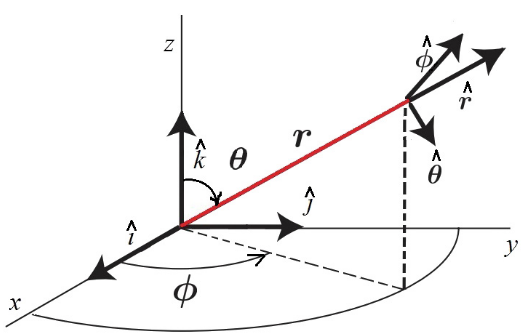

Figure B.3. Spherical coordinates.

Appendix C

Noether's theorem

Noether's theorem was presented in 1918 CE by Emmy Noether and is one of the most beautiful theorems in physics. Noether's theorem is very useful in providing insights into general theories in physics. Noether's theorem allows us to relate symmetries of a theory with its laws of conservation. For example:

- If a system is invariant (does not change) under spatial translation we obtain the law of conservation of linear momentum.
- If a system is invariant under rotation we obtain the law of conservation of angular momentum.
- If a system does not change over time we obtain the law of conservation of energy.

Why do we even expect conservation laws to be related to symmetry?
At the outset, it is not at all clear what the relation between conservation laws and symmetries is. We resort to a simple example. Consider Newton's second law: the rate of change of linear momentum equals the net external force acting on a body, i.e.

$$\vec{F} = \frac{dp}{dt}. \tag{C.1}$$

Suppose that the force field $\vec{F}$ is derived from an external potential, i.e. $\vec{F} = -\Delta V$, and the problem in 1D is then

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x} = \lim_{\varepsilon \to 0} \frac{V(x+\varepsilon) - U(x)}{\varepsilon}. \tag{C.2}$$

Note that when potential (V) is a constant then $V(x+\varepsilon) = V(x)$. This means that $\vec{F} = 0$ and hence from Newton's law, we can write $\frac{dp}{dt} = 0$ with linear momentum (p) = constant.

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Next, when we say $V(x) = V(y)$ it is the same as saying that potential V is invariant under translations $x \to x'$, i.e. that translation in space is a symmetry of the function V. Hence we have found that the translational symmetry of V implies conservation of linear momentum.

Noether's theorem establishes such relationships between symmetries of the action and quantities conserved along the trajectory.

Appendix D

Configuration space

Configuration space is also called *C-space*. In classical mechanics, the parameters that define the configuration of a system are called generalized coordinates and the vector space defined by these coordinates is called the *configuration space* of the physical system. Configuration space is a way to visualize the state of an entire system as a single point in a higher dimensional space:

- A single particle's position in a 3D Euclidean 3-space is

$$\vec{r} = (x, y, z), \tag{D.1}$$

  then its C-space = R^3.

  If a particle is constrained in some way, for example, if it can only move along a wire, then the C-space is the subset of R^3 that defines point on the wire, i.e. S^1.

  If the particle can only move in 2D, for example when we play air-hockey, the puck can only move on the table-top. Then the C-space is given as S^2.

  Similarly if the particle moves on a sphere, the C-space is also given as S^2.
- For a system made up of n-particles, the C-space will be given by R^{3n}.

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