Techniques of Classical Mechanics

From Lagrangian to Newtonian mechanics

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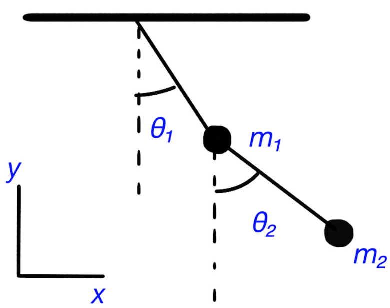

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To my husband, who is my rock, and
to my family, who have been putting up with me since either my birth or theirs!

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Preface

Techniques of Classical Mechanics: From Newtonian to Lagrangian Mechanics is, as the name implies, intended to be a gentle transition from introductory freshman physics to more advanced topics in mechanics. This book is the outcome of an undergraduate physics course given at Susquehanna University to Junior and Senior physics and math majors. There is sufficient material for a one-semester four-credit hour long course. The underlying philosophy of this book is that physics is a cohesive set of ideas, knowledge and experience that needs to be understood in its entirety and should not be compartmentalized.

Mechanics is one of the earliest physics topics to be developed and historically forms the basis of many fields in physics and engineering. So a deep understanding and appreciation of the field is essential for any serious scholar in the field. I will introduce Lagrangian and Hamiltonian mechanics quite early in the text and then address physics problems utilizing as many techniques as I can. My hope is that this text will be the bridge that helps students transition from introductory physics to graduate level physics.

I recognize that mathematics is fundamental and indispensable to the study of physics. A fair amount of familiarity with calculus up to multivariate calculus and deferential equations is assumed. Generally, I will assume that students will recognize Newton's force law as a second order differential equation and will be able to recognize its solutions as sinusoidal and exponential functions. However, where possible I will avoid deriving proofs in favor of problem solving methods. For details of mathematical proofs of concepts please check other sources.

Main ideas for understanding

In this text I have put physics concepts before the historic development of physics topics. Where necessary I have prioritized concepts according to their fundamental significance rather than historic development.

When I started writing this book I pictured sitting and talking with a student. I have tried to keep it as conversational and down-to-earth as possible. Students should be able to pick it up and learn on their own just as easily as they would in a classroom environment. Worked examples are included as part of the text in nearly every section. Additional concept questions and exercises are included for further practice and understanding.

Wherever possible I will introduce physics concepts by showing various methods that could be used to solve them. For example, I will introduce a physics problem then show how to solve it using many different techniques, say by solving it by energy conservation, Newtonian mechanics, Lagrangian mechanics, Hamiltonian mechanics, etc. I hope that by doing this students/readers will develop a feel for the subject and when required they will have in their basket of knowledge the correct tools to approach the various kinds of physics problems.

The main ideas that we will tackle in the course of this textbook include the following topics:

Big idea 1: Kinematics
Big idea 2: Conservation laws
Big idea 3: Simple harmonic motion
Big idea 4: Gravitation
Big idea 5: Circular motion
Big idea 6: Fluids
Big idea 7: Resistance due to fluids
Big idea 8: Solids
Big idea 9: Motion of rigid bodies
Big idea 10: System of particles
Big ideal 11: Scattering theory

Mainly for instructors

I realize that you may see some concepts repeated in multiple chapters. This repetition is intentional. I feel that reviews and re-reviews of these important concepts can never hurt. Some questions at the end may go just beyond the material in the chapter. Such content is intended for students to apply the concepts that they have learnt during the course of the chapter and the book overall.

Mainly for students

Before we begin, I just have a few additional things to comment on. Remember, physics must *not* be memorized but rather understood. Know that understanding does not come quickly or easily, it takes patience and resilience. Remember how you learnt to play an instrument? Did you become an expert overnight? Or did it take hours and hours of practice? Physics is just the same, you need to actively learn it, cramming overnight before the exam is not be the ticket to understanding and hence success.

Easy how-tos to succeed in physics:

1. Manage your time.
2. Commit yourself whole-heartedly.
3. Find a calendar-method that works for you. Get organized.
4. Find a study partner; motivate your partner and yourself.
5. Actively seek help when needed.
6. Set a scheduled dedicated time to study physics every day. Thirty minutes a day, every day will get you more than pulling-an-all-nighter the day of the quiz.
7. Prepare before coming to class, be a good listener and take good notes.
8. And the most important of all—remember, someone very wise once said: ‘None of the tricks of success will work unless you do’.

Half your battle is—just show up. So read the book, do the examples and exercises, come to class, ask questions, share your knowledge, communicate and most important of all, work hard and enjoy the ride.

Samya Zain
30 August 2018

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I am indebted to many people who have been invaluable during this whole process. First and foremost is Dr Grosse, who sat through the course and when I first mentioned the idea of writing this book, encouraged me to put words to paper, and Andrew Steely who was responsible for the first read, poor guy!

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My heartfelt gratitude also extends to my family, my dad Dr Iqbal and mom Safia Bano, and my brother Dr Aamir Iqbal, who have always believed in me and have encouraged me to be my best self. To my very patient husband Zain, for putting up with the long hours of me typing away and for 'understanding' without once attempting to actually read my manuscripts! My kids, Hareem, Sami and Eman, for being (nearly always!) willing to postpone the Hershey park trip and not complaining about lots and lots of pizza nights.

Author biography

Samya Zain

Samya Zain grew up in Lahore, Pakistan. In seventh grade she decided that she wanted to learn more physics, since physics explains how the universe works, and additionally allowed her not to go into the medical field like the rest of her family. She completed her undergraduate work at the University of the Punjab, Lahore, Pakistan and was awarded the gold medal for the highest score in the annual final exam. Then with her husband and four-year-old son she went on to graduate work in physics at the university at Albany, State University of New York in 2001. She received her PhD in experimental particle physics in 2006 and continued as a post-doctoral fellow at the Albany High Energy Physics group. In the fall of 2008, she joined Susquehanna University as an assistant professor, where she currently serves as an associate professor and chair of the department of physics.

When not working on physics she can often be found with her nose in a book. She makes time to read every day (even when she has to hide from her family in order to do so). She paints when the mood strikes her, loves visiting used bookstores and collecting books and absolutely refuses to let any of her collection go.

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Chapter 1

Foundations

1.1 The nature of science

Science is the observation and precise description of natural phenomena. Math is the language of all sciences, for example, physics, chemistry, biology, geology, astronomy, engineering, etc

Physics serves as a foundation for other sciences. Physics is fundamentally the study of matter and waves, from the smallest particles, such as quarks and electrons, to the largest bodies, such as galaxies. The difference in scale between the smallest and largest objects is of the order of 10^{40}!

Physicists use the language of mathematics to construct physical theories to describe the world around us. The description might not be perfect, but it is remarkable how well we still do with our descriptions. So in general we can think of physics as a *toolbox of ideas* which can be used in building scientific models to describe the world around us.

Before we begin a formal discussion let us define a few important concepts that we will use:

- *Coordinate system*. A coordinate system is a way of using sets of numbers to identify points in space. It is an imaginary grid of lines that fills all space. A coordinate system is used to identify the location of a particular particle or an event.
- *Origin*. An origin is an agreed upon reference point in a particular coordinate system. All measurements are made with respect to the origin in the coordinate system.
- *Systems of coordinates*. The most common coordinates are *rectangular*, *spherical* and *cylindrical* coordinates, the details of each are given in section 1.7.
- *Event*. If an action occurs at a particular point of space at a particular time then in physics we call it an *event*. For example, the first instantaneous contact between colliding cars in an accident is called an event.

  However, for the complete description of the entire collision we need to define many points that occur over multiple units of time. We call this complicated process a *collection* of events. To simplify our systems we often consider mass no matter how small or large to occupy one point in space.

- *Ideal particle*. In classical physics an ideal particle is an approximation and is regarded as a point-like object. In most cases we will regard complicated objects, such as a driver and car where the driver may be drinking coffee, as a single object with a particular mass (m). Classical physics often simplifies systems by considering objects to occupy a single point of space in our coordinate system.
- *Complete description*. In classical physics we need to know the position of a particle at all times. Hence, for a *complete description* of the particle state, we need to know the mass of the object and the position of it in space at any given time. This means that measurements of position, time and mass are fundamentally important to physics and understanding physical systems.
- *Three dimensions of space*. Since we live in a three-dimensional space, it is obvious that we will require three coordinates to locate an object relative to a user-defined origin. In general we use Cartesian coordinates (x, y, z), but we also use spherical and cylindrical coordinates. The specifics and details of each coordinate system are given in section 1.7. In order to measure an event we have to do the following:
    1. Set an origin, called a user-defined origin.
    2. Define a coordinate system with respect to the user-defined origin.
    3. Measure the coordinates of the event with respect to the user-defined origin and coordinate system.

    This is called a *relative measurement* since it is based on the definitions of the origin and coordinate system. If we change the origin or the orientation of the coordinate system it will appear that the event has changed, which cannot happen. To overcome this issue we have to define different coordinate systems called *frames of reference*, which can be transformed to each other such that the event under consideration remains physically constant and unchanged. Measurement of position is always relative and only becomes meaningful once a coordinate system is defined.
- *Reference frames*. A coordinate system and a synchronized time together make up a reference frame. Both of them determine the location of a particular event. When the coordinates of the particle are constant over time, the reference frame is called the *rest frame* of the particle. For example, we are on planet Earth, hence Earth is a rest frame for us and we call it *Earth's rest frame* (abbreviated as ERF). In Earth's rest frame we can ignore the motion of the Earth with respect to the Sun and only concentrate on the motion of the car driving down a highway at 55 miles/hour.

1.2 Units

At the heart of it all, physics is an experimental science and experiments involve measurements. Measurements in physics involve many varied and sometimes complex quantities such as force, energy, momentum, magnetic field, capacitance, etc.

The surprising thing is that all these complex qualities can be expressed using a few fundamental quantities, for example, the measurement of time (T), length (L) and mass (M). Historically, the scientific community decided that these units, e.g. seconds, meters and kilograms, are convenient to use in physical experiments and decided that these will be called *fundamental units*, or SI units, which is abbreviated from the French, Le Système International d'Unités (the International System of Units). However, everyone does not agree on the standard units and the history of these debates is quite interesting.

We should always be able to convert from one system of units to another given the *dictionary* of units.

1.2.1 Different systems of units

There are three systems of units currently in use around the world. These are:

- The *MKS system*, also called the metric system, which uses meter (m), kilogram (kg) and seconds (s) as the base units for measurements. This system is the most commonly used around the world.
- The *CGS system*, also called the Gaussian system, which uses centimeter (cm), grams (gm) and seconds (s) as base units for measurements.
- The *US-engineering system*, also called the British system, is most commonly used in the US. In this system the unit of length is a foot and the unit of time is a second. This system does not have a base unit for mass, instead it uses the unit of the pound-mass, symbolized as (lb-mass).

Units of...	MKS	CGS	US-engineering
Length	meter (m)	centimeter (cm)	foot (ft)
Mass	kilogram (kg)	gram (gm)	pound-mass (lb-mass)
Time	second (s)	second (s)	second (s)
Temperature	kelvin (K)	kelvin (K)	Fahrenheit (°F)

1.3 International system of units (SI)

The International System of Units (SI) specifies a set of base units and derived units. The names of base SI units are written in lowercase, for example, meter (m), kilogram (kg) and time (s).

Derived units are associated with derived quantities such as the newton, which is a unit of weight or force ($\mathrm{kg\,m\,s^{-2}}$), and watt, which is a unit of power ($\mathrm{kg\,m^2\,s^{-3}}$). Interesting fact: the symbols for units named after scientists are often capitalized.

1.3.1 Base SI units

In this section let us discuss the base units of time, mass and length.

1.3.1.1 Time: the second

What is time and what is the arrow of time? This is a deep philosophical question, and that is a discussion for a later time. At this point all we have to remember is that time is measured in SI units: seconds. The dictionary of conversion units of time includes:

1 millisecond	= 1 ms	= 10^{-3} s
1 microsecond	= 1 μs	= 10^{-6} s
1 nanosecond	= 1 ns	= 10^{-9} s

1.3.1.2 Mass: the kilogram

Mass is something we seem to be very familiar with in everyday life, however, the concept of mass has a very deep relationship to both length and time.

Mass is the only base unit that is defined by an artifact, a platinum–iridium cylinder stored at the International Bureau of Weights and Measures, in France. The mass of this cylinder is defined to be exactly one *kilogram*. It is expected that in November 2018 the kilogram will be defined in terms of the quantum mechanical Planck's constant (h), and will be given an exact value based on measurements from around the world. Please note the physical kilogram mass will not be eliminated as they are the most practical method of everyday measurements. But the difference will be that the SI mass unit will be based on a fundamental constant of nature rather than an artifact.

Conversion of the kilogram is given in the table below.

1 kilogram	= 1 kg	= 10^{3} g
1 gram	= 1 gm	= 10^{-3} kg
1 milligram	= 1 mg	= 10^{-6} kg
1 microgram	= 1 μg	= 10^{-9} kg
1 nanogram	= 1 ng	= 10^{-12} kg

1.3.1.3 Length: the meter

Despite the fact that length and time appear independent to us, there is a deep connection between them. We will discuss this briefly when we discuss gravitation, but you will understand it in detail when you study special relativity in an undergraduate course in topics of modern physics. For now the dictionary of conversion of units of length includes:

1 kilometer	= 1 km	= 10^{3} m
1 centimeter	= 1 cm	= 10^{-2} m
1 millimeter	= 1 mm	= 10^{-3} m
1 micrometer	= 1 μm	= 10^{-6} m
1 nanometer	= 1 nm	= 10^{-9} m

Example 1.1

The given speed on a highway is 55 miles per hour. Convert this speed to meters per second. (*Hint*: 1 mile = 1.6 km.)

Solution

$$
\begin{aligned}
55\,\frac{\text{miles}}{\text{hour}} &= \frac{55}{1}\,\frac{\text{miles}}{\text{hour}} \times \frac{1609}{1}\,\frac{\text{m}}{\text{miles}} \times \frac{1}{3600}\,\frac{\text{hours}}{\text{second}} \\
&= 24.58\frac{\text{m}}{\text{s}}.
\end{aligned}
$$

1.3.2 Derived units

The International System of Units (SI) specifies a set of seven base units. From these seven base units other SI units of measurement are derived. We call these *derived units*. Each of the derived units can be expressed as a product of one or more base units. For example, the SI derived unit of area is the square meter (m^2) and of density is kilogram per meter cubed (kg m^{-3}).

Some important facts about derived quantities include:

- The dimension of any derived physical quantity is some product of the base quantities, for example, density is kilogram per meter cubed.
- When the values of the base quantities change, the derived quantities will also change by the same factor, for example, when mass is doubled, the density of the object also doubles.
- When two or more derived quantities of the same units are added, their sum will also have the same units, for example sum of two speeds, given in (m s^{-1}) will also have units of (m s^{-1}).
- Products and ratios of derived quantities usually have units that are not the same as the original quantities. For example, force (N) per unit area (m^2) is N m^{-2}, which is a pascal (Pa), a unit of pressure.

Example 1.2

Density is a derived unit given in kg m^{-3} in the MKS system. The density of water is 997 kg m^{-3}, convert this to:

(a) CGS units.
(b) US-engineering or British units.

Solution

(a) In the CGS unit the base units are grams and centimeters, and 1 kg = 1000 g and 1 m = 100 cm. Hence:

$$
\begin{aligned}
997\,\frac{\text{kg}}{\text{m}^3} &= \frac{997}{1}\,\frac{\text{kg}}{\text{m}^3} \times \frac{1000}{1}\,\frac{\text{g}}{\text{kg}} \times \frac{1}{1000^3}\,\frac{\text{m}^3}{\text{cm}^3} \\
&= 0.997\,\frac{\text{g}}{\text{cm}^3}.
\end{aligned}
$$

(b) Similarly 0.454 kg = 1 lb-mass and 1 foot = 0.305 m. Hence:

$$997\,\frac{\text{kg}}{\text{m}^3} = \frac{997}{1}\,\frac{\text{kg}}{\text{m}^3} \times \frac{1}{0.454}\frac{\text{lb} - \text{mass}}{\text{kg}} \times \frac{0.305^3}{1}\,\frac{\text{m}^3}{\text{ft}^3}$$
$$= 62\frac{\text{lb} - \text{mass}}{\text{ft}^3}.$$

Concept questions 1.1

1. Suppose a really good thief stole the standard kilogram from the town of Sèvres, France, where it is kept. Would this be the end of the metric system?
2. Name some of the ways that ancient people kept time, from the Egyptians to the Romans to the Arabs and the Chinese.
3. Could you take length, mass and density as your three fundamental units of measurement? If so, what are the advantages or disadvantages to this system?
4. Could you think of setting up your own system of fundamental units? Which units do you think you can use if you are not allowed to use the standard length, time and mass?

Exercises 1.1

1. Calculate your height in each of the three systems of units, the MKS, CGS and British units.
2. What is the difference between a nautical and a regular mile? In which cases are each used and does it change the distance you have to travel if you use one over the other?
3. The accepted age of planet Earth is 4.5×10^9 yr. Convert the age of Earth to months, hours and seconds.
4. Most of our body is made up of water. If we assume that the density of a human body is the same as density of water, can you approximate your volume?

1.4 Dimensional analysis

Dimensional analysis, also called the factor-label method or the unit factor method, is a very useful problem-solving technique.

The concept of physical dimension was introduced by the French mathematician Joseph Fourier in 1822 CE. The actual origins of dimensional analysis have been disputed by historians; but most agree that the main credit still goes to Fourier. Dimensional analysis is based on the idea that physical laws should be independent of the units used to measure the physical variables. For example $\vec{F} = m\vec{a}$ is independent of the units employed to measure m and $\vec{a}$. This led to the conclusion that meaningful laws must be independent of the units of measurement. At the most basic level, a collection of basic physical dimension includes dimensions for mass = $[M]$, length = $[L]$ and time = $[T]$.

- *Position* (x) has the dimension $[L]$ (length).
- *Velocity*, which is the derivative of position with respect to time (velocity = dx/dt = m s^{-1}), has dimension $[L][T]^{-1}$ where $[L]$ represents length (dx) with SI unit (m), $[T]$ represents time (dt) with SI unit (s).
- *Acceleration*, the second derivative of position (acceleration = $\frac{dv}{dt} = \frac{d^2x}{dt^2}$), has dimension $[L][T]^{-2}$.
- *Force* ($F = ma = m\frac{d^2x}{dt^2}$) has dimension $[M][L][T]^{-2}$.

Dimensional analysis is used to understand and describe relationships between the physical quantities involved in a particular phenomenon without being bogged down with the various system of units. Dimensional analysis was first used by Lord Rayleigh in 1872 CE to illustrate why the sky is blue. Rayleigh first published the technique in his 1877 book *The Theory of Sound.*

Example 1.3. Centripetal force

How to set up an equation for force on a mass attached to a string revolving in a circle? What does it depend on? Ignore drag forces in this case.

Solution

The force (F_c) on a mass attached to a string rotating in a circle is called a centripetal force[1]. To find the equation of force, let us first discuss the quantities that a rotating mass must depend on. We know from experience a heavier mass will require more force to keep it moving in a circle, similarly if the movement is faster then more force will be needed. Lastly, we know from observation that the length of the string will also be a factor. This means F_c will depend on the following quantities, attached mass (m), speed of rotation (v) and length of string (r), as

$$F_c \propto [m]^{\alpha}[v]^{\beta}[r]^{\gamma}. \tag{1.1}$$

The left-hand side (lhs) of the above equation (1.1) is force. The unit of force is a newton which is given as

$$\text{newton} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} = [M][L][T]^{-2},$$

and the right-hand side (rhs) of equation (1.1) will be given as

$$[M]^{\alpha}\left[\frac{L}{T}\right]^{\beta}[L]^{\gamma} = [M]^{\alpha}[L]^{\beta-\gamma}[T]^{-\beta}.$$

Equating the above equations we can write

[1] Centripetal force is a force that acts on a body moving in a circular path and is directed toward the center around which the body is moving.

$$[M]^1[L]^1[T]^{-2} = [M]^{\alpha}[L]^{\beta+\gamma}[T]^{-\beta}.$$

Equating powers,

$$\begin{aligned}
&\text{mass } [M]\text{: } 1 = \alpha \\
&\text{length } [L]\text{: } 1 = \beta + \gamma \\
&\text{time } [T]\text{: } -2 = -\beta.
\end{aligned}$$

Solving the above expressions ($\alpha = 1$; $\beta = 2$; $\gamma = -1$) and putting values in equation (1.1), we obtain

$$\begin{aligned}
F_c &= [m]^1[v]^2[r]^{-1} \\
F_c &= \frac{mv^2}{r}.
\end{aligned} \tag{1.2}$$

Example 1.4. Period of a mass attached to a spring

Use dimensional analysis to find an equation for a period[2] (T) for a mass (m) attached to an idealized massless spring. Assume drag plays no role in this case.

Solution

Let us first find all the parameters that could play a role. We have a mass (m) attached to a spring, we will see in chapter 7 that this system will undergo a simple harmonic motion, and under this motion the factors that are important will be:

- Mass (m).
- Restoring constant (k)—given dimensions of restoring force constant are $[M][T]^{-2}$.

These dependences can be expressed mathematically as

$$\begin{aligned}
T &\propto m^{\alpha}k^{\beta} \\
T &\propto [M]^{\alpha}[M]^{\beta}[T]^{-2\beta}.
\end{aligned} \tag{1.3}$$

Equating powers,

$$\begin{aligned}
&\text{mass } [M]\text{: } 0 = \alpha + \beta \\
&\text{time } [T]\text{: } 1 = -2\beta.
\end{aligned}$$

Solving the above expressions ($\alpha = \frac{1}{2}$; $\beta = -\frac{1}{2}$) and putting values in equation (1.3) we obtain

$$\begin{aligned}
T &\propto m^{\frac{1}{2}}k^{-\frac{1}{2}} \\
T &\propto \sqrt{\frac{m}{k}}.
\end{aligned}$$

[2] The wave period is the time it takes to complete one cycle. The standard unit of a wave period is in seconds.

Note that in this case dimensional analysis does not give us the constant of proportionality but it does tell us how the period (T) depends on the mass and spring constant. Given a particular value of mass we can easily find the constant of proportionality by measuring the period.

1.4.1 Planck length (l_p)

Planck length is a unit named after Max Planck, who was the first to propose it. Planck length is a unit of length in Planck units and is represented by the formula and given numerically as

$$l_p \equiv \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}, \tag{1.4}$$

where $\hbar$ is the reduced Planck constant, mathematically given as $\hbar = \frac{h}{2\pi}$, with a numerical value of (6.63×10^{-34}kg m^2 s^{-1}); G is the universal gravitational constant, with a numerical value of (6.67×10^{-11} m^3 kg^{-1} s^{-2}), as discussed in detail in chapter 8, and c is the speed of light (3×10^8 m s^{-1}).

Note that $l_p \approx 10^{-20}$ times smaller than the diameter of a proton, which is unbelievably small. Planck length is very important for theoretical calculations, however, it is not very important for us. It is just the shortest possible length that can be measured.

- In some forms of quantum gravity, Planck length is the length scale at which the structure of space–time becomes dominated by quantum effects and hence it becomes impossible to find the difference between two locations one Planck length apart.
- Precise effects of quantum gravity are unknown; however, we think that space–time might have a discrete structure at these lengths. What does that even mean? We really do not know.

Example 1.5. Planck length

The Planck length depends on the following fundamental constants:

- G, the universal gravitational constant, with units m^3 kg^{-1} s^{-2}.
- c, the speed of light, with units m s^{-1}.
- $\hbar$, the reduced Planck constant, with units kg m^2 s^{-1}.

We can therefore write the Plank length in terms of these dependences as

$$l_p \propto c^{\alpha} G^{\beta} \hbar^{\gamma}, \tag{1.5}$$

whose dimensional relationship must be

$$[L]^1 \propto [LT^{-1}]^{\alpha}[L^3T^{-2}M^{-1}]^{\beta}[ML^2T^{-1}]^{\gamma}$$
$$[L]^1[T]^0[M]^0 \propto [L]^{\alpha+3\beta+2\gamma}[T]^{-\alpha-2\beta-\gamma}[M]^{-\beta+\gamma}.$$

Equating powers,

$$\begin{aligned} \text{length } [L]&: 1 = \alpha + 3\beta + 2\gamma \\ \text{time } [T]&: 0 = -\alpha - 2\beta - \gamma \\ \text{mass } [M]&: 0 = -\beta + \gamma. \end{aligned}$$

Solving the above expressions ($\alpha = -\frac{3}{2}$; $\beta = \frac{1}{2}$; $\gamma = \frac{1}{2}$) and putting the values in equation (1.5) we obtain

$$l_p \propto c^{-\frac{3}{2}} G^{\frac{1}{2}} \hbar^{\frac{1}{2}}$$

$$l_p \propto \sqrt{\frac{G\hbar}{c^3}}.$$

Use the values to find the numerical value of Planck length:

$$l_p \approx 10^{-35}\text{m}.$$

1.4.2 Planck time (t_p)

Planck time is also named after Max Planck. It is the time required for light to travel in a vacuum a distance of 1 Planck length (which is 1.6×10^{-35} m or about 10^{-20} times the size of a proton). It is the unit of time in the system of natural units called *Planck units*,

$$t_p \equiv \sqrt{\frac{\hbar G}{c^5}} \approx 5.39 \times 10^{-44} \text{ s}. \tag{1.6}$$

Planck time represents a rough time scale at which quantum gravitational effects are likely to become important. Human and hence scientific effects are billions of billions of billions of Planck times, so these effects are nearly impossible to detect. So far the smallest direct time measurement interval is $\approx 1.2 \times 10^{-17}$s, or 12 as, which is about 2.2×10^{26} Planck times.

The term t_p is called the *quantum of time*, the smallest measurement of time that has any meaning. Within the framework of what we understand today, we can only say that theoretically the Universe came into existence when it already had an age of 10^{-43} s!

1.4.3 Planck's mass (m_p)

Planck's mass is nature's maximum allowed mass for point particles (called *quanta*). In other words, an elementary particle capable of holding a single elementary charge. Unlike the other Planck base units, Planck mass has a scale more or less conceivable to humans. Traditionally Planck mass ≅ the mass of a flea egg.

Concept questions 1.2

1. Suppose the diameter of a penny is 0.75 inches. How many pennies will you need to line up end to end to cover 1 mile in length?
2. Suppose we have a quantity $y = y(t)$ having units $[M][L]^{-1}$, given as

$$y(t) = 4e^{-2t},$$

where t has units of time. What are the units associated with the numbers 4 and 2 in the equation?
3. Given the density of water is approximately $\rho_{\text{water}} = 1.0\ \text{g cm}^{-3}$ and $R_{\text{Earth}} = 6.4 \times 10^3$ km, estimate the mass of water on Earth. *Hint*: assume oceans cover 75% of the Earth's surface and assume the average depth of the ocean to be approximately 1 km.

Exercises 1.2

1. Use dimensional analysis to determine the speed of sound in a gas. *Hint*: the speed of sound in gas is proportional to the pressure (p), density (ρ) and the volume (V) of the gas.
2. Find the equation for the time taken to drop a ball from height h using dimensional analysis.
3. What is the Planck mass? Determine using dimensional analysis:
   (a) The formula for the Planck mass.
   (b) The numerical value for the Planck mass.

1.5 A quick review of vectors

1.5.1 Vectors

According to the Merriam-Webster dictionary, a vector is a quantity, for example velocity or acceleration, that has a size and direction. Vector can be formally defined as follows,

> 'A quantity that has a magnitude and a direction relative to a chosen reference frame. Vector quantities are represented pictorially by an arrow whose head represents the direction of the quantity and whose length represents the magnitude of the quantity'.

Familiar examples of a vector include velocity, acceleration and force.

A given vector $\vec{A}$ is specified by defining its magnitude and direction relative to a chosen reference frame. A typical example of a vector in three dimensions in unit vector notation[3] is written as

$$\vec{A} = (a_x, a_y, a_z) = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}. \tag{1.7}$$

If $\vec{A}$ is a velocity vector then $\vec{a}_x$ is the velocity of the particle along the x-direction, $\vec{a}_y$ is the velocity of the particle along the y-direction and $\vec{a}_z$ is the velocity of the particle along the z-direction.

1.5.2 Scalars

A scalar, on the other hand, is a physical quantity that is able to be completely specified by a number. Familiar examples include temperature, density, volume, etc. Mathematically, scalars are treated as ordinary real numbers. They obey the rules of regular algebraic addition, subtraction, multiplication, division, etc, along with the rules for dimensioned quantities.

1.5.3 Formal definitions and rules of vector algebra

Vectors have the following properties:

- *Vector equality*: $\vec{A} = \vec{B}$ iff $\mathrm{a}_x = \mathrm{b}_x$; $\mathrm{a}_y = \mathrm{b}_y$; $\mathrm{a}_z = \mathrm{b}_z$.
- *Vector magnitude*: $|\vec{A}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$.
- *Vector addition*: $\vec{A} + \vec{B} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k}$.
- *Vector scalar multiplication*: $c\vec{A} = ca_x\hat{i} + ca_y\hat{j} + ca_z\hat{k}$.
- *Null vector*: $\vec{0} = (0, 0, 0)$ direction undetermined.
- *Commutative law*: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.
- *Associate law*: $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$.
- *Distributive law*: $c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$.
- *Unit coordinate vectors*: vectors with unit magnitude usually denoted with the caret symbol (^). For Cartesian coordinates:

$$\hat{i} = e_x = (1, 0, 0)$$
$$\hat{j} = e_y = (0, 1, 0)$$
$$\hat{k} = e_z = (0, 0, 1).$$

The unit vectors are also called *basis vectors*.

[3] A unit vector in vector space is a vector of length 1. A general vector can be represented as a sum of multiples of unit vectors. A unit vector is often denoted by an addition of a *hat* on the vector: example, $\hat{i}$ usually represents the x-direction in Cartesian coordinates, whereas $\hat{j}$ represents the y-direction and $\hat{k}$ represents the z-direction.

1.5.4 Scalar product for vectors

Given two vectors $\vec{A}$ and $\vec{B}$, the scalar product $(\vec{A} \cdot \vec{B})$ is defined as

$$\vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z = |A||B|\cos\theta, \tag{1.8}$$

here θ is the angle between the two vectors $\vec{A}$ and $\vec{B}$. The scalar product is also called the *dot product* or *inner product*. Some properties of the scalar product are as follows:

- The scalar product is commutative:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}.$$

- The scalar product is distributive:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}.$$

- For scalar products of basis vectors:

$$\begin{gathered} \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \text{otherwise } (\hat{i} \cdot \hat{j}) = (\hat{i} \cdot \hat{k}) = (\hat{k} \cdot \hat{j}) = 0 \end{gathered}.$$

1.5.5 Law of cosines

Let $|\vec{A}|$, $|\vec{B}|$ and $|\vec{C}|$ be the magnitudes of three sides of a triangle as shown in figure 1.1. and the opposite angles be $\hat{a}, \hat{b}$ and $\hat{c}$, then the law of cosines can be written as

$$\begin{aligned} A^2 &= B^2 + C^2 + 2BC\cos\hat{a} \\ B^2 &= A^2 + C^2 + 2AC\cos\hat{b} \\ C^2 &= A^2 + B^2 + 2AB\cos\hat{c}. \end{aligned} \tag{1.9}$$

Example 1.6

Find the angle between the vectors $\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + 2\hat{k}$.

Solution

Since $\vec{A} \cdot \vec{B} = AB\cos\theta$, where θ is the angle between the two vectors, then

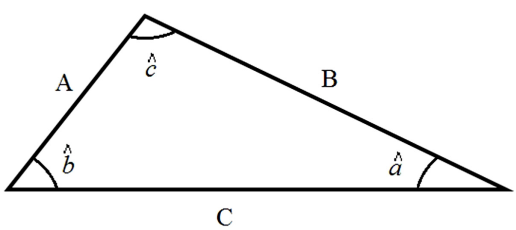

Figure 1.1. Law of cosines.

$$\begin{aligned}\cos\theta &= \frac{\vec{A}\cdot\vec{B}}{AB}\\ &= \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}}\\ &= \frac{(2)(1) + (1)(-1) + (1)(2)}{\sqrt{3}\sqrt{6}}\\ &= \frac{3}{4.2}\\ \cos\theta &= 0.71\end{aligned}$$

Then,

$$\begin{aligned}\theta &= \cos^{-1}(0.71)\\ \theta &= 44^\circ\end{aligned}$$

1.5.6 Vector product for vectors

Given two vectors $\vec{A}$ and $\vec{B}$ the vector product ($\vec{A}\times\vec{B}$) is defined as

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta. \tag{1.10}$$

In terms of its components the vector product will be defined as

$$\vec{A}\times\vec{B} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$\vec{A}\times\vec{B} = (a_y b_z - a_z b_y)\hat{i} - (a_x b_z - a_z b_x)\hat{j} + (a_x b_y - a_y b_x)\hat{k}. \tag{1.11}$$

The vector product is also called the *cross product*. Some properties of the vector product are as follows:

- The vector product is anti-commutative: $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$.
- $n(\vec{A}\times\vec{B}) = (n\vec{A})\times\vec{B} = \vec{A}\times(n\vec{B})$.
- The vector product is distributive: $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$.
- For vector products of basis vectors:

$$\begin{aligned}&\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0\\ &\hat{i}\times\hat{j} = \hat{k};\ \hat{j}\times\hat{k} = \hat{i};\ \hat{k}\times\hat{i} = \hat{j}\\ &\hat{i}\times\hat{k} = -\hat{j};\ \hat{k}\times\hat{j} = -\hat{i};\ \hat{j}\times\hat{i} = -\hat{k}.\end{aligned}$$

Torque ($\vec{\tau}$)[4] is a common example where cross products find a home. Torque is mathematically defined as

[4] Torque is a twisting force that tends to cause rotation in any object that it is applied to.

$$\vec{\tau} = \vec{r} \times \vec{F} = rF \sin \theta \hat{\tau}, \tag{1.12}$$

where r is called the moment arm, $\vec{F}$ is applied force and $\hat{\tau}$ is a unit vector perpendicular to $\vec{r}$ and $\vec{F}$, given by the right-hand rule. I have only defined torque here, for more details see section 1.12.

Example 1.7

Given the two vectors $\vec{A} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$, find $\vec{A} \times \vec{B}$.

Solution

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ 1 & -2 & 2 \end{vmatrix}$$
$$\vec{A} \times \vec{B} = \hat{i}(-4 - 2) - \hat{j}(4 + 1) + \hat{k}(-4 - 2)$$
$$\vec{A} \times \vec{B} = -6\hat{i} - 5\hat{j} - 6\hat{k}.$$

Example 1.8

Find a unit vector *normal*[5] to the plane containing the two vectors $\vec{A}$ and $\vec{B}$; $\vec{A} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + 2\hat{k}$.

Solution

$$\begin{aligned} \hat{n} &= \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} \\ &= \frac{\hat{i} - 5\hat{j} - 3\hat{k}}{\sqrt{1^2 + 5^2 + 3^2}} \\ &= \frac{1}{\sqrt{35}}(1\hat{i} - 5\hat{j} - 3\hat{k}). \end{aligned}$$

1.6 Derivatives of vectors

Suppose we are given a vector ($\vec{A}$) whose components are a function of one variable, say time (t). Then we can express $\vec{A}$ mathematically as

$$\vec{A} = (a_x(t), a_y(t), a_z(t)) = a_x(t)\hat{i} + a_y(t)\hat{j} + a_z(t)\hat{k}. \tag{1.13}$$

Then the first derivative of the vector $\vec{A}$ with respect to (w.r.t.) the variable (t) will be given as

[5] Given a particular surface in 3D space, a normal is a vector that is perpendicular to the surface. The normal vector is commonly denoted as $\hat{n}$. When a normal vector has magnitude 1, it is called a unit normal vector.

$$\frac{d\vec{A}}{dt} = \frac{da_x(t)}{dt}\hat{i} + \frac{da_y(t)}{dt}\hat{j} + \frac{da_z(t)}{dt}\hat{k} = \dot{a}_x(t)\hat{i} + \dot{a}_y(t)\hat{j} + \dot{a}_z(t)\hat{k}, \tag{1.14}$$

whereas the second derivative of the vector $\vec{A}$ w.r.t. the variable (t) will be given as

$$\frac{d^2\vec{A}}{dt^2} = \frac{d^2a_x(t)}{dt^2}\hat{i} + \frac{d^2a_y(t)}{dt^2}\hat{j} + \frac{d^2a_z(t)}{dt^2}\hat{k} = \ddot{a}_x(t)\hat{i} + \ddot{a}_y(t)\hat{j} + \ddot{a}_z(t)\hat{k}. \tag{1.15}$$

Concept questions 1.3

1. Can a zero magnitude vector have a direction?
2. Four vectors have the same magnitude. Under what conditions will their sum equal zero?
3. Is it possible to obtain a vector with a negative magnitude?
4. Given that the dot product between vectors $\vec{A}$ and $\vec{B}$ is positive, what can we conclude about the angle between the two vectors?
5. We have talked about vector addition, vector subtraction and vector products. Explain why we have not talked about vector division?

Exercises 1.3

1. Suppose we have a cube whose edges are of unit length. Let one corner coincide with an origin of x, y, z. Find the vector
   (a) that begins at 0 and ends along a major diagonal of the cube; call this vector $\vec{A}$.
   (b) that begins at 0 and ends along the diagonal of the lower face of the cube; call this vector $\vec{B}$.
   (c) Find $\vec{C} = \vec{A} \times \vec{B}$.
   (d) Find the angle between $\vec{A}$ and $\vec{B}$.
2. Given the following vectors, $\vec{a} = 3\hat{i} - 3\hat{j} + 1\hat{k}$ and $\vec{b} = 4\hat{i} + 9\hat{j} + 2\hat{k}$, find:
   (a) $\vec{a} + \vec{b}$.
   (b) $\vec{a} - \vec{b}$.
   (c) The dot product between $\vec{a}$ and $\vec{b}$.
   (d) The cross product between $\vec{a}$ and $\vec{b}$.
3. Given the time-varying vector $\vec{A} = at\hat{i} + bt^2\hat{j} + ct^3\hat{k}$, where a, b and c are constants and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the x, y and z coordinate axes, respectively. Find the first and second order derivatives.

1.7 Position vector

A coordinate system uses one or more coordinates (numbers) to uniquely specify the position of a point-particle in space. We employ various coordinate systems to help visualize physics problems in the three-dimensional (3D) space we live in. The three most commonly used coordinate systems are the Cartesian, spherical and cylindrical coordinate systems.

The position vector ($\vec{r}$) of a particle is a vector drawn from a user-defined origin to the particle. It always points radially outwards from the origin and expresses the displacement from the origin. The position vector is also called a *location vector* or *radius vector*;

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}. \tag{1.16}$$

The magnitude of the position vector ($\vec{r}$) of a particle is the distance from the origin and is mathematically expressed as

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2} \tag{1.17}$$

and the unit vector is given as

$$\hat{r} = \hat{\vec{r}} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}, \tag{1.18}$$

which is a unit vector pointing radially outwards.

Most of classical physics deals with finding the position vector $\vec{r}$ of things, may it be quarks or planets. The term $\vec{r}$ defines the motion of a particle in a particular coordinate system (figure 1.2).

If we take the origin at (0, 0, 0) in each of the coordinate systems, then the position vector $\vec{r}$ can be defined as follows:

- Cartesian: $(x, y, z) = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
- Polar: (r, ϕ).
- Spherical: (r, θ, ϕ).
- Cylindrical: (r, ϕ, z).

1.7.1 Separation vector

In classical mechanics, we may sometimes encounter problems where the tail of the displacement vector does not coincide with the user-defined origin. In this case there

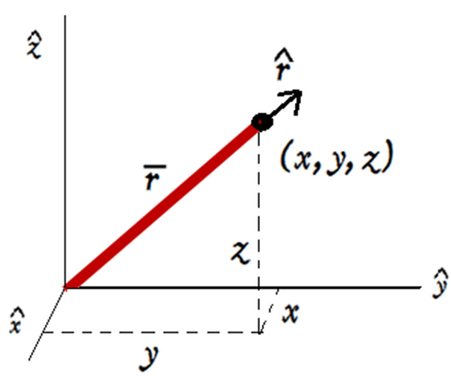

Figure 1.2. Position vector example.

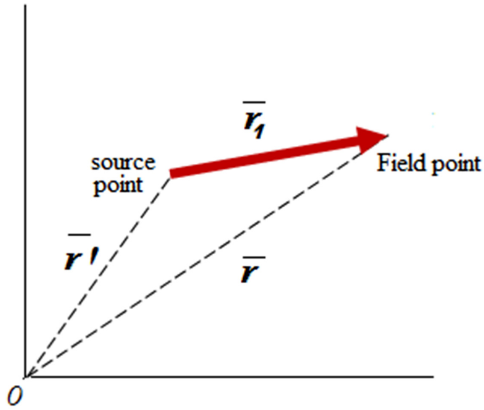

Figure 1.3. A separation vector example, as defined in electrodynamics.

will be two points needed to define the vector, the head-point and tail-point in the coordinate system. This is especially useful in electrodynamics, with the two points being the source point ($\vec{r}'$) and field point ($\vec{r}$). In this case the source point is where the electric charge is located and the field point is where you are calculating the electric or magnetic field strength, as shown in figure 1.3.

In this case the separation vector from source point to field point will be given by

$$\vec{r}_1 = \vec{r} - \vec{r}'. \tag{1.19}$$

Its magnitude is $|\vec{r}_1| = |\vec{r} - \vec{r}'|$, and a unit vector in the direction from $\vec{r}'$ to $\vec{r}'$ is

$$\hat{r}_1 = \frac{\vec{r}_1}{|\vec{r}_1|} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}. \tag{1.20}$$

1.8 Transformation between various coordinate systems

Geometrical figures can be described in many different possible coordinate systems. Hence it is important to understand how different possible coordinate systems are related. Relationships between coordinate systems are defined by using transformation formulas for the coordinates in one system in terms of the coordinates in another system.

1.8.1 Polar coordinates (r, ϕ)

The term polar was used by eighteenth-century Italian writers and is historically attributed to Gregorio Fontana. A polar coordinate system is a two-dimensional (2D) coordinate system where each point is determined by measuring the distance from a user-defined origin (O), also called the *pole*, and an angle from a reference direction, usually taken to lie along the x-axis in a 2D Cartesian coordinate plane. In general we consider polar coordinates to lie on a flat plane, as shown in figure 1.4, and the origin (O) of the system is where all coordinates of the system have zero value.

The magnitude of the distance from the origin (pole) is called the radial coordinate or radius (r), the angle (ϕ) is taken from the reference direction and is called the angular coordinate, polar angle, or azimuth.

In other texts the radial coordinate is also sometimes denoted by ρ, and the angular coordinate by θ. Angles in polar notation are expressed either in degrees or in radians (2π rad being equal to 360°). A positive angular coordinate means that the angle is measured counterclockwise from the reference direction (polar axis). Unit vectors in polar coordinates are $\hat{r}$ and $\hat{\phi}$ (figures 1.4 and 1.5).

Conversion between Cartesian and polar coordinates

In a plane the Cartesian coordinates are given by (x, y) and the polar coordinates are given by (r, ϕ). If we suppose that both coordinates have the same origin, and the polar axis aligns with the positive x-axis, then the coordinate transformation from polar to Cartesian coordinates is given by

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}, \tag{1.21}$$

whereas the transformation from Cartesian to polar coordinates is given by

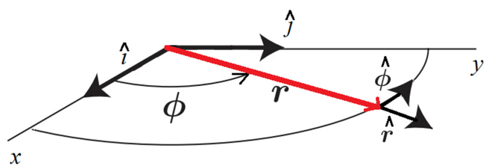

Figure 1.4. Polar coordinates (r, ϕ).

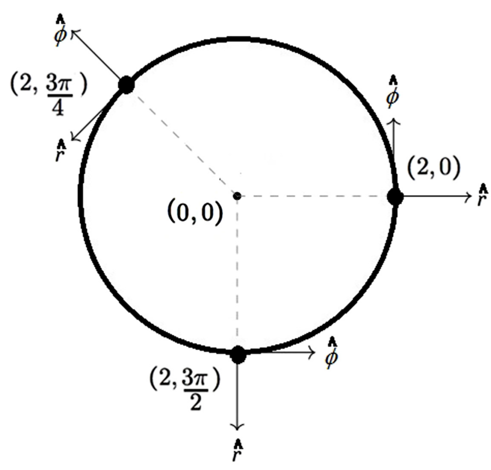

Figure 1.5. Overhead view of various points in polar coordinates (r, ϕ).

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}. \tag{1.22}$$

1.8.2 Conversion between Cartesian and spherical coordinates (r, θ, ϕ)

A spherical coordinate system is a way to extend two-dimensional (2D) polar coordinates to three dimensions (3D). It is a coordinate system for 3D space, where the position of a point is specified by three numbers, given as:

- *Radial distance* (r) from a fixed origin, also called the radius, where $r \in [0, \infty)$.
- *Polar angle* (θ) measured from a fixed zenith direction, also called co-latitude, zenith angle, normal angle, or inclination angle, where $\theta \in [0, \pi]$.
- *Azimuth angle* (ϕ) is defined in the in the xy-plane. It is the projection in the xy-plane that passes through the origin and is perpendicular to the zenith and is measured from a fixed reference direction (usually the x-axis) on that plane; $\phi \in [0, 2\pi]$.

It can be seen as the 3D version of the polar coordinate system (figure 1.6). The spherical coordinates on Earth are essentially the *latitude* angle (θ), the *longitudinal* angle (ϕ) and the radial distance (r) from the origin. Unit vectors in spherical coordinates are $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$.

To convert between Cartesian and spherical coordinates we use the following equations:

$$\text{spherical coordinates } (r, \theta, \phi)\text{:} \begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta \end{cases}. \tag{1.23}$$

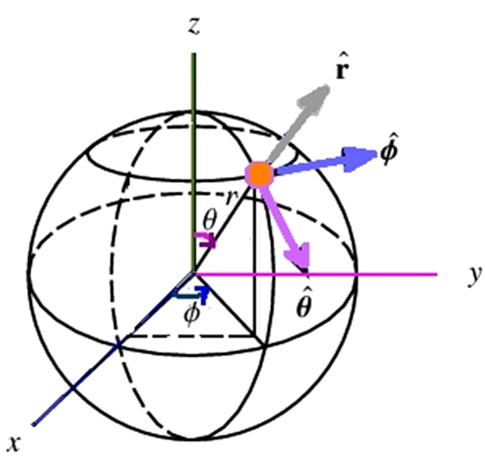

Figure 1.6. Spherical coordinates.

Example 1.9

Convert the point (1, −1, −1) from Cartesian to spherical coordinates.

Solution

Spherical coordinates are given as (r, θ, ϕ). To find the point in spherical coordinates we can take the following steps:

1. Find r:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 1} = \sqrt{3}.$$

2. Find θ:

$$z = r\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 125° \approx \frac{2\pi}{3}.$$

3. Find ϕ:

$$y = r\sin\theta\sin\phi \Rightarrow \phi = \sin^{-1}\left(\frac{y}{r\sin\theta}\right) = 45° = \frac{\pi}{4} \text{ or}$$
$$x = r\sin\theta\cos\phi \Rightarrow \phi = \cos^{-1}\left(\frac{x}{r\sin\theta}\right) = 45° = \frac{\pi}{4}.$$

Hence, (1, −1, −1) in spherical coordinates are given as $(r, \theta, \phi) = (\sqrt{3}, \frac{2\pi}{3}, \frac{\pi}{4})$.

1.8.3 Conversion between Cartesian and cylindrical coordinates (r, ϕ, z)

Cylindrical coordinates are also an extension of the two-dimensional polar coordinates to three dimensions, also called *cylindrical polar coordinates* or *polar cylindrical coordinates*.

The origin of the cylindrical coordinate system is the point where all three coordinates are zero. This is the intersection between the reference plane and the vertical axis. In the cylindrical coordinate system a point position is specified by the following three points:

- The distance (r) from a chosen reference axis, called the polar axis.
- The direction (ϕ) from the axis relative to a chosen reference direction.
- The distance (z) from a chosen reference plane perpendicular to the axis, called the cylindrical or longitudinal axis. It is also called height or altitude (figure 1.7).

The cylindrical axis is the ray that lies in the reference plane, starting at the origin (0, 0, 0) and pointing in the reference direction. The cylindrical axis can be positive or negative depending on the choice of the origin. The radius and the azimuth correspond to a 2D polar coordinate system and together we can call them polar coordinates.

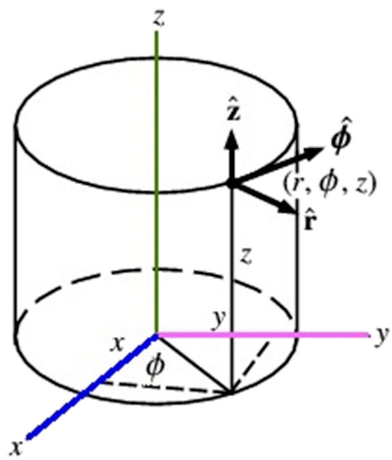

Figure 1.7. Cylindrical coordinates.

The cylindrical coordinates are essentially the *longitudinal* angle (ϕ), the radial distance (r) and height (h) from the origin. The unit vectors in cylindrical coordinates are $\hat{r}$, $\hat{\phi}$ and $\hat{z}$. To convert between Cartesian and cylindrical coordinates we use the following equations:

$$\text{cylindrical coordinates } (r, \phi, z): \begin{cases} x = r\cos\phi \\ y = r\sin\phi\,, \\ z = z \end{cases} . \tag{1.24}$$

Example 1.10

Convert the point (2, −1, −1) from Cartesian to cylindrical coordinates.

Solution

Spherical coordinates are given as (r, ϕ, z). To find the point in spherical coordinates we can take the following steps:

1. Find r:

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 1} = \sqrt{5}\,.$$

2. Find ϕ:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{1}{2}\right) = -26.5° \approx -\frac{\pi}{7}.$$

3. Find h:

$$h = -1.$$

Hence, (1, −1, −1) in cylindrical coordinates are given as $(r, \phi, h) = (\sqrt{5}, -\frac{\pi}{7}, -1)$.

1.9 Velocity and acceleration

1.9.1 Velocity and acceleration in Cartesian coordinates

As we saw in section 1.7, the position of a particle can be specified by a single vector, called the position vector or displacement relative to the origin. In Cartesian coordinates, the position vector $(x(t), y(t), z(t))$ is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}. \tag{1.25}$$

If the vector is the position vector $\vec{r}$ of a moving particle and the parameter is the time (t), then the derivative of $\vec{r}$ with respect to t is called the *velocity*, and is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \frac{dx}{dt}\hat{i} + \frac{dx}{dt}\hat{j} + \frac{dx}{dt}\hat{k} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}, \tag{1.26}$$

where the dots indicate differentiation with respect to t. The magnitude of the velocity is called the *speed*. Speed is a scalar quantity and hence only has magnitude but no direction and is mathematically given as

$$\text{speed} = |\vec{v}| = \frac{d|\vec{r}|}{dt}. \tag{1.27}$$

The time derivative of the velocity is called the *acceleration* and is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{r} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}, \tag{1.28}$$

where the double dots indicate differentiation with respect to t twice. The magnitude of the acceleration is given by

$$|\vec{a}| = \frac{d|\vec{v}|}{dt}. \tag{1.29}$$

1.10 Velocity and acceleration in various coordinates

As we discussed in section 1.8, the components in each coordinate system are given as:

- Cartesian (x, y, z).
- Polar (r, ϕ).
- Spherical (r, θ, ϕ).
- Cylindrical (r, ϕ, z).

Then, in each coordinate system the velocity and acceleration vectors can be written as:

- Position vector: $\vec{r}$.
- Velocity: $\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$.
- Acceleration: $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$.

Next, let us briefly discuss each coordinate system individually. See appendix B for details of the derivations for the formulas.

1. Cartesian coordinates (x, y, z):

$$\begin{aligned}\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k}\\ \vec{v} &= \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\\ \vec{a} &= \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}.\end{aligned} \tag{1.30}$$

2. Polar coordinates (r, ϕ):

$$\begin{aligned}\vec{r} &= r\hat{r}\\ \vec{v} &= \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\\ \vec{a} &= (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}.\end{aligned} \tag{1.31}$$

3. Spherical coordinates (r, θ, ϕ):

$$\begin{aligned}\vec{r} &= r\hat{r}\\ \vec{v} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}\\ \vec{a} &= (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta}\\ &\quad + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi}.\end{aligned} \tag{1.32}$$

4. Cylindrical coordinates (r, ϕ, z):

$$\begin{aligned}\vec{r} &= r\hat{r} + z\hat{z}\\ \vec{v} &= \dot{r}\hat{r} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{z}\\ \vec{a} &= (\ddot{r} - R\dot{\phi}^2)\hat{r} + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}.\end{aligned} \tag{1.33}$$

Example 1.11

Suppose the position vector of a moving car is given by the equation, $\vec{r}(t) = t^3 + 5t - 3$. Calculate the corresponding velocity and the acceleration functions.

Solution

Velocity function:

$$v(t) = \dot{\vec{r}} = 3t^2 + 5.$$

Acceleration function:

$$a(t) = \ddot{\vec{r}} = 6t.$$

Example 1.12

A rocket is fired vertically from rest from the ground and starts accelerating at the rate of $4t$ m/s^2. If it maintains this rate of acceleration where will the rocket be after 5 s?

Solution

Given the initial conditions, $v_0 = 0\,\mathrm{m\,s^{-1}}$ and $a(t) = 4t\,\mathrm{m\ s^{-2}}$, we need to find the displacement $y(t = 5\,\mathrm{s})$. Velocity function:

$$v(t) = \int \vec{r}\,dt = \int 4t\,dt = 2t^2 + v_0.$$

Displacement function:

$$y(t) = \int 2t^2 dt = 2\frac{t^3}{3} + y_0.$$

Then at $t = 5$ s, the rocket starts on the ground, hence $y_0 = 0\,\mathrm{m}$. Put the values into the equation:

$$x(5) = 2\left(\frac{5^3}{3}\right) = 83\ \mathrm{m}.$$

Hence the rocket will be 83 m above the ground.

Concept questions 1.4

1. Does the speedometer in your car give speed or velocity? Does the speedometer care whether you are going north or south at 55 mph?
2. Can you think of a situation where a particle moving at a constant speed experiences an acceleration?
3. Suppose at a particular instance of time the velocity of an object is zero. Can this object have a non-zero acceleration at that instance? Give an example of such a motion.

Exercises 1.4

1. Blue whales are the longest known mammals and they can grow to be the length of a basketball court, about 30 m, and can weigh up to 180 metric tons. If the speed of nerve impulses for a whale is taken to be about $100\,\mathrm{m\,s^{-1}}$, how long will it take for the whale to know if its tail gets hurt?
2. Suppose a mass is hung by a spring which stretches the spring by 3 cm to some initial state, y_0. We then displace the spring by pulling on the mass at time $t = 0$.

Releasing the mass sets it in motion. Then at any given time the displacement is given by

$$y(t) = 4\sin(\omega t),$$

where ω is the value of angular frequency and can be considered as a constant. Find:

(a) The velocity and acceleration of the mass as a function of time.
(b) The velocity and acceleration of the mass at $t = 3$ s.

3. Given that the acceleration of a certain particle is given by

$$a(t) = \frac{1}{4}t + \cos(2t): \tag{1.34}$$

(a) Find the velocity function and the displacement as a function of time.
(b) Given $v(t = 0) = 2\,\mathrm{m\,s^{-1}}$, find the velocity at $t = 2$ s.

Further reading

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