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Advanced topics

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Appendix D

Lie algebra representation theory: a primer

In this primer I follow the excellent pedagogical presentation of Zuber [1].

D.1 The Cartan subalgebra

We consider a semi-simple[1] Lie algebra $\mathbf{g}$ of finite dimension over the complex numbers $\mathbf{C}$. The Cartan subalgebra is the maximal abelian subalgebra $\mathbf{h}$ of $\mathbf{g}$ such that all its elements are diagonalizable in the adjoint representation. The elements of $\mathbf{g}$ can be chosen to be hermitian. The Cartan subalgebra is not unique and different choices are related by an automorphism of the algebra, i.e. as $h \longrightarrow ghg^{-1}$. The dimension l of $\mathbf{h}$ is called the rank of $\mathbf{g}$. For example, for $su(n)$ the Cartan subalgebra can be generated by the following $l = n - 1$ diagonal traceless matrices

$$H_1 = (1, -1, \dots, 0),\ \ H_2 = (0, 1, -1, \dots, 0), \dots,\ \ H_{n-1} = (0, \dots, 1, -1). \quad \text{(D.1)}$$

Let H_i, $i = 1, \dots, l$ be the basis elements of $\mathbf{h}$ which are chosen to be hermitian. By definition we have

$$[H_i, H_j] = 0 \Leftrightarrow [\mathrm{ad}\, H_i, \mathrm{ad}\, H_j] = 0. \quad \text{(D.2)}$$

Thus, from $\mathrm{ad}\, H_i H_j = [H_i, H_j] = 0$ we see that H_j are eigenvectors of $\mathrm{ad}\, H_i$ with eigenvalues 0. The other linearly independent eigenvectors with eigenvalues α_i, not all vanishing, will be denoted by E_α, viz

$$\mathrm{ad}\, H_i E_\alpha = [H_i, E_\alpha] = \alpha_i E_\alpha. \quad \text{(D.3)}$$

[1] A semi-simple algebra has no *abelian ideal* but $\{0\}$, whereas a simple algebra has no *ideal* but $\{0\}$. In other words, the semi-simple is more general since it can have ideals. Thus, any semi-simple algebra can be decomposed into a direct sum of simple algebras.

These eigenvalues are real since ad H_i are hermitian after multiplication by i. We compute for $H = \sum_i h^i H_i$ that

$$\text{ad } HE_\alpha = \alpha(H)E_\alpha, \;\; \alpha(H) = \sum_i h^i \alpha_i. \tag{D.4}$$

Obviously, $\alpha(H) = \sum_i h^i \alpha_i$ is a linear form on $\mathbf{h}$. Linear forms on a vector space E form the dual vector space E^*. Thus the set of all $\alpha(H)$ form the dual space $\mathbf{h}^*$ of $\mathbf{h}$. This dual space is called the root space and $\alpha(H)$ is called a root. We note that

$$\alpha(H_i) = \alpha_i. \tag{D.5}$$

The total number of the eigenvectors H_j and E_α of H_i is equal to the dimension d of the adjoint representation of the Lie algebra $\mathbf{g}$. Since the number of the H_j is equal to the rank l and the roots are not degenerate we conclude that the number of the roots is $d - l$. This number is even since if α is a root $-\alpha$ is also a root.

The Killing form in the basis (H_i, E_α) of $\mathbf{g}$ is given by

$$(H_i, E_\alpha) = 0, \;\; (E_\alpha, E_\beta) = 0 \text{ unless } \alpha + \beta = 0. \tag{D.6}$$

The Killing form is defined by the trace in the adjoint representation in an obvious way, viz

$$(H_i, E_\alpha) = \text{tr ad } H_i \text{ ad } E_\alpha. \tag{D.7}$$

The Killing form on $\mathbf{h}$ is non-degenerate, i.e. there exists an isomorphism between $\mathbf{h}$ and $\mathbf{h}^*$ which allows us to associate to every element $\alpha \in \mathbf{h}^*$ a unique element $H_\alpha = \sum_i h_\alpha^i H_i \in \mathbf{H}$ as follows

$$\begin{aligned}(H_\alpha, H) &= \sum_i h_\alpha^i \sum_j h^j (H_i, H_j) = \sum_i h_\alpha^i \sum_j h^j g_{ij} \\ &= \sum_j h^j \alpha_j = \alpha(H) \equiv \alpha.\end{aligned} \tag{D.8}$$

In the above equation, we have used the fact that the metric $g_{ij} = (H_i, H_j)$ is invertible since the Killing form was assumed to be non-degenerate, and $\alpha_j = \sum_i h_\alpha^i g_{ij}$. The bilinear form on the space of roots $\mathbf{H}^*$ can then be defined by

$$\langle \alpha, \beta \rangle = (H_\alpha, H_\beta). \tag{D.9}$$

In order to compute the commutation relations of the E_α, we compute using Jacobi identity the following

$$\begin{aligned}\text{ad } H_i[E_\alpha, E_\beta] &= [H_i, [E_\alpha, E_\beta]] \\ &= -[E_\beta, [H_i, E_\alpha]] - [E_\alpha, [E_\beta, H_i]] \\ &= (\alpha + \beta)_i [E_\alpha, E_\beta].\end{aligned} \tag{D.10}$$

We have three possibilities. If $\alpha + \beta$ is a root then $[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$. If $\alpha + \beta \neq 0$ is not a root then $[E_\alpha, E_\beta] = 0$. If $\alpha + \beta = 0$ then $[E_\alpha, E_{-\alpha}] = H \in \mathbf{h}$. We compute

$$\begin{aligned}(H_i, [E_\alpha, E_{-\alpha}]) &= \mathrm{tr\ ad}\ H_i[\mathrm{ad}\ E_\alpha, \mathrm{ad}\ E_{-\alpha}]\\ &= \alpha_i\ \mathrm{tr\ ad}\ E_\alpha\ \mathrm{ad}\ E_{-\alpha}\\ &= \alpha_i(E_\alpha, E_{-\alpha})\\ &= (H_i, H_\alpha)(E_\alpha, E_{-\alpha}) \Rightarrow [E_\alpha, E_{-\alpha}]\\ &= (E_\alpha, E_{-\alpha})H_\alpha.\end{aligned} \tag{D.11}$$

In summary, we have obtained the commutation relations

$$[H_i, H_j] = 0,\ \ [H_i, E_\alpha] = \alpha_i E_\alpha \tag{D.12}$$

$$\begin{aligned}[E_\alpha, E_\beta] &= N_{\alpha\beta}E_{\alpha+\beta},\ \ \alpha+\beta = \mathrm{root}\\ [E_\alpha, E_{-\alpha}] &= (E_\alpha, E_{-\alpha})H_\alpha,\ \ \alpha+\beta = 0\\ [E_\alpha, E_{-\alpha}] &= 0,\ \ \mathrm{otherwise}.\end{aligned} \tag{D.13}$$

Since the restriction of the Killing form to $\mathbf{h}$ is positive definite we can choose the normalization

$$(H_i, H_j) = \delta_{ij},\ \ (E_\alpha, E_\beta) = \delta_{\alpha+\beta,0} \tag{D.14}$$

Thus $g_{ij} = \delta_{ij}$, $\alpha_i = h_\alpha^i$ and hence

$$H_\alpha = \sum_i \alpha_i H_i,\ \ \langle\alpha, \alpha\rangle = (H_\alpha, H_\alpha) = \sum_i \alpha_i^2. \tag{D.15}$$

We get finally the commutation relations

$$[H_\alpha, E_{\pm\alpha}] = \pm\langle\alpha, \alpha\rangle E_{\pm\alpha},\ \ [E_\alpha, E_{-\alpha}] = H_\alpha. \tag{D.16}$$

Thus we get an $su(2)$ Lie algebra for every root.

D.2 Roots, Cartan matrix and Dynkin diagrams

We have established that we have $d - l$ roots. Their inner product is inherited from the Killing form, viz $\langle\alpha, \beta\rangle = (H_\alpha, H_\beta) = \sum_i \alpha_i\beta_i$. It is obvious that $H_\alpha/\langle\alpha, \alpha\rangle$ plays the role of J_z in $su(2)$, whereas the E_α and $E_{-\alpha}$ play the role of the raising and lowering operators J_+ and J_-, respectively, in $su(2)$. We also compute

$$\mathrm{ad}\ H_\alpha E_\beta = [H_\alpha, E_\beta] = \langle\alpha, \beta\rangle E_\beta. \tag{D.17}$$

Let $p \leqslant 0$ be the smallest integer such that $(\mathrm{ad}\ E_{-\alpha})^{|p|}E_\beta$ is non-zero, i.e. $(\mathrm{ad}\ E_{-\alpha})^{|p|+1}E_\beta = 0$. This means that $E_{-k\alpha+\beta}$, $k = 1, \ldots, |p|$, correspond to non-zero roots $-k\alpha + \beta$. Let $q \geqslant 0$ be the largest integer such that $(\mathrm{ad}\ E_\alpha)^q E_\beta$ is non-zero, i.e. $(\mathrm{ad}\ E_\alpha)^{q+1}E_\beta = 0$. This means that $E_{k\alpha+\beta}$, $k = 1, \ldots, q$, correspond to non-zero roots $k\alpha + \beta$. The set $\{E_{\beta'};\ \beta' = \beta + q\alpha, \ldots, \beta, \ldots, \beta + p\alpha\}$ is called the α-chain through β. Similarly to $su(2)$, the $E_{\beta'}$ form a representation of the $su(2)$ algebra generated by $E_{\pm\alpha}$ and H_α. Furthermore, the eigenvalues corresponding to the highest state $E_{\beta+q\alpha}$ and the lowest state $E_{\beta+p\alpha}$ given, respectively, by

$$
\begin{aligned}
[H_\alpha, E_{\beta+q\alpha}] &= \langle\alpha, \beta + q\alpha\rangle E_{\beta+q\alpha}, \\
[H_\alpha, E_{\beta+p\alpha}] &= \langle\alpha, \beta + p\alpha\rangle E_{\beta+p\alpha},
\end{aligned}
\tag{D.18}
$$

are opposite to each other, i.e.

$$
\begin{aligned}
\langle\alpha, \beta + q\alpha\rangle = -\langle\alpha, \beta + p\alpha\rangle \Rightarrow 2\frac{\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} \\
= -q - p = m \in \mathbf{Z}.
\end{aligned}
\tag{D.19}
$$

The set of all roots, which are not linearly independent in $\mathbf{h}^*$, is denoted by Δ and it is decomposed into positive and negative roots. Obviously, the opposite of a positive root is a negative root and vice versa. A basis in Δ can be given by the so-called simple roots denoted by α_i, $i = 1, \ldots, l$, such that the positive (negative) roots are linear combinations of simple roots with positive (negative) integer coefficients. A simple root cannot be rewritten as the sum of two positive roots. The choice of simple roots is not unique and different choices are related by the Weyl group, which leaves the set of roots globally invariant.

Let α and β be two simple roots. Then obviously $\alpha - \beta$ cannot be a root and as a consequence $p = 0$ and $m = -q \leqslant 0$. In other words, $\langle\alpha, \beta\rangle \leqslant 0$. The Cartan matrix is defined in terms of the simple roots by

$$
C_{ij} = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_j, \alpha_j\rangle}.
\tag{D.20}
$$

This is not a symmetric matrix. We compute for $i \neq j$, using $\langle\alpha, \beta\rangle = |\alpha||\beta| \cos\theta$, that

$$
C_{ij} = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_j, \alpha_j\rangle} = 2\frac{|\alpha_i|}{|\alpha_j|}\cos\theta_{ij} = m_i \leqslant 0
\tag{D.21}
$$

$$
C_{ji} = 2\frac{\langle\alpha_j, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} = 2\frac{|\alpha_j|}{|\alpha_i|}\cos\theta_{ij} = m_j \leqslant 0.
\tag{D.22}
$$

By multiplying the last two equations and using the Schwarz inequality $\langle\alpha, \beta\rangle^2 < \langle\alpha, \alpha\rangle\langle\beta, \beta\rangle$ we obtain

$$
m_i m_j < 4.
\tag{D.23}
$$

The value $m_i m_j = 4$ is forbidden because $\alpha_i \neq -\alpha_j$. We also compute by dividing the above two equations the following

$$
\frac{|\alpha_i|}{|\alpha_j|} = \sqrt{\frac{m_i}{m_j}}, \;\; \cos\theta_{ij} = -\frac{1}{2}\sqrt{m_i m_j}.
\tag{D.24}
$$

The only allowed angles are, therefore, $\pi/2$, $2\pi/3$, $3\pi/4$ and $5\pi/6$ corresponding to the cosine equals $0, -1/2, -\sqrt{2}/2, -\sqrt{3}/2$. The allowed ratios of lengths of roots are, therefore, 0, 1, $\sqrt{2}$ and $\sqrt{3}$.

If the set of roots decomposes into two mutually orthogonal subsets then the corresponding semi-simple algebra **g** decomposes into the direct sum of two simple algebras corresponding to the orthogonal sets. In the following we will only consider simple algebras following [1].

For rank 1 there is one complex simple Lie algebra which is $A_1 = sl(2, \mathbf{C})$. For rank 2 there are two complex simple Lie algebras which are $A_2 = sl(3, \mathbf{C})$ and $B_2 = so(5, \mathbf{C})$ and one complex semi-simple Lie algebra which is $D_2 = so(4, \mathbf{C})$. There is also another rank 2 complex simple algebra G_2 which is an exceptional algebra of dimension 14. For higher ranks, the Cartan classification of complex simple algebra is given by

$$\begin{aligned} &A_l = sl(l+1, \mathbf{C}), \;\; B_l = so(2l+1, \mathbf{C}), \\ &C_l = sp(2l, \mathbf{C}), \;\; D_l = so(2l, \mathbf{C}). \end{aligned} \tag{D.25}$$

These four infinite families are the so-called classical Lie algebra. The unique real compact forms of these algebras are given, respectively, by $A_l = su(l+1)$, $B_l = so(2l+1)$, $C_l = usp(2l)$, $D_l = so(2l)$.

Furthermore, the full Cartan classification of the exceptional complex simple algebra is given by the following five cases

$$E_6(78), \;\; E_7(133), \;\; E_8(248), \;\; F_4(52), \;\; G_2(14). \tag{D.26}$$

The number in brackets is the dimension.

For higher ranks we use the Dynkin diagram which encodes the Cartan matrix to visualize the root system. The Dynkin diagram is constructed as follows

- Each simple root is represented by a vertex.
- Any two roots such that $\langle \alpha_i, \alpha_j \rangle \neq 0$ are linked by a line.
- The line is simple if

$$C_{ij} = C_{ji} = -1, \;\; \left(\theta_{ij} = \frac{2\pi}{3}, \;\; \frac{|\alpha_i|}{|\alpha_j|} = 1\right). \tag{D.27}$$

- The line is double if

$$C_{ij} = -2, = C_{ji} = -1, \;\; \left(\theta_{ij} = \frac{3\pi}{4}, \;\; \frac{|\alpha_i|}{|\alpha_j|} = \sqrt{2}\right). \tag{D.28}$$

- The line is triple if

$$C_{ij} = -3, = C_{ji} = -1, \;\; \left(\theta_{ij} = \frac{5\pi}{6}, \;\; \frac{|\alpha_i|}{|\alpha_j|} = \sqrt{3}\right). \tag{D.29}$$

- The line carries an arrow from i to j if $|\alpha_i| > |\alpha_j|$.

D.3 Weights, Dynkin labels and representations

The roots studied so far provide one particular finite dimensional irreducible unitary representation of the algebra **g** known as the adjoint representation. A general finite dimensional irreducible unitary representation of the algebra $\mathbf{g}$ is given by the so-called weights. We start with the Cartan subalgebra generated by the commuting elements H_i. These can be obviously diagonalized simultaneously. Let $|\lambda_a\rangle$ be the eigenvectors of the Cartan elements H_i with eigenvalues λ_i, viz

$$H_i|\lambda_a\rangle = \lambda_i|\lambda_a\rangle. \tag{D.30}$$

The vector of eigenvalues $\lambda = (\lambda_1, \dots, \lambda_l)$ is called a weight and is associated with the vectors $|\lambda_a\rangle$ where the index a denotes possible degeneracy of the eigenvalue λ. Obviously, for $H = \sum_i h^i H_i$ we have

$$H|\lambda_a\rangle = \lambda(H)|\lambda_a\rangle,\ \lambda(H) = \sum_i h^i\lambda_i. \tag{D.31}$$

Thus, λ is a linear form on $\mathbf{h}$ and thus an element in $\mathbf{h}^*$ which is the space of roots. Since H is hermitian the weights are real. The set of weights in a given representation forms what we call a weight diagram.

Let us call the representation space E. The dimension of the representation space E is equal to the total number of λ_a including their multiplicities. Since there is an $su(2)$ Lie algebra for every root α, the space E contains subspaces corresponding to the $d-l$ $su(2)$ Lie algebras $\{H_\alpha, E_\alpha, E_{-\alpha}\}$ satisfying $[H_\alpha, E_{\pm\alpha}] = \pm\langle\alpha, \alpha\rangle E_{\pm\alpha}$, $[E_\alpha, E_{-\alpha}] = H_\alpha$.

We compute

$$H_iE_{\pm\alpha}|\lambda_a\rangle = (\lambda \pm \alpha)_i E_{\pm\alpha}|\lambda_a\rangle. \tag{D.32}$$

Thus, the vector $E_\alpha|\lambda_a\rangle$ is an eigenvector associated with the eigenvalue $\lambda + \alpha$. In other words, all vectors in the representation space E are obtained from each other by the action of the $E_{\pm\alpha}$. We also conclude that any two weights of the same representation can only differ by a linear combination of roots with integer coefficients.

The weight λ will be associated with the operator E_λ. Let $p' \leqslant 0$ be the smallest integer such that $(E_{-\alpha})^{|p|}|\lambda_a\rangle$ is non-zero, i.e. $(E_{-\alpha})^{|p'|+1}|\lambda_a\rangle = 0$. This means that $(E_{-\alpha})^k|\lambda_a\rangle$, $k = 1, \dots, |p'|$, correspond to non-zero weights $-k\alpha + \lambda$. Let $q' \geqslant 0$ be the largest integer such that $(E_\alpha)^{q'}|\lambda_a\rangle$ is non-zero, i.e. $(E_\alpha)^{q'+1}|\lambda_a\rangle = 0$. This means that $E_{k\alpha+\lambda}|\lambda_a\rangle$, $k = 1, \dots, q'$, correspond to non-zero weights $k\alpha + \lambda$. The weights $\{\beta' = \lambda - |p'|\alpha, \dots, \lambda, \dots, \lambda + q'\alpha\}$ will be associated with the operator $E_{\beta'}$. Similar to before, the operators $\{E_{\beta'}\}$ form a representation of the $su(2)$ algebra generated by $E_{\pm\alpha}$ and H_α. Furthermore, the eigenvalues corresponding to the highest state $E_{\lambda+q'\alpha}$ and the lowest state $E_{\lambda+p'\alpha}$ given, respectively, by

$$\begin{aligned}
[H_\alpha, E_{\lambda+q'\alpha}] &= \langle\alpha, \lambda + q'\alpha\rangle E_{\lambda+q'\alpha},\\
[H_\alpha, E_{\lambda+p'\alpha}] &= \langle\alpha, \lambda + p'\alpha\rangle E_{\lambda+p'\alpha},
\end{aligned} \tag{D.33}$$

are opposite to each other, i.e.

$$\langle\alpha, \lambda + q'\alpha\rangle = -\langle\alpha, \lambda + p'\alpha\rangle \Rightarrow 2\frac{\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle} = -q' - p' = m' \in \mathbf{Z}. \tag{D.34}$$

Since we are dealing with an $su(2)$ algebra, the number of the states $\{E_{\beta'}\}$ is exactly given by $q' - p' + 1 = 2j + 1$, i.e. $q' - p' = 2j$.

In the above space we can clearly introduce an ordering given by $\lambda' > \lambda$ if $\lambda' - \lambda = \sum_i n_i\alpha_i$ where n_i are positive integers. There exists, therefore, a highest weight state denoted by $|\Lambda\rangle$ such that if α is any positive root we have

$$E_\alpha|\Lambda\rangle = 0. \tag{D.35}$$

In this case $q' = 0$ and $p' = -2j$ and hence

$$2\langle\Lambda, \alpha\rangle = j\langle\alpha, \alpha\rangle > 0. \tag{D.36}$$

We define the Dynkin labels of a weight λ by

$$\lambda_i = 2\frac{\langle\lambda, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} \in \mathbf{Z}. \tag{D.37}$$

The α are simple roots and, therefore, there are l Dynkin labels. If we choose the weight λ to be the highest weight Λ we obtain

$$\lambda_i = 2\frac{\langle\Lambda, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} = j \in \mathbf{N}. \tag{D.38}$$

We define the fundamental weights Λ_i by the formula

$$\delta_{ij} = 2\frac{\langle\Lambda_i, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}. \tag{D.39}$$

Clearly, there are l of them and they provide a basis in $\mathbf{h}^*$. Also, it is obvious that any highest weight state can be rewritten as a linear combination of the fundamental weights with coefficients given by the Dynkin labels, viz

$$\Lambda = \sum_{i=1}^{l}\lambda_i\Lambda_i,\ \ \lambda_i \in \mathbf{N}. \tag{D.40}$$

Any irreducible unitary representation is completely characterized by its highest weight state. The fundamental weights define the so-called fundamental representations. Hence, we have l fundamental representations characterized by Λ_i.

The dimension and the Casimir operators of a given irreducible representation characterized by a highest weight state Λ are given in terms of the Weyl vector defined by the half sum of the positive roots or by the sum of the fundamental roots as follows

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_i \Lambda_i. \tag{D.41}$$

The dimension and the quadratic Casimir of the irreducible representation Λ are given respectively by the Weyl formulas

$$\dim(\Lambda) = \prod_{\alpha>0}\frac{\langle\Lambda+\rho,\,\alpha\rangle}{\langle\rho,\,\alpha\rangle} \tag{D.42}$$

$$C_2(\Lambda) = \frac{1}{2}\langle\Lambda,\,\Lambda+2\rho\rangle. \tag{D.43}$$

D.4 Explicit construction of Lie algebra representations

D.4.1 $A_l = su(l+1)$

In this case the dimension of $\mathbf{h}^*$ is $l = n-1$, i.e. $\mathbf{h}^* = \mathbf{R}^{n-1}$. Let $\hat{e}_i$, $i = 1, \ldots, n$, be an orthonormal basis in $\mathbf{R}^n$ satisfying

$$\langle\hat{e}_i,\,\hat{e}_j\rangle = \delta_{ij}. \tag{D.44}$$

We define any hyperplane in $\mathbf{R}^n$ by the vector normal to it. We consider the hyperplane given by the normal vector

$$\hat{\rho} = \sum_{i=1}^{n}\hat{e}_i. \tag{D.45}$$

We project the $\hat{e}_i$ on this hyperplane to obtain vectors e_i given by

$$e_i = \hat{e}_i - \frac{1}{n}\hat{\rho}. \tag{D.46}$$

Indeed, we check that $\langle\hat{\rho},\,e_i\rangle = 0$. Furthermore, we check that

$$\langle e_i,\,e_j\rangle = \delta_{ij} - \frac{1}{n}. \tag{D.47}$$

We also check that $\sum_i e_i = 0$. In other words, the e_i are not linearly independent and they form a basis in $\mathbf{R}^{n-1}$. We define the $l = n-1$ simple roots and the $(d-l)/2 = n(n-1)/2$ positive roots, respectively, by

$$\alpha_i = \alpha_{ii+1} = e_i - e_{i+1},\ \ i = 1, \ldots, l \tag{D.48}$$

$$\alpha_{ij} = e_i - e_j,\ \ i < j = 1, \ldots, n. \tag{D.49}$$

The simple roots satisfy

$$\langle\alpha_i,\,\alpha_j\rangle = 2\delta_{ij} - \delta_{ij+1} - \delta_{i+1j}. \tag{D.50}$$

From this equation, we can infer directly the value of the Cartan matrix. We have

$$C_{ij} = \langle\alpha_i, \alpha_j\rangle = 2\delta_{ij} - \delta_{ij+1} - \delta_{i+1j}. \tag{D.51}$$

In order to compute the Weyl vector we divide the positive roots as $\{\alpha_{1i} = e_1 - e_i,\ i = 2, \ldots, n\}$ ($n - 1$ roots), $\{\alpha_{2i} = e_2 - e_i,\ i = 3, \ldots, n\}$ ($n - 2$ roots), $\{\alpha_{3i} = e_3 - e_i,\ i = 4, \ldots, n\}$ ($n - 3$ roots), …, $\{\alpha_{n-2i} = e_{n-2} - e_i,\ i = n - 1, n\}$ (2 roots), $\{\alpha_{n-1i} = e_{n-1} - e_i,\ i = n\}$ (1 root). We have then

$$\begin{aligned} 2\rho = \sum_{\alpha>0}\alpha = \sum_{i<j}\alpha_{ij} \\ = (n-1)e_1 + (n-3)e_2 + (n-5)e_3 + \cdots \\ + (n-2i+1)e_i + \cdots -(n-1)e_n. \end{aligned} \tag{D.52}$$

The index i in the above formula goes from $i = 1$ to n. Now, if we re-express the e_i in terms of the simple roots α_i we find that the coefficient of each α_i is the sum of the coefficients of e_j with $j \leqslant i$, viz

$$\sum_{j=1}^{i}(n - 2j + 1) = i(n - i), \quad i = 1, \ldots, n - 1 \tag{D.53}$$

We obtain then

$$\begin{aligned} 2\rho = (n-1)\alpha_1 + 2(n-2)\alpha_2 + 3(n-3)\alpha_3 + \cdots \\ + i(n-i)\alpha_i + \cdots + (n-1)\alpha_{n-1}. \end{aligned} \tag{D.54}$$

By using $\sum_i e_i = 0$ we can also rewrite the Weyl vector as

$$\rho = \sum_{i=1}^{l}(n - i)e_i. \tag{D.55}$$

From equation (D.50) we have $\langle\alpha_i, \alpha_i\rangle = 2$. Thus, the fundamental weights must be defined by the condition $\langle\Lambda_j, \alpha_i\rangle = \delta_{ij}$. A solution is given by

$$\Lambda_i = \sum_{j=1}^{i} e_j. \tag{D.56}$$

We verify that $\langle\Lambda_j, \alpha_i\rangle = \sum_{k=1}^{j}\delta_{ki} - \sum_{k=1}^{j-1}\delta_{ki}$. For $i = j$ we get $1 - 0$, for $i > j$ we get $0 - 0$, whereas for $i < j$ we get $1 - 1$. Hence

$$2\frac{\langle\Lambda_j, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} = \delta_{ij}. \tag{D.57}$$

Explicitly, we have

$$\begin{aligned} e_1 &= \Lambda_1 \\ e_i &= \Lambda_i - \Lambda_{i-1}, \ \ i = 2, \dots, n-1 \\ e_n &= -\Lambda_{n-1}. \end{aligned} \tag{D.58}$$

Furthermore, we compute

$$\langle \Lambda_j, \alpha_i \rangle = \frac{i(n-j)}{n}, \ \ i \leqslant j. \tag{D.59}$$

We need now to compute the dimension of the irreducible representation with highest weight state $\Lambda = \sum_{i=1}^{l} \lambda_i \Lambda_i$. We have

$$\begin{aligned} \langle \Lambda, \alpha_{ij} \rangle &= \langle \Lambda, e_i \rangle - \langle \Lambda, e_j \rangle \\ &= \sum_{k=1}^{l} \lambda_k \sum_{k'=1}^{k} \left(\delta_{k'i} - \frac{1}{n} \right) - (i \longrightarrow j) \\ &= \sum_{p=1}^{l} f_p \left(\delta_{pi} - \frac{1}{n} \right) - (i \longrightarrow j) \\ &= f_i - f_j. \end{aligned} \tag{D.60}$$

The f_i is defined by the equation

$$f_i = \sum_{q=i}^{l} \lambda_q, \ \ i = 1, \dots, l; \ f_n = 0. \tag{D.61}$$

Further, we compute in the same way

$$\begin{aligned} \langle \rho, \alpha_{ij} \rangle &= \langle \rho, e_i \rangle - \langle \rho, e_j \rangle \\ &= \frac{1}{2} \sum_{k=1}^{l} (n - 2k + 1) \langle e_k, e_i \rangle - (i \longrightarrow j) \\ &= j - i. \end{aligned} \tag{D.62}$$

Hence, we find the dimension

$$\dim(\Lambda) = \prod_{i<j} \frac{f_i - f_j + j - i}{j - i}. \tag{D.63}$$

The Casimir operator is given by

$$C_2(\Lambda) = \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} \lambda_i (\lambda_j + 2) \langle \Lambda_i, \Lambda_j \rangle. \tag{D.64}$$

We compute

$$\langle \Lambda_i, \Lambda_j \rangle = \frac{i(n-j)}{n}, \;\; i \leqslant j \tag{D.65}$$

$$\langle \Lambda_i, \Lambda_j \rangle = \frac{j(n-i)}{n}, \;\; i > j. \tag{D.66}$$

Example: $su(4)$ In this case $n = 4$, $d = n^2 - 1 = 15$, $l = 3$ and thus $\mathbf{h}^* = \mathbf{R}^3$. We have $(d - l)/2 = 6$ positive roots. The simple roots are

$$\begin{aligned} &\alpha_1 = \alpha_{12} = e_1 - e_2, \;\; \alpha_2 = \alpha_{23} = e_2 - e_3, \\ &\alpha_3 = \alpha_{34} = e_3 - e_4. \end{aligned} \tag{D.67}$$

The other positive roots are

$$\alpha_{13} = e_1 - e_3, \;\; \alpha_{14} = e_1 - e_4, \;\; \alpha_{24} = e_2 - e_4. \tag{D.68}$$

The Weyl vector is (using also $\sum_i e_i = 0$)

$$\begin{aligned} 2\rho &= 3e_1 + e_2 - e_3 - 3e_4 = 6e_1 + 4e_2 + 2e_3 \\ &= 3\alpha_1 + 4\alpha_2 + 3\alpha_3. \end{aligned} \tag{D.69}$$

The weights are

$$\Lambda_1 = e_1, \;\; \Lambda_2 = e_1 + e_2, \;\; \Lambda_3 = e_1 + e_2 + e_3. \tag{D.70}$$

The Weyl vector can also be rewritten as

$$\rho = \Lambda_1 + \Lambda_2 + \Lambda_3. \tag{D.71}$$

The Casimir operator and the dimension are given by

$$\begin{aligned} C_2(\Lambda) = &\frac{1}{8}\lambda_1(3\lambda_1 + 2\lambda_2 + \lambda_3 + 12) + \frac{1}{4}\lambda_2(\lambda_1 + 2\lambda_2 + \lambda_3 + 8) \\ &+ \frac{1}{8}\lambda_3(\lambda_1 + 2\lambda_2 + 3\lambda_3 + 12) \end{aligned} \tag{D.72}$$

$$\begin{aligned} \dim(\Lambda) = &\frac{1}{12}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1)(\lambda_1 + \lambda_2 + 2) \\ &\times (\lambda_2 + \lambda_3 + 2)(\lambda_1 + \lambda_2 + \lambda_3 + 3). \end{aligned} \tag{D.73}$$

For example $C_2(n, 0, n) = n(n + 3)$ and $\dim(n, 0, n) = (n + 1)^2(n + 2)^2(2n + 3)/12$.

D.4.2 $B_l = so(2l + 1)$

In this case the dimension is $d = l(2l + 1)$ and the rank is l, i.e. $\mathbf{h}^* = \mathbf{R}^l$. We choose a basis in $\mathbf{R}^l$ given by e_i, $i = 1, \dots, l$, such that $\langle e_i, e_j \rangle = \delta_{ij}$. The l simple roots are

$$\alpha_i = e_i - e_{i+1}, \;\; i = 1, \dots, l - 1; \;\; \alpha_l = e_l. \tag{D.74}$$

There are $(d - l)/2 = l^2$ positive roots. Here they are

$$e_i = \sum_{k=i}^{l} \alpha_k, \ \ i = 1, \dots, l \tag{D.75}$$

$$e_i - e_j = \sum_{k=i}^{j-1} \alpha_k, \ \ 1 \leqslant i < j \leqslant l \tag{D.76}$$

$$e_i + e_j = \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{l} \alpha_k, \ \ 1 \leqslant i < j \leqslant l. \tag{D.77}$$

The Weyl vector is the half sum of the positive roots. We get

$$\begin{aligned} 2\rho &= \sum_{i=1}^{l} e_i + \sum_{i<j}(e_i - e_j) + \sum_{i<j}(e_i + e_j) \\ &= \sum_{i=1}^{l} e_i(2l - 2i + 1) \\ &= \sum_{i=1}^{l}\sum_{k=i}^{l} \alpha_k(2l - 2i + 1) \\ &= \sum_{i=1}^{l}\sum_{k=1}^{i} \alpha_i(2l - 2k + 1) \\ &= \sum_{i=1}^{l} \alpha_i \cdot i(2l - i). \end{aligned} \tag{D.78}$$

The non-zero elements of the Cartan matrix are given by

$$\langle \alpha_i, \alpha_i \rangle = 2, \ \ i = 1, \dots, l - 1 \tag{D.79}$$

$$\langle \alpha_l, \alpha_l \rangle = 1 \tag{D.80}$$

$$\langle \alpha_i, \alpha_j \rangle = -1, \ \ 1 \leqslant i = j + 1, \ j = i + 1 \leqslant l. \tag{D.81}$$

The fundamental weights are defined by

$$\Lambda_i = \sum_{j=1}^{i} e_j, \ \ i = 1, \dots, l - 1; \ \ \Lambda_l = \frac{1}{2}\sum_{j=1}^{l} e_j. \tag{D.82}$$

We can re-express the basis vectors in terms of the fundamental weights as

$$e_1 = \Lambda_1; \ \ e_i = \Lambda_i - \Lambda_{i-1}, \ \ 2 \leqslant i \leqslant l - 1; \ \ e_l = 2\Lambda_l - \Lambda_{l-1}. \tag{D.83}$$

The Dynkin labels of the roots are given by

$$\alpha_1 = 2\Lambda_1 - \Lambda_2 = (2, -1, \ldots) \tag{D.84}$$

$$\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} = (\ldots, -1, 2, -1, \ldots), \quad i = 2, \ldots, l-2 \tag{D.85}$$

$$\alpha_{l-1} = -\Lambda_{l-2} + 2\Lambda_{l-1} - 2\Lambda_l = (\ldots, -1, 2, -2) \tag{D.86}$$

$$\alpha_l = -\Lambda_{l-1} + 2\Lambda_l = (\ldots, -1, 2). \tag{D.87}$$

In general, irreducible representations of $so(2l+1)$ are characterized by the highest weight vectors Λ which can be rewritten as $\Lambda = \sum_i n_i e_i = (n_1, \ldots, n_l)$ with $n_1 \geqslant \cdots \geqslant n_{l-1} \geqslant n_l \geqslant 0$ with dimensions (see [2] page 407)

$$\begin{gathered}\dim(n_1, \ldots, n_l) = \prod_{i<j} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \prod_i \frac{l_i}{m_i} \\ l_i = n_i + l - i + \frac{1}{2}, \, m_i = l - i + \frac{1}{2}.\end{gathered} \tag{D.88}$$

Example: $so(5)$ In this case $l = 2$ and thus $\mathbf{h}^* = \mathbf{R}^2$. The simple roots are

$$\alpha_1 = e_1 - e_2, \;\; \alpha_2 = e_2. \tag{D.89}$$

There are four positive roots. These are the two simple roots plus

$$e_1 = \alpha_1 + \alpha_2, \;\; e_1 + e_2 = \alpha_1 + 2\alpha_2. \tag{D.90}$$

The Weyl vector is

$$\rho = \frac{3}{2}e_1 + \frac{1}{2}e_2 = \frac{3}{2}\alpha_1 + 2\alpha_2. \tag{D.91}$$

The fundamental weights are

$$\begin{aligned}\Lambda_1 &= e_1 = \alpha_1 + \alpha_2 \\ \Lambda_2 &= \frac{1}{2}(e_1 + e_2) = \frac{1}{2}\alpha_1 + \alpha_2.\end{aligned} \tag{D.92}$$

A straightforward calculation gives the dimension and the Casimir

$$\dim(\Lambda) = \frac{1}{6}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)(2\lambda_1 + \lambda_2 + 3) \tag{D.93}$$

$$C_2(\Lambda) = \frac{1}{4}\lambda_1(2\lambda_1 + \lambda_2 + 6) + \frac{1}{4}\lambda_2(\lambda_1 + \lambda_2 + 4). \tag{D.94}$$

The highest weight state $\Lambda = \sum_i \lambda_i \Lambda_i$ is usually expressed as $\Lambda = \sum_i n_i e_i$ where the spin quantum numbers n_i are defined in terms of the Dynkin labels λ_i by $\lambda_1 = n_1 - n_2$

and $\lambda_2 = 2n_2$. The dimension and the Casimir operator become in terms of n_i given by

$$\dim(\Lambda) = \frac{1}{6}(2n_1 + 3)(2n_2 + 1)(n_1 - n_2 + 1)(n_1 + n_2 + 2) \tag{D.95}$$

$$C_2(\Lambda) = \frac{1}{2}n_1(n_1 + 3) + \frac{1}{2}n_2(n_2 + 1). \tag{D.96}$$

D.4.3 $D_l = so(2l)$

In this case the dimension is $d = l(2l - 1)$ and the rank is l, i.e. $\mathbf{h}^* = \mathbf{R}^l$. We choose a basis in $\mathbf{R}^l$ given by e_i, $i = 1, \dots, l$, such that $\langle e_i, e_j\rangle = \delta_{ij}$. The l simple roots are

$$\alpha_i = e_i - e_{i+1},\ \ i = 1, \dots, l - 1;\ \ \alpha_l = e_{l-1} + e_l. \tag{D.97}$$

There are $(d - l)/2 = l(l - 1)$ positive roots. Here they are

$$e_i - e_j = \sum_{k=i}^{j-1}\alpha_k,\ \ 1 \leqslant i < j \leqslant l \tag{D.98}$$

$$e_i + e_j = \sum_{k=i}^{j-1}\alpha_k + 2\sum_{k=j}^{l-2}\alpha_k + \alpha_{l-1} + \alpha_l,\ \ 1 \leqslant i < j \leqslant l - 1 \tag{D.99}$$

$$e_i + e_l = \sum_{k=i}^{l-2}\alpha_k + \alpha_l,\ \ 1 \leqslant i \leqslant l - 1. \tag{D.100}$$

The Weyl vector is the half sum of the positive roots. We get

$$\begin{aligned} 2\rho &= \sum_{i=1}^{l-1} 2(l - i)e_i \\ &= \sum_{i=1}^{l-2}\alpha_i \cdot i(2l - i - 1) + \frac{l(l - 1)}{2}(\alpha_{l-1} + \alpha_l). \end{aligned} \tag{D.101}$$

The non-zero elements of the Cartan matrix are given by

$$\langle \alpha_i, \alpha_i\rangle = 2,\ \ i = 1, \dots, l \tag{D.102}$$

$$\langle \alpha_i, \alpha_j\rangle = -1,\ \ i = l,\ j = l - 2;\ \ i = l - 2,\ j = l \tag{D.103}$$

$$\langle \alpha_i, \alpha_j\rangle = -1,\ \ 1 \leqslant i = j + 1,\ j = i + 1 \leqslant l - 2. \tag{D.104}$$

The fundamental weights are defined by

$$\begin{aligned}\Lambda_i &= \sum_{j=1}^{i} e_j \\ &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{l-2}) \\ &\quad + \frac{i}{2}(\alpha_{l-1} + \alpha_l), \;\; i = 1, \ldots, l-2\end{aligned} \tag{D.105}$$

$$\begin{aligned}\Lambda_{l-1} &= \frac{1}{2}(e_1 + \cdots + e_{l-1} - e_l) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2}) + \frac{l}{4}\alpha_{l-1} + \frac{l-2}{4}\alpha_l\end{aligned} \tag{D.106}$$

$$\begin{aligned}\Lambda_l &= \frac{1}{2}(e_1 + \cdots + e_{l-1} + e_l) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2}) + \frac{l-2}{4}\alpha_{l-1} + \frac{l}{4}\alpha_l.\end{aligned} \tag{D.107}$$

In general, irreducible representations of $so(2l)$ are characterized by the highest weight vectors Λ which can be rewritten as $\Lambda = \sum_i n_i e_i = (n_1, \ldots, n_l)$ with $n_1 \geqslant \cdots \geqslant n_{l-1} \geqslant |n_l| \geqslant 0$ with dimensions (see [2] page 409)

$$\begin{gathered}\dim(n_1, \ldots, n_l) = \prod_{i<j} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \\ l_i = n_i + l - i, \; m_i = l - i.\end{gathered} \tag{D.108}$$

Example: $so(6)$ In this case $l = 3$ and thus $\mathbf{h}^* = \mathbf{R}^3$. The simple roots are

$$\alpha_1 = e_1 - e_2, \;\; \alpha_2 = e_2 - e_3, \;\; \alpha_3 = e_2 + e_3. \tag{D.109}$$

There are six positive roots. These are the three simple roots plus

$$\begin{aligned}&e_1 - e_3 = \alpha_1 + \alpha_2, \;\; e_1 + e_3 = \alpha_1 + \alpha_3, \\ &e_1 + e_2 = \alpha_1 + \alpha_2 + \alpha_3.\end{aligned} \tag{D.110}$$

The Weyl vector is

$$\rho = 2e_1 + e_2 = 2\alpha_1 + \frac{3}{2}\alpha_2 + \frac{3}{2}\alpha_3. \tag{D.111}$$

The Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \tag{D.112}$$

The fundamental weights are

$$\begin{aligned}
\Lambda_1 &= e_1 = \alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 \\
\Lambda_2 &= \frac{1}{2}(e_1 + e_2 - e_3) = \frac{1}{2}\alpha_1 + \frac{3}{4}\alpha_2 + \frac{1}{4}\alpha_3 \\
\Lambda_3 &= \frac{1}{2}(e_1 + e_2 + e_3) = \frac{1}{2}\alpha_1 + \frac{1}{4}\alpha_2 + \frac{3}{4}\alpha_3.
\end{aligned} \tag{D.113}$$

A straightforward calculation gives the dimension and the Casimir

$$\begin{aligned}
\dim(\Lambda) = \frac{1}{12}&(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1)(\lambda_1 + \lambda_2 + 2) \\
&\times (\lambda_1 + \lambda_3 + 2)(\lambda_1 + \lambda_2 + \lambda_3 + 3)
\end{aligned} \tag{D.114}$$

$$\begin{aligned}
C_2(\Lambda) = &\frac{1}{4}\lambda_1(2\lambda_1 + \lambda_2 + \lambda_3 + 8) + \frac{1}{8}\lambda_2(2\lambda_1 + 3\lambda_2 + \lambda_3 + 12) \\
&+ \frac{1}{8}\lambda_3(2\lambda_1 + \lambda_2 + 3\lambda_3 + 12).
\end{aligned} \tag{D.115}$$

In terms of the spin quantum numbers n_i defined by $\lambda_1 = n_1 - n_2$, $\lambda_2 = n_2 - n_3$ and $\lambda_3 = n_2 + n_3$ we have

$$\begin{aligned}
\dim(\Lambda) = \frac{1}{12}&((n_1 + 2)^2 - n_3^2)((n_1 + 2)^2 \\
&- (n_2 + 1)^2)((n_2 + 1)^2 - n_3^2)
\end{aligned} \tag{D.116}$$

$$C_2(\Lambda) = \frac{1}{2}n_1(n_1 + 4) + \frac{1}{2}n_2(n_2 + 2) + \frac{1}{2}n_3^2. \tag{D.117}$$

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Advanced topics

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Appendix E

On homotopy theory

- **Compactification:** In one dimension we can compactify $\mathbf{R}$ by adding one point (at) ∞ to obtain the one-sphere S^1. Conversely, by removing one point from the circle we get essentially $\mathbf{R}$. This is called Alexandroff one-point compactification. Generalization to higher dimensions is obvious.
- **Topological spaces and manifolds:** A topological space is a collection of open subsets (they do not contain any of their boundary points) with certain properties which allow the introduction of the concept of continuity (smoothness). Manifolds have the added property of differentiability.
- **Homeomorphic:** Two spaces are said to be homeomorphic if they can be mapped continuously and bijectively onto each other. Two homeomorphic spaces are topologically identical and posses the same connectedness properties (i.e. they are homotopically equivalent).
- **Homotopic equivalence:** Two spaces X and Y are homotopically equivalent if there exist continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f = \mathbf{1}_X$, $f \circ g = \mathbf{1}_Y$. A very important example is the sphere S^n and the punctured $\mathbf{R}^{n+1}$ (i.e. $\mathbf{R}^{n+1}/\{0\}$)[1].
- **Homotopy:** A homotopy (or a deformation) of a smooth map f between two smooth manifolds X and Y is a smooth map $F : X \times I \longrightarrow Y$, $I = [0, 1]$ with the property $F(x, 0) = f(x)$. The maps $f_t(x) = F(x, t)$ are said to be homotopic.
- **Homotopy classes:** The relation homotopy divides the set of smooth maps between two smooth manifolds X and Y into equivalence classes called homotopy classes.

[1] This can be shown using the stereographic projection.

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- **Connected spaces:** An arcwise connected space is one in which every two points are connected by some path.
- **Loops:** A loop is a closed path. A loop through a point $x_0 \in M$ is a map $\alpha : [0, 1] \longrightarrow M$ such that $\alpha(0) = \alpha(1) = x_0$.

  A product of two loops α and β is a loop $\gamma = \alpha * \beta$ which corresponds to traversing the original loops consecutively, viz $\gamma(t) = \alpha(2t)$, $0 \leqslant t \leqslant 1/2$ and $\gamma(t) = \beta(2t - 1)$, $1/2 \leqslant t \leqslant 1$. The inverse loop α^{-1} corresponds to traversing the loop α in the opposite direction. The constant loop is obviously given by $c(t) = x_0$ for all t.

  Two loops are said to be homotopic, and we write $\alpha \sim \beta$, if they can be continuously deformed into each other. Thus, there must exist a mapping $H : [0, 1] \times [0, 1] \longrightarrow M$ which satisfy $H(s, 0) = \alpha(s)$, $H(s, 1) = \beta(s)$ and $H(0, t) = H(1, t) = x_0$.
- **The fundamental group:** The fundamental group of M, denoted $\pi_1(M, x_0)$, consists of all the equivalence (homotopy) classes of loops through $x_0 \in M$. The product of homotopy classes is given by the product of their representatives and thus $\pi_1(M, x_0)$ is a group where the neutral element is given by the constant loop. For arcwise connected space the fundamental group is independent of the base point x_0.
- **Homotopic equivalence:** Homotopically equivalent spaces have the same fundamental group.
- **Simply connected:** In a simply connected space every loop can be contracted to a point. The fundamental group in this case is trivial, viz $\pi_1 = 0$.
- **The circle:** Let us consider $M = S^1$. The maps $\theta : S^1 \longrightarrow S^1$ are phases. We can divide the first S^1 by 2π to get the interval $[0, 1]$. An arbitrary phase on S^1 always satisfies $\theta(0) = 0$, $\theta(2\pi) = 2\pi m$. This can be continuously deformed to the linear function $m\phi$. To see this consider the map

$$H(\phi, t) = (1 - t)\theta(\phi) + t\phi\frac{\theta(2\pi)}{2\pi}. \quad \text{(E.1)}$$

  This satisfies $H(0, t) = \theta(0) = 0$, $H(2\pi, t) = \theta(2\pi)$. In other words, $H(\phi, t)$ is a homotopy and as a consequence $\theta(\phi)$ is in the same equivalence class as $m\phi$. The set of homotopy (equivalence) classes is therefore Z. The fundamental group of the circle is

$$\pi_1(S^1) = Z. \quad \text{(E.2)}$$

- **Higher spheres:** All higher spheres are simply connected and as a consequence

$$\pi_1(S^n) = 0, \; n \geqslant 2. \quad \text{(E.3)}$$

  The reason is very simple. Any loop in $\mathbf{R}^{n+1}/\{0\}$ can always avoid the point defect at the origin and be shrunk to a point.
- **Torus:** The fundamental group of a product of spaces X and Y is $\pi_1(X \otimes Y) = \pi_1(X) \otimes \pi_1(Y)$. Thus for a two-dimensional torus we have

$$\pi_1(T) = Z \otimes Z. \quad \text{(E.4)}$$

- **Higher homotopy groups:** The fundamental group uses loops and their behavior under deformations to characterize topological properties. Furthermore, the fundamental group (using loops) cannot detect point defects in dimensions higher than two. The higher homotopy groups uses generalizations of the one-dimensional loop to detect point defects in higher dimensions.
- **The *n*-cubes and *n*-loops:** An n-cube is defined by

$$I^n = \{(s_1, \ldots, s_n)|0 \leqslant s_i \leqslant 1 \text{ all } s_i\}. \tag{E.5}$$

The boundary is defined by

$$\partial I^n = \{(s_1, \ldots, s_n) \in I^n|s_i = 0 \text{ or } s_i = 1\}. \tag{E.6}$$

An n-loop is a continuous map from the n-cube to the topological space X, viz

$$\alpha : I^n \longrightarrow X, \tag{E.7}$$

which satisfies

$$\alpha(s) = x_0, \;\; s \in \partial I^n. \tag{E.8}$$

In other words, all the points on the boundary are mapped to a single point $x_0 \in X$. Thus, n-loops are topologically equivalent to n-spheres.

As before a homotopy is a continuous deformation of the above n-loop. We define

$$F : I^n \times I \longrightarrow X. \tag{E.9}$$

We demand

$$\begin{aligned} F(s_1, s_2, \ldots, 0) &= \alpha(s_1, s_2, \ldots, s_n), \\ F(s_1, s_2, \ldots, 0) &= \beta(s_1, s_2, \ldots, s_n) \end{aligned} \tag{E.10}$$

$$F(s_1, s_2, \ldots, t) = x_0, \;\; (s_1, s_2, \ldots, s_n) \in \partial I^n. \tag{E.11}$$

The two n-loops α, β are therefore homotopic, viz $\alpha \sim \beta$.
- **Higher homotopy groups:** Again the homotopy relation defines an equivalence relation and as a consequence the space of n-loops is turned into a set of equivalence classes. For arcwise connected spaces (i.e. the base point x_0 is irrelevant) the set of equivalence classes is denoted by $\pi_n(X)$ and it is a group. The higher homotopy groups for $n > 1$ are all abelian as opposed to the fundamental group which can be non-abelian. Some of the most important examples are

$$\pi_n(S^n) = Z \tag{E.12}$$

$$\pi_m(S^n) = 0, \;\; m < n. \tag{E.13}$$