Chapter 1

Relativistic quantum mechanics

This chapter contains standard preparatory material. We will present an overview of special relativity [2], relativistic Klein–Gordon and Dirac wave equations and the convention in this book for Dirac spinors [3], and a self-contained discussion of representation theory of the rotation and Lorentz groups [1].

1.1 The rotation groups *SO*(3) and *SO(n)*

1.1.1 The Lie algebra *so*(3) and *so(n)*

The line element dl^2 in the physical space $\mathbf{R}^3$, which measures the distance between any two points $\vec{x}$ and $\vec{x} + d\vec{x}$, is given by the Euclidean formula

$$dl^2 = d\vec{x}^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx^2 + dy^2 + dz^2. \tag{1.1}$$

This is a particular instance of the scalar product on $\mathbf{R}^3$ defined by $\vec{x}\vec{y} = x_1y_1 + x_2y_2 + x_3y_3$. This scalar product (and as a consequence the line element) is invariant under the linear transformations

$$\vec{x} \longrightarrow \vec{x}' = R\vec{x} \tag{1.2}$$

provided the matrices R are orthogonal, viz

$$R \cdot R^T = R^T \cdot R = \mathbf{1}_3. \tag{1.3}$$

We can immediately show that either $\det R = +1$, which corresponds to proper orthogonal transformations which are precisely the rotations in the physical space $\mathbf{R}^3$, or that $\det R = -1$ which corresponds to improper orthogonal transformations such as space reflection or parity.

The set of all proper orthogonal transformations form the group of rotations denoted by $SO(3)$ where ‘S’ stands for ‘special’ meaning those transformations R with determinant equal to $+1$. The set of all orthogonal transformations form the

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group $O(3)$. Clearly, the group of rotations $SO(3)$ is a subgroup of the orthogonal group $O(3)$.

This generalizes to n dimensions (rotations and orthogonal transformations acting in $\mathbf{R}^n$) to obtain the groups $SO(n)$ and $O(n)$ as the set of linear transformations which are $n \times n$ matrices R satisfying the orthogonality condition

$$R \cdot R^T = R^T \cdot R = \mathbf{1}_n. \tag{1.4}$$

In general, a group is a set G equipped with an operation $*$ (composition law or matrix multiplication) which satisfy the following four natural axioms:

- **Closure:** The composition $(g_1 * g_2)$ of any two elements g_1 and g_2 of G is another element of G.
- **Associativity:** We must have $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
- **Identity:** There exists an element $e \in G$ such that $e * g = g * e = g$.
- **Invertibility:** There exists for every $g \in G$ an inverse $g^{-1} \in G$ such that $g * g^{-1} = e$.

The group can be infinite (the rotation group $SO(3)$) or finite (the reflection group). It can also be continuous (the rotation group $SO(3)$) or discrete (the reflection group). It can be abelian when the composition law $*$ is commutative otherwise it is non-abelian. For example, the groups $SO(3)$ and $O(3)$ are called non-abelian since their elements do not commute, i.e. the order of composition of two orthogonal transformations is important and thus we have $R * R' \neq R' * R$.

The dimension of a group G is the number of independent parameters required to define or characterize a general element g in this group. In the case of the rotation group $SO(3)$ it is obvious that a general rotation (about an arbitrary axis with an arbitrary angle) is the composition $R_1 \cdot R_2 \cdot R_3$ of three rotations R_1, R_2, R_3 about the axes x_1, x_2, x_3 with angles θ_1, θ_2, θ_3 respectively. Thus, in this case the independent parameters required to characterize a general element (rotation) in $SO(3)$ are precisely the angles θ_1, θ_2, θ_3 and the dimension of the group is three, viz

$$d_{SO(3)} = 3. \tag{1.5}$$

This result can also be shown by solving equations (1.4). There are n^2 variables *a priori* in the matrix R which are constrained by the $n(n+1)/2$ independent equations contained in equation (1.4) leaving therefore $n(n-1)/2$ independent variables. Hence, the dimension of the rotation group $SO(n)$ in n dimensions is given by

$$d_{SO(n)} = \frac{n(n-1)}{2}. \tag{1.6}$$

By substituting $n = 3$ we obtain $d_{SO(3)} = 3$.

If the group is also a manifold then it is a Lie group. Indeed, continuous groups of finite dimension are actually Lie groups. The rotation groups $SO(n)$ are examples of Lie groups. They are in fact compact Lie groups. The tangent space at the identity e of the group G is called the Lie algebra of the group which is a vector space. The Lie algebra of the Lie group $SO(n)$ is denoted $so(n)$.

In general, the Lie algebra L of a Lie group G is the tangent vector space at the identity which is a set of elements satisfying the following axioms:

- If $X \in L$ and $Y \in L$ then $X + Y \in L$.
- If $X \in L$ then $\alpha X \in L$ for any complex number α.
- If $X \in L$ and $Y \in L$ then $[X, Y] \in L$ and $[X, Y] = -[Y, X]$.
- If $X \in L, Y \in L$ and $Z \in L$ then $[X, Y + Z] = [X, Y] + [X, Z]$.
- If $X \in L, Y \in L$ and $Z \in L$ then $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. This is called the Jacobi identity.

The Lie algebra $so(3)$ of the three-dimensional rotation group $SO(3)$ can be constructed as follows. The R_1, R_2, R_3 rotations and their infinitesimal forms can be written explicitly as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow R_1 = \mathbf{1}_3 + i\theta_1 L_1,$$
$$L_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{1.7}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow R_2 = \mathbf{1}_3 + i\theta_2 L_2,$$
$$L_2 = -i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{1.8}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow R_3 = \mathbf{1}_3 + i\theta_3 L_3,$$
$$L_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

The operators L_1, L_2, L_3 are called the generators of the Lie algebra $so(3)$ of the rotation group. We can easily check that they satisfy the angular momentum algebra

$$[L_1, L_2] = iL_3, \quad [L_3, L_1] = iL_2,$$
$$[L_2, L_3] = iL_1 \Leftrightarrow [L_i, L_j] = i\varepsilon_{ijk}L_k. \tag{1.10}$$

We know from quantum mechanics that the generators L_i commute with the squared angular moment operator $\vec{L}^2 = L_1^2 + L_2^2 + L_3^2$. We have then

$$[L_i, \vec{L}^2] = 0, \quad \vec{L}^2 = L_1^2 + L_2^2 + L_3^2. \tag{1.11}$$

The operators $\vec{L}^2$ and L_3 can then be diagonalized simultaneously with eigenvalues given by

$$\begin{aligned} \vec{L}^2|lm_3\rangle &= l(l+1)|lm_3\rangle \\ L_3|lm_3\rangle &= m_3|lm_3\rangle. \end{aligned} \tag{1.12}$$

The eigenvalues l and m_3 take the values $l = 1$ and $m_3 = +1, 0, -1$. In other words, $\vec{L}$ is the orbital angular momentum operator.

1.1.2 Representations of *SO*(3) and *so*(3)

A representation U of a group G on a vector space V over the field **C** is a map (a group homomorphism) from the group G to the general linear group GL(V) (denoted also as Aut(V)) consisting of all bijective linear operators (automorphisms) acting in V. The group operation, which we will also denote by *, is the functional composition of linear operations. We write

$$\begin{aligned} U : G &\longrightarrow \mathrm{GL}(V) \\ g &\longrightarrow U(g). \end{aligned} \tag{1.13}$$

Thus, every element g in G is associated with a linear operator $U(g)$ in GL(V) such that the composition law is maintained, i.e. if $g_1 \in G$ and $g_2 \in G$ then

$$U(g_1 * g_2) = U(g_1) * U(g_2). \tag{1.14}$$

We will also have

$$U(e) = \mathbf{1}, \quad U(g^{-1}) = U(g)^{-1}. \tag{1.15}$$

The vector space V is called the representation space and its dimension is called the dimension of the representation U. If V is n-dimensional then $\mathrm{GL}(V) = \mathrm{GL}(n, \mathbf{C})$. In this case we are dealing with a finite dimensional matrix representation. We may use the map U or the vector space V to refer to the representation.

A subspace V_1 of V is called an invariant subspace with respect to the representation U if for every $v \in V_1$ we have $U(g)v \in V_1$ for every $g \in G$. A representation $g \longrightarrow U(g)$ is called irreducible if and only if the only invariant subspace with respect to U is the vector space V itself. Otherwise the representation is called reducible. For finite groups it can be shown that an arbitrary representation V will break up into irreducible representations V_i. We write V as a direct sum of the V_i as follows

$$V = \oplus_i V_i = V_1 \oplus V_2 \oplus \cdots. \tag{1.16}$$

This means that the representation operator $U(g)$, which is usually a matrix, is a block diagonal matrix where each block $U_i(g)$ corresponds to a vector space V_i.

We are therefore only interested in irreducible representations which are also not equivalent. Indeed, it is almost obvious that if two representations are related by a unitary transformation then they are necessarily equivalent.

Furthermore, for Lie groups it can be shown that representations of the Lie algebra determine the representation of the group uniquely.

A representation T of the Lie algebra L is a map from L to $M(V)$ which consists of all linear transformations of a vector space V. Clearly, if $V = \mathbf{R}^n$, then $M(V)$ is the set of $n \times n$ square matrices and the representation T is a matrix representation. Again we may use the map T or the vector space V to refer to the representation. We write

$$\begin{aligned} T : L &\longrightarrow M(V) \\ X &\longrightarrow T(X). \end{aligned} \tag{1.17}$$

Thus, every element X in L is associated with a linear operator $T(X)$ in $M(V)$ such that if $X \in L$ and $Y \in L$ then

$$T(X + Y) = T(X) + T(Y), \quad T(\alpha X) = \alpha T(X), \quad \alpha \in \mathbf{C}. \tag{1.18}$$

More importantly we have

$$T([X, Y]) = [T(X), T(Y)]. \tag{1.19}$$

The simplest and most basic irreducible representation is called the fundamental representation, which for the rotation group $SO(3)$, is a spinor representation. The adjoint or vector representation is an irreducible representation provided by the group elements directly.

For the rotation Lie algebra $so(3)$ the adjoint representation (also called the vector representation) is a three-dimensional irreducible representation given precisely by the generators L_1, L_2 and L_3. An infinitesimal rotation was found to be given by

$$R(\delta\theta) = \mathbf{1}_3 + i\delta\theta_i L_i. \tag{1.20}$$

A finite rotation can then be found by integration to be given by

$$R(\theta) = \exp(i\theta_i L_i). \tag{1.21}$$

This adjoint or vector representation is three-dimensional. A general N-dimensional representation operator of the above infinitesimal rotation should be given by

$$U(\delta\theta) = \mathbf{1}_N + i\delta\theta_i J_i. \tag{1.22}$$

Similarly, the general N-dimensional representation of the above finite rotation can also be found by integration to be given by

$$U(\theta) = \exp(i\theta_i J_i). \tag{1.23}$$

The generators J_i are the N-dimensional representation operators of the Lie algebra $so(3)$ in the same way that the generators L_i are the three-dimensional representation operators of this Lie algebra. They must therefore be angular momentum operators satisfying the angular momentum algebra (1.10), viz

$$[J_i, J_j] = i\varepsilon_{ijk}J_k. \tag{1.24}$$

As it turns out, finding all sets $\{J_1, J_2, J_3\}$ which solve this condition (1.24) is equivalent to the problem of finding all irreducible representations of the rotation group $SO(3)$.

This is in accord with Shur's lemma which guarantees that a representation U is irreducible if and only if the only matrices which commute with the representation operators $U(g)$ for all $g \in G$ are matrices proportional to the identity matrix. These matrices are called the Casimir operators (corresponding to conserved quantities). The number of Casimir operators in a Lie algebra is called the rank of the group and their eigenvalues characterize the irreducible representations of the Lie algebra.

Hence, by finding the set of all Casimir operators (which by construction commute among themselves and therefore can be diagonalized simultaneously) we can obtain irreducible representations by (1) computing their eigenvalues and then by (2) restricting each time to a given eigenspace with a fixed eigenvalue which by Shur's lemma is guaranteed to correspond to an irreducible representation.

From quantum mechanics we know that the angular momentum generators J_i commute with the squared angular momentum operator $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$. This is precisely the (single) Casimir operator of the rotation group $SO(3)$. We have then

$$[J_i, \vec{J}^2] = 0, \quad \vec{J}^2 = J_1^2 + J_2^2 + J_3^2. \tag{1.25}$$

The operators $\vec{J}^2$ and J_3 can be then diagonalized simultaneously with eigenvalues given by

$$\begin{aligned} &\vec{J}^2|jm\rangle = j(j+1)|jm \\ &J_3|jm\rangle = m|jm\rangle \\ &\qquad m = j, j-1, \ldots, -j+1, -j, \quad j = 0, 1/2, 1, 3/2, \ldots \end{aligned} \tag{1.26}$$

The spin (integer or half-integer) j characterizes then the irreducible representations of the Lie algebra $so(3)$ and the rotation group $SO(3)$ which are obviously $(2j+1)$-dimensional (this is the number of independent states $|jm\rangle$ for some j since m varies from $-j$ to $+j$ with step equal 1). Hence, the dimension of these irreducible representations is $N = 2j + 1$. The representations with integer spin are called tensor representation (bosons), whereas those with half-integer spin are called spinor representations (fermions). These are all unitary representations.

The adjoint or vector representation given by the generators L_i corresponds therefore to spin one, i.e. $j = 1$ and $N = 3$.

The fundamental representation corresponds to spin one-half, i.e. $j = 1/2$ and $N = 2$, and it is generated by Pauli matrices, viz

$$J_i = \frac{\sigma_i}{2}. \tag{1.27}$$

A finite rotation about an axis $\vec{n}$ with an angle θ is given in the fundamental representation $j = 1/2$ by

$$U(\vec{n}, \theta) = \exp(i\theta\vec{n}\vec{\sigma}/2). \tag{1.28}$$

This acts on spinor wave functions, or spinors for short, which under a rotation with a 2π angle acquires an overall minus sign (spin-statistic theorem).

The reducible representations of the rotation group $SO(3)$ are easily obtained by taking tensor products of the irreducible representations j. The main result is already known from quantum mechanics and is given by

$$\begin{aligned} j_1 \otimes j_2 &= (j_1 + j_2) \oplus (j_1 + j_2 - 1) \\ &\oplus (j_1 + j_2 - 2) \oplus \cdots \oplus |j_1 - j_2| = \sum_{\oplus} j. \end{aligned} \tag{1.29}$$

The Lie algebra $o(3)$ and the orthogonal group $O(3)$ will involve an additional Casimir operator. Indeed, the generators L_i commute also with the reflection operator, i.e.

$$[L_i, R_0] = 0, \quad R_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.30}$$

Similarly, in the N-dimensional representation of the orthogonal Lie algebra $o(3)$ the generators J_i commute with representation operator U_0 of the reflection operator, i.e.

$$[J_i, U_0] = 0. \tag{1.31}$$

As a consequence, the irreducible representations of $o(3)$ and $O(3)$ are characterized by the pair (l, r) where r is the eigenvalue of the reflection operator U_0 which can only take the two values $r = \pm 1$.

1.2 Special relativity

1.2.1 Postulates

Classical mechanics obeys the principle of relativity which states that the laws of nature take the same form in all inertial frames. An inertial frame is any frame in which Newton's first law holds. Therefore, all other frames which move with a constant velocity with respect to a given inertial frame are also inertial frames.

Any two inertial frames O and O' can be related by a Galilean transformation which is of the general form

$$\begin{aligned} t' &= t + \tau \\ \vec{x}' &= R\vec{x} + \vec{v}t + \vec{d}. \end{aligned} \tag{1.32}$$

In the above R is a constant orthogonal matrix, $\vec{d}$ and $\vec{v}$ are constant vectors and τ is a constant scalar. Thus the observer O' sees the coordinates axes of O rotated by R, moving with a velocity $\vec{v}$, translated by $\vec{d}$ and it sees the clock of O running behind by the amount τ. The set of all transformations of the form (1.32) forms a 10-parameter group called the Galilean group.

The invariance/covariance of the equations of motion under these transformations, which is called Galilean invariance/covariance, is the precise statement of the principle of Galilean relativity.

In contrast to the laws of classical mechanics, the laws of classical electrodynamics do not obey the Galilean principle of relativity. Before the advent of the theory of special relativity the laws of electrodynamics were thought to hold only in the inertial reference frame which is at rest with respect to an invisible medium filling all space known as the ether. For example, electromagnetic waves were thought to propagate through the vacuum at a speed relative to the ether, equal to the speed of light $c = 1/\sqrt{\mu_0 \epsilon_0} = 3 \times 10^8$ m s^{-1}.

The motion of the Earth through the ether creates an ether wind. Thus, only by measuring the speed of light in the direction of the ether wind can we get the value c, whereas measuring it in any other direction will give a different result. In other words we can detect the ether by measuring the speed of light in different directions which is precisely what Michelson and Morley tried to do in their famous experiments. The outcome of these experiments was always negative in the sense that the speed of light was found to be exactly the same, equal to c in all directions.

The theory of special relativity was the first to accommodate this empirical finding by postulating that the speed of light is the same in all inertial reference frames, i.e. there is no ether. Furthermore, it postulates that classical electrodynamics (and physical laws in general) must hold in all inertial reference frames. This is the principle of relativity, although now its precise statement cannot be given in terms of the invariance/covariance under Galilean transformations but in terms of the invariance/covariance under Lorentz transformations which we will discuss further in the next section.

Einstein's original motivation behind the principle of relativity comes from the physics of the electromotive force. The interaction between a conductor and a magnet in the reference frame where the conductor is moving and the magnet is at rest is known to result in an motional emf. The charges in the moving conductor will experience a magnetic force given by the Lorentz force law. As a consequence, a current will flow in the conductor with an induced motional emf given by the flux rule $\mathcal{E} = -d\Phi/dt$. In the reference frame where the conductor is at rest and the magnet is moving there is no magnetic force acting on the charges. However, the moving magnet generates a changing magnetic field which by Faraday's law induces an electric field. As a consequence in the rest frame of the conductor the charges experience an electric force which causes a current to flow with an induced transformer emf given precisely by the flux rule, viz $\mathcal{E} = -d\Phi/dt$.

So, in summary, although the two observers associated with the states of rest of the conductor and the magnet have different interpretations of the process, their predictions are in perfect agreement. This indeed suggests, as pointed out first by Einstein, that the laws of classical electrodynamics are the same in all inertial reference frames.

The two fundamental postulates of special relativity are therefore:

- The principle of relativity: The laws of physics take the same form in all inertial reference frames.
- The constancy of the speed of light: The speed of light in vacuum is the same in all inertial reference frames.

1.2.2 Relativistic effects

The Gedanken experiments we will discuss here might be called 'The train-and-platform thought experiments'.

Relativity of simultaneity

We consider an observer O' in the middle of a freight car moving at a speed v with respect to the ground and a second observer O standing on a platform. A light bulb hanging in the center of the car is switched on just as the two observers pass each other.

It is clear that with respect to the observer O' light will reach the front end A and the back end B of the freight car at the same time. The two events 'light reaches the front end' and 'light reaches the back end' are simultaneous.

According to the second postulate light propagates with the same velocity with respect to the observer O. This observer sees the back end B moving toward the point at which the flash was given off and the front end A moving away from it. Thus light will reach B before it reaches A. In other words with respect to O the event 'light reaches the back end' happens before the event 'light reaches the front end'.

Time dilation

Let us now ask the question: How long does it take a light ray to travel from the bulb to the floor?

Let us call h the height of the freight car. It is clear that with respect to O' the time spent by the light ray between the bulb and the floor is

$$\Delta t' = \frac{h}{c}. \tag{1.33}$$

The observer O will measure a time Δt during which the freight car moves a horizontal distance $v\Delta t$. The trajectory of the light ray is not given by the vertical distance h but by the hypotenuse of the right triangle with h and vdt as the other two sides. Thus with respect to O the light ray travels a longer distance given by $\sqrt{h^2 + v^2\Delta t^2}$ and therefore the time spent is

$$\Delta t = \frac{\sqrt{h^2 + v^2\Delta t^2}}{c}. \tag{1.34}$$

Solving for Δt we get

$$\Delta t = \gamma \frac{h}{c} = \gamma \Delta t'. \tag{1.35}$$

The factor γ is known as Lorentz factor and it is given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{1.36}$$

Hence we obtain

$$\Delta t' = \sqrt{1 - \frac{v^2}{c^2}} \Delta t \leqslant \Delta t. \tag{1.37}$$

The time measured on the train is shorter than the time measured on the ground. In other words moving clocks run slow. This is called time dilation.

Lorentz contraction

We now place a lamp at the back end B of the freight car and a mirror at the front end A. Then we ask the question: How long does it take a light ray to travel from the lamp to the mirror and back?

Again with respect to the observer O' the answer is simple. If $\Delta x'$ is the length of the freight car measured by O' then the time spent by the light ray in the round trip between the lamp and the mirror is

$$\Delta t' = 2\frac{\Delta x'}{c}. \tag{1.38}$$

Let Δx be the length of the freight car measured by O and Δt_1 be the time for the light ray to reach the front end A. Then clearly

$$c\Delta t_1 = \Delta x + v\Delta t_1. \tag{1.39}$$

The term $v\Delta t_1$ is the distance traveled by the train during the time Δt_1. Let Δt_2 be the time for the light ray to return to the back end B. Then

$$c\Delta t_2 = \Delta x - v\Delta t_2. \tag{1.40}$$

The time spent by the light ray in the round trip between the lamp and the mirror is therefore

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{\Delta x}{c - v} + \frac{\Delta x}{c + v} = 2\gamma^2 \frac{\Delta x}{c}. \tag{1.41}$$

The time intervals Δt and $\Delta t'$ are related by time dilation, viz

$$\Delta t = \gamma \Delta t'. \tag{1.42}$$

This is equivalent to

$$\Delta x' = \gamma \Delta x \geqslant \Delta x. \tag{1.43}$$

The length measured on the train is longer than the length measured on the ground. In other words moving objects are shortened. This is called Lorentz contraction.

We point out here that only the length parallel to the direction of motion is contracted while lengths perpendicular to the direction of the motion remain not contracted.

1.2.3 Lorentz transformations: boosts

Any physical process consists of a collection of events. Any event takes place at a given point (x, y, z) of space at an instant of time t. Lorentz transformations relate the coordinates (x, y, z, t) of a given event in an inertial reference frame O to the coordinates (x', y', z', t') of the same event in another inertial reference frame O'.

Let (x, y, z, t) be the coordinates in O of an event E. The projection of E onto the x-axis is given by the point P which has the coordinates $(x, 0, 0, t)$. For simplicity we will assume that the observer O' moves with respect to the observer O at a constant speed v along the x-axis. At time $t = 0$ the two observers O and O' coincide. After time t the observer O' moves a distance vt on the x-axis. Let d be the distance between O' and P as measured by O. Then clearly

$$x = d + vt. \tag{1.44}$$

Before the theory of special relativity the coordinate x' of the event E in the reference frame O' is taken to be equal to the distance d. We get therefore the transformation laws

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned} \tag{1.45}$$

This is a Galilean transformation. Indeed this is a special case of equation (1.32).

As we have already seen, Einstein's postulates lead to Lorentz contraction. In other words the distance between O' and P measured by the observer O', which is precisely the coordinate x', is larger than d. More precisely

$$x' = \gamma d. \tag{1.46}$$

Hence

$$x' = \gamma(x - vt). \tag{1.47}$$

Einstein's postulates also lead to time dilation and relativity of simultaneity. Thus, the time of the event E measured by O' is different from t. Since the observer O moves with respect to O' at a speed v in the negative x-direction we must have

$$x = \gamma(x' + vt'). \tag{1.48}$$

Thus we get

$$t' = \gamma\left(t - \frac{v}{c^2}x\right). \tag{1.49}$$

In summary we get the transformation laws

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right). \end{aligned} \tag{1.50}$$

This is a special Lorentz transformation which is a boost along the x-axis.

Let us look at the clock found at the origin of the reference frame O'. We set $x' = 0$ in the above equations. We then get the time dilation effect, viz

$$t' = \frac{t}{\gamma}. \tag{1.51}$$

At time $t = 0$ the clocks in O' read different times depending on their location since

$$t' = -\gamma\frac{v}{c^2}x. \tag{1.52}$$

Hence, moving clocks cannot be synchronized.

We consider now two events A and B with coordinates (x_A, t_A) and (x_B, t_B) in O and coordinates (x_A', t_A') and (x_B', t_B') in O'. We can compute

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right). \tag{1.53}$$

Thus, if the two events are simultaneous with respect to O, i.e. $\Delta t = 0$, they are not simultaneous with respect to O' since

$$\Delta t' = -\gamma\frac{v}{c^2}\Delta x. \tag{1.54}$$

1.2.4 Spacetime

The above Lorentz boost transformation can be rewritten as

$$\begin{aligned} x^{0'} &= \gamma(x^0 - \beta x^1) \\ x^{1'} &= \gamma(x^1 - \beta x^0) \\ x^{2'} &= x^2 \\ x^{3'} &= x^3. \end{aligned} \tag{1.55}$$

In the above equation

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \tag{1.56}$$

$$\beta = \frac{v}{c}, \quad \gamma = \sqrt{1 - \beta^2}. \tag{1.57}$$

This can also be rewritten as

$$x^{\mu'} = \sum_{\nu=0}^{4} \Lambda^{\mu}_{\nu} x^{\nu} \tag{1.58}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.59}$$

The matrix Λ is the Lorentz boost transformation matrix. A general Lorentz boost transformation can be obtained if the relative motion of the two inertial reference frames O and O' is along an arbitrary direction in space. The transformation law of the coordinates x^{μ} will still be given by equation (1.58) with a more complicated matrix Λ. A general Lorentz transformation can be written as a product of a rotation and a boost along a direction $\hat{n}$ given by

$$\begin{aligned} x^{'0} &= x^0 \cosh\alpha - \hat{n}\vec{x} \sinh\alpha \\ \vec{x}' &= \vec{x} + \hat{n}((\cosh\alpha - 1)\hat{n}\vec{x} - x^0 \sinh\alpha) \end{aligned} \tag{1.60}$$

$$\frac{\vec{v}}{c} = \tanh\alpha \;\; \hat{n}. \tag{1.61}$$

Indeed, the set of all Lorentz transformations contains rotations as a subset.

The set of coordinates (x^0, x^1, x^2, x^3) which transforms under Lorentz transformations as $x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu}$ will be called a 4-vector in analogy with the set of coordinates (x^1, x^2, x^3) which is called a vector because it transforms under rotations as $x^{a'} = R^{a}_{b} x^{b}$. Thus, in general, a 4-vector a is any set of numbers (a^0, a^1, a^2, a^3) which transforms as (x^0, x^1, x^2, x^3) under Lorentz transformations, viz

$$a^{\mu'} = \sum_{\nu=0}^{4} \Lambda^{\mu}_{\nu} a^{\nu}. \tag{1.62}$$

For the particular Lorentz transformation (1.59) we have

$$\begin{aligned} a^{0'} &= \gamma(a^0 - \beta a^1) \\ a^{1'} &= \gamma(a^1 - \beta a^0) \\ a^{2'} &= a^2 \\ a^{3'} &= a^3. \end{aligned} \tag{1.63}$$

The numbers a^μ are called the contravariant components of the 4-vector a. We define the covariant components a_μ by

$$a_0 = a^0, \quad a_1 = -a^1, \quad a_2 = -a^2, \quad a_3 = -a^3. \tag{1.64}$$

By using the Lorentz transformation (1.63) we verify any two 4-vectors a and b the identity

$$a^{0'}b^{0'} - a^{1'}b^{1'} - a^{2'}b^{2'} - a^{3'}b^{3'} = a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3. \tag{1.65}$$

In fact we can show that this identity holds for all Lorentz transformations. We recall that under rotations the scalar product $\vec{a}\vec{b}$ of any two vectors $\vec{a}$ and $\vec{b}$ is invariant, i.e.

$$a^{1'}b^{1'} + a^{2'}b^{2'} + a^{3'}b^{3'} = a^1b^1 + a^2b^2 + a^3b^3. \tag{1.66}$$

The four-dimensional scalar product must therefore be defined by the Lorentz invariant combination $a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3$, namely

$$\begin{aligned} ab &= a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 \\ &= \sum_{\mu=0}^{3} a_\mu b^\mu \\ &= a_\mu b^\mu. \end{aligned} \tag{1.67}$$

In the last equation we have employed the so-called Einstein summation convention, i.e. a repeated index is summed over.

We define the separation 4-vector Δx between two events A and B occurring at the points $(x_A^0, x_A^1, x_A^2, x_A^3)$ and $(x_B^0, x_B^1, x_B^2, x_B^3)$ by the components

$$\Delta x^\mu = x_A^\mu - x_B^\mu. \tag{1.68}$$

The distance squared between the two events A and B, which is called the interval between A and B, is defined by

$$\Delta s^2 = \Delta x_\mu \Delta x^\mu = c^2\Delta t^2 - \Delta\vec{x}^2. \tag{1.69}$$

This is a Lorentz invariant quantity. However, it could be positive, negative or zero.

In the case $\Delta s^2 > 0$ the interval is called timelike. There exists an inertial reference frame in which the two events occur at the same place and are only separated temporally.

In the case $\Delta s^2 < 0$ the interval is called spacelike. There exists an inertial reference frame in which the two events occur at the same time and are only separated in space.

In the case $\Delta s^2 = 0$ the interval is called lightlike. The two events are connected by a signal traveling at the speed of light.

1.2.5 Metric

The interval ds^2 between two infinitesimally close events A and B in spacetime with position 4-vectors x_A^μ and $x_B^\mu = x_A^\mu + dx^\mu$ is given by

$$\begin{aligned} ds^2 &= \sum_{\mu=0}^{3}(x_A - x_B)_\mu(x_A - x_B)^\mu \\ &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= c^2(dt)^2 - (d\vec{x})^2. \end{aligned} \tag{1.70}$$

We can also write this interval as (using also Einstein's summation convention)

$$\begin{aligned} ds^2 &= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu \\ &= \sum_{\mu,\nu=0}^{3} \eta^{\mu\nu}dx_\mu dx_\nu = \eta^{\mu\nu}dx_\mu dx_\nu. \end{aligned} \tag{1.71}$$

The 4×4 matrix η is called the metric tensor and it is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.72}$$

Clearly we can also write

$$ds^2 = \sum_{\mu,\nu=0}^{3} \eta_\mu^\nu dx^\mu dx_\nu = \eta_\mu^\nu dx^\mu dx_\nu. \tag{1.73}$$

In this case

$$\eta_\mu^\nu = \delta_\mu^\nu. \tag{1.74}$$

The metric η is used to lower and raise Lorentz indices, viz

$$x_\mu = \eta_{\mu\nu}x^\nu. \tag{1.75}$$

The interval ds^2 is invariant under Poincaré transformations which combine translations a with Lorentz transformations Λ:

$$x^\mu \longrightarrow x'^\mu = \Lambda_\nu^\mu x^\nu + a^\mu. \tag{1.76}$$

We compute

$$ds^2 = \eta_{\mu\nu}dx'^\mu dx'^\nu = \eta_{\mu\nu}dx^\mu dx^\nu. \tag{1.77}$$

This leads to the condition

$$\eta_{\mu\nu}\Lambda_\rho^\mu\Lambda_\sigma^\nu = \eta_{\rho\sigma} \Leftrightarrow \Lambda^T\eta\Lambda = \eta. \tag{1.78}$$

1.3 Klein–Gordon equation

The non-relativistic energy–momentum relation reads

$$E = \frac{\vec{p}^2}{2m} + V. \tag{1.79}$$

The correspondence principle is

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}, \quad \vec{p} \longrightarrow \frac{\hbar}{i}\vec{\nabla}. \tag{1.80}$$

This yields the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{1.81}$$

We will only consider the free case, i.e. $V = 0$. We have then

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{1.82}$$

The energy–momentum 4-vector is given by

$$p^\mu = (p^0, p^1, p^2, p^3) = \left(\frac{E}{c}, \vec{p}\right). \tag{1.83}$$

The relativistic momentum and energy are defined by

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.84}$$

The energy–momentum 4-vector satisfies

$$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m^2c^2. \tag{1.85}$$

The relativistic energy–momentum relation is therefore given by

$$\vec{p}^2c^2 + m^2c^4 = E^2. \tag{1.86}$$

Thus the free Schrödinger equation will be replaced by the relativistic wave equation

$$(-\hbar^2c^2\nabla^2 + m^2c^4)\phi = -\hbar^2\frac{\partial^2\phi}{\partial t^2}. \tag{1.87}$$

This can also be rewritten as

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 - \frac{m^2c^2}{\hbar^2}\right)\phi = 0. \tag{1.88}$$

This is the Klein–Gordon equation. In contrast with the Schrödinger equation the Klein–Gordon equation is a second-order differential equation. In relativistic notation we have

$$E \longrightarrow i\hbar\frac{\partial}{\partial t} \Leftrightarrow p_0 \longrightarrow i\hbar\partial_0, \quad \partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t} \tag{1.89}$$

$$\vec{p} \longrightarrow \frac{\hbar}{i}\vec{\nabla} \Leftrightarrow p_i \longrightarrow i\hbar\partial_i, \quad \partial_i = \frac{\partial}{\partial x^i}. \tag{1.90}$$

In other words

$$p_\mu \longrightarrow i\hbar\partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \tag{1.91}$$

$$p_\mu p^\mu \longrightarrow -\hbar^2\partial_\mu\partial^\mu = \hbar^2\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right). \tag{1.92}$$

The covariant form of the Klein–Gordon equation is

$$\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi = 0. \tag{1.93}$$

Free solutions are of the form

$$\phi(t, \vec{x}) = e^{-\frac{i}{\hbar}px}, \quad px = p_\mu x^\mu = Et - \vec{p}\,\vec{x}. \tag{1.94}$$

Indeed we compute

$$\partial_\mu\partial^\mu\phi(t, \vec{x}) = -\frac{1}{c^2\hbar^2}(E^2 - \vec{p}^{\,2}c^2)\phi(t, \vec{x}). \tag{1.95}$$

Thus we must have

$$E^2 - \vec{p}^{\,2}c^2 = m^2c^4. \tag{1.96}$$

In other words

$$E^2 = \pm\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}. \tag{1.97}$$

There exists therefore negative-energy solutions. The energy gap is $2mc^2$. As it stands the existence of negative-energy solutions means that the spectrum is not bounded from below and as a consequence an arbitrarily large amount of energy can be extracted. This is a severe problem for a single-particle wave equation. However, these negative-energy solutions, as we will see shortly, will be related to antiparticles.

From the two equations

$$\phi^*\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi = 0, \tag{1.98}$$

$$\phi\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi^* = 0, \tag{1.99}$$

we get the continuity equation

$$\partial^\mu J_\mu = 0, \tag{1.100}$$

where

$$J_\mu = \frac{i\hbar}{2m}[\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*]. \tag{1.101}$$

We have included the factor $i\hbar/2m$ in order that the zero component J_0 has the dimension of a probability density. The continuity equation can also be put in the form

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J} = 0, \tag{1.102}$$

where

$$\rho = \frac{J_0}{c} = \frac{i\hbar}{2mc^2}\left[\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right] \tag{1.103}$$

$$\vec{J} = -\frac{i\hbar}{2mc}[\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*]. \tag{1.104}$$

Clearly the zero component J_0 is not positive definite and hence it can be a probability density. This is due to the fact that the Klein–Gordon equation is second-order.

The Dirac equation is a relativistic wave equation which is a first-order differential equation. The corresponding probability density will therefore be positive definite. However negative-energy solutions will still be present.

1.4 Dirac equation

The Dirac equation is a first-order differential equation of the same form as the Schrödinger equation, viz

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi. \tag{1.105}$$

In order to derive the form of the Hamiltonian H we go back to the relativistic energy–momentum relation

$$p_\mu p^\mu - m^2c^2 = 0. \tag{1.106}$$

The only requirement on H is that it must be linear in spatial derivatives since we want space and time to be on equal footing. We thus factor out the above equation as follows

$$\begin{aligned} p_\mu p^\mu - m^2c^2 &= (\gamma^\mu p_\mu + mc)(\beta^\nu p_\nu - mc) \\ &= \gamma^\mu \beta^\nu p_\mu p_\nu - mc(\gamma^\mu - \beta^\mu)p_\mu - m^2c^2. \end{aligned} \tag{1.107}$$

We must therefore have $\beta^\mu = \gamma^\mu$, i.e.

$$p_\mu p^\mu = \gamma^\mu \gamma^\nu p_\mu p_\nu. \tag{1.108}$$

This is equivalent to

$$\begin{aligned} p_0^2 - p_1^2 - p_2^2 - p_3^2 = {} & (\gamma^0)^2 p_0^2 + (\gamma^1)^2 p_1^2 + (\gamma^2)^2 p_2^2 + (\gamma^3)^2 p_3^2 \\ & + (\gamma^1\gamma^2 + \gamma^2\gamma^1)p_1p_2 + (\gamma^1\gamma^3 + \gamma^3\gamma^1)p_1p_3 \\ & + (\gamma^2\gamma^3 + \gamma^3\gamma^2)p_2p_3 + (\gamma^1\gamma^0 + \gamma^0\gamma^1)p_1p_0 \\ & + (\gamma^2\gamma^0 + \gamma^0\gamma^2)p_2p_0 + (\gamma^3\gamma^0 + \gamma^0\gamma^3)p_3p_0. \end{aligned} \tag{1.109}$$

Clearly the objects γ^μ cannot be complex numbers since we must have

$$\begin{aligned} (\gamma^0)^2 = 1, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0. \end{aligned} \tag{1.110}$$

These conditions can be rewritten in a compact form as

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \tag{1.111}$$

This algebra is an example of a Clifford algebra and the solutions are matrices γ^μ which are called Dirac matrices. In four-dimensional Minkowski space the smallest Dirac matrices must be 4×4 matrices. All 4×4 representations are unitarily equivalent. We choose the so-called Weyl or chiral representation given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{1.112}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.113}$$

Note that

$$(\gamma^0)^+ = \gamma^0, \quad (\gamma^i)^+ = -\gamma^i \Leftrightarrow (\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0. \tag{1.114}$$

The relativistic energy–momentum relation becomes

$$p_\mu p^\mu - m^2c^2 = (\gamma^\mu p_\mu + mc)(\gamma^\nu p_\nu - mc) = 0. \tag{1.115}$$

Thus, either $\gamma^\mu p_\mu + mc = 0$ or $\gamma^\mu p_\mu - mc = 0$. The convention is to take

$$\gamma^\mu p_\mu - mc = 0. \tag{1.116}$$

By applying the correspondence principle $p_\mu \longrightarrow i\hbar\partial_\mu$ we obtain the relativistic wave equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{1.117}$$

This is the Dirac equation in a covariant form. Let us introduce the Feynman 'slash' defined by

$$\partial\!\!\!/ = \gamma^\mu\partial_\mu \tag{1.118}$$

$$(i\hbar\partial\!\!\!/ - mc)\psi = 0. \tag{1.119}$$

Since the γ matrices are 4×4 the wave function ψ must be a four-component object which we call a Dirac spinor. Thus we have

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{1.120}$$

The Hermitian conjugate of the Dirac equation (1.131) is

$$\psi^+(i\hbar(\gamma^\mu)^+ \overleftarrow{\partial}_\mu + mc) = 0. \tag{1.121}$$

In other words

$$\psi^+(i\hbar\gamma^0\gamma^\mu\gamma^0 \overleftarrow{\partial}_\mu + mc) = 0. \tag{1.122}$$

The Hermitian conjugate of a Dirac spinor is not ψ^+ but it is defined by

$$\bar{\psi} = \psi^+\gamma^0. \tag{1.123}$$

Thus the Hermitian conjugate of the Dirac equation is

$$\bar{\psi}(i\hbar\gamma^\mu \overleftarrow{\partial}_\mu + mc) = 0. \tag{1.124}$$

Equivalently

$$\bar{\psi}(i\hbar \overleftarrow{\partial\!\!\!/} + mc) = 0. \tag{1.125}$$

Putting equations (1.119) and (1.125) together we obtain

$$\bar{\psi}(i\hbar\overleftarrow{\not\partial}+i\hbar\overrightarrow{\not\partial})\psi = 0. \quad (1.126)$$

We obtain the continuity equation

$$\partial_\mu J^\mu = 0, \quad J^\mu = \bar{\psi}\gamma^\mu\psi. \quad (1.127)$$

Explicitly we have

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J} = 0 \quad (1.128)$$

$$\rho = \frac{J^0}{c} = \frac{1}{c}\bar{\psi}\gamma^0\psi = \frac{1}{c}\psi^+\psi \quad (1.129)$$

$$\vec{J} = \bar{\psi}\vec{\gamma}\psi = \psi^+\vec{\alpha}\psi. \quad (1.130)$$

The probability density ρ is positive definite as desired.

1.5 Free solutions of the Dirac equation

We seek solutions of the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \quad (1.131)$$

The plane-wave solutions are of the form

$$\psi(x) = a \;\, e^{-\frac{i}{\hbar}px}u(p). \quad (1.132)$$

Explicitly

$$\psi(t, \vec{x}) = a \;\, e^{-\frac{i}{\hbar}(Et-\vec{p}\,\vec{x})}u(E, \vec{p}\,). \quad (1.133)$$

The spinor $u(p)$ must satisfy

$$(\gamma^\mu p_\mu - mc)u = 0. \quad (1.134)$$

We write

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}. \quad (1.135)$$

We compute

$$\gamma^\mu p_\mu - mc = \begin{pmatrix} -mc & \dfrac{E}{c} - \vec{\sigma}\vec{p} \\ \dfrac{E}{c} + \vec{\sigma}\vec{p} & -mc \end{pmatrix}. \quad (1.136)$$

We then get

$$u_A = \frac{\frac{E}{c} - \vec{\sigma}\vec{p}}{mc} u_B \tag{1.137}$$

$$u_B = \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc} u_A. \tag{1.138}$$

A consistency condition is

$$u_A = \frac{\frac{E}{c} - \vec{\sigma}\vec{p}}{mc} \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc} u_A = \frac{\frac{E^2}{c^2} - (\vec{\sigma}\vec{p})^2}{m^2c^2} u_A. \tag{1.139}$$

Thus one must have

$$\frac{E^2}{c^2} - (\vec{\sigma}\vec{p})^2 = m^2c^2 \Leftrightarrow E^2 = \vec{p}^2c^2 + m^2c^4. \tag{1.140}$$

Therefore we have a single condition

$$u_B = \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc} u_A. \tag{1.141}$$

There are four possible solutions. They are

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad u^{(1)} = N^{(1)} \begin{pmatrix} 1 \\ 0 \\ \dfrac{\frac{E}{c} + p^3}{mc} \\ \dfrac{p^1 + ip^2}{mc} \end{pmatrix} \tag{1.142}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad u^{(4)} = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ \dfrac{p^1 - ip^2}{mc} \\ \dfrac{\frac{E}{c} - p^3}{mc} \end{pmatrix} \tag{1.143}$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(3)} = N^{(3)} \begin{pmatrix} \dfrac{\frac{E}{c} - p^3}{mc} \\ -\dfrac{p^1 + ip^2}{mc} \\ 1 \\ 0 \end{pmatrix} \tag{1.144}$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(2)} = N^{(2)} \begin{pmatrix} -\dfrac{p^1 - ip^2}{mc} \\ \dfrac{\frac{E}{c} + p^3}{mc} \\ 0 \\ 1 \end{pmatrix}. \tag{1.145}$$

The first and the fourth solutions will be normalized such that

$$\bar{u}u = u^+\gamma^0 u = u_A^+ u_B + u_B^+ u_A = 2mc. \tag{1.146}$$

We obtain

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}. \tag{1.147}$$

Clearly one must have $E \geqslant 0$ otherwise the square root will not be well defined. In other words $u^{(1)}$ and $u^{(2)}$ correspond to positive-energy solutions associated with particles. The spinors $u^{(i)}(p)$ can be rewritten as

$$u^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi^i \\ \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^i \end{pmatrix}. \tag{1.148}$$

The two-dimensional spinors ξ^i satisfy

$$(\xi^r)^+ \xi^s = \delta^{rs}. \tag{1.149}$$

The remaining spinors $u^{(3)}$ and $u^{(4)}$ must correspond to negative-energy solutions which must be reinterpreted as positive-energy antiparticles. Thus we flip the signs of the energy and the momentum such that the wave function (1.133) becomes

$$\psi(t, \vec{x}) = a\ e^{\frac{i}{\hbar}(Et - \vec{p}\,\vec{x})} u(-E, -\vec{p}). \tag{1.150}$$

The solutions u^3 and u^4 become

$$v^{(1)}(E, \vec{p}) = u^{(3)}(-E, -\vec{p}) = N^{(3)} \begin{pmatrix} -\dfrac{\dfrac{E}{c} - p^3}{mc} \\ \dfrac{p^1 + ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}$$

$$v^{(2)}(E, \vec{p}) = u^{(4)}(-E, -\vec{p}) = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ -\dfrac{p^1 - ip^2}{mc} \\ -\dfrac{\dfrac{E}{c} - p^3}{mc} \end{pmatrix}. \tag{1.151}$$

We impose the normalization condition

$$\bar{v}v = v^{+}\gamma^0 v = v_A^{+} v_B + v_B^{+} v_A = -2mc. \tag{1.152}$$

We obtain

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\dfrac{E}{c} - p^3}}. \tag{1.153}$$

The spinors $v^{(i)}(p)$ can be rewritten as

$$v^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \eta^i \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\, \eta^i \end{pmatrix}. \tag{1.154}$$

Again the two-dimensional spinors η^i satisfy

$$(\eta^r)^{+}\eta^s = \delta^{rs}. \tag{1.155}$$

1.6 Lorentz covariance: first look

In this section we will refer to the Klein–Gordon wave function ϕ as a scalar field and to the Dirac wave function ψ as a Dirac spinor field although we are still thinking of them as quantum wave functions and not classical fields.

1.6.1 Scalar fields

Let us recall that the set of all Lorentz transformations form a group called the Lorentz group. An arbitrary Lorentz transformation acts as

$$x^{\mu} \longrightarrow x^{'\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{1.156}$$

In the inertial reference frame O the Klein–Gordon wave function is $\phi = \phi(x)$. It is a scalar field. Thus in the transformed reference frame O' the wave function must be $\phi' = \phi'(x')$ where

$$\phi'(x') = \phi(x). \tag{1.157}$$

For a one-component field this is the only possible linear transformation law. The Klein–Gordon equation in the reference frame O' if it holds is of the form

$$\left(\partial'_{\mu}\partial^{'\mu} + \frac{m^2c^2}{\hbar^2}\right)\phi'(x') = 0. \tag{1.158}$$

It is not difficult to show that

$$\partial'_{\mu}\partial^{'\mu} = \partial_{\mu}\partial^{\mu}. \tag{1.159}$$

The Klein–Gordon equation (1.158) becomes

$$\left(\partial_{\mu}\partial^{\mu} + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0. \tag{1.160}$$

1.6.2 Vector fields

Let $V^{\mu} = V^{\mu}(x)$ be an arbitrary vector field (for example $\partial^{\mu}\phi$ and the electromagnetic vector potential A^{μ}). Under Lorentz transformations it must transform as a 4-vector, i.e. as in equation (1.156) and hence

$$V^{'\mu}(x') = \Lambda^{\mu}_{\nu} V^{\nu}(x). \tag{1.161}$$

This should be contrasted with the transformation law of an ordinary vector field $V^{i}(x)$ under rotations in three-dimensional space given by

$$V^{'i}(x') = R^{ij} V^{j}(x). \tag{1.162}$$

The group of rotations in three-dimensional space is a continuous group. The set of infinitesimal transformations (the transformations near the identity) form a vector space which we call the Lie algebra of the group. The basis vectors of this vector space are called the generators of the Lie algebra and they are given by the angular momentum operators L^{i} which satisfy the commutation relations

$$[L^{i}, L^{j}] = i\hbar\epsilon^{ijk}L^{k}. \tag{1.163}$$

A rotation with an angle $|\theta|$ about the axis $\hat{\theta}$ is obtained by exponentiation from the Lie algebra, viz

$$R = \exp(-i\theta^{i}L^{i}). \tag{1.164}$$

The angular momentum operators J^i are given by (our convention is $\varepsilon_{123} = +1$)

$$L^i = -i\hbar\varepsilon^{ijk}x^j\partial^k. \tag{1.165}$$

This is equivalent to

$$L^{ij} = \varepsilon^{ijk}L^k = -i\hbar(x^i\partial^j - x^j\partial^i). \tag{1.166}$$

Generalization of this result to four-dimensional Minkowski space yields the six generators of the Lorentz group given by

$$L^{\mu\nu} = -i\hbar(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{1.167}$$

We compute the commutation relations

$$[L^{\mu\nu}, L^{\rho\sigma}] = i\hbar(\eta^{\nu\rho}L^{\mu\sigma} - \eta^{\mu\rho}L^{\nu\sigma} - \eta^{\nu\sigma}L^{\mu\rho} + \eta^{\mu\sigma}L^{\nu\rho}). \tag{1.168}$$

A solution of equation (1.168) is given by the 4×4 matrices

$$(L^{\mu\nu})_{\alpha\beta} = i\hbar\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right). \tag{1.169}$$

Equivalently we can write this solution as

$$(L^{\mu\nu})^\alpha{}_\beta = i\hbar\left(\eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}\right). \tag{1.170}$$

This representation is the four-dimensional vector representation of the Lorentz group which is denoted by (1/2, 1/2). It is an irreducible representation of the Lorentz group. A scalar field transforms in the trivial representation of the Lorentz group denoted by (0, 0). It remains to determine the transformation properties of spinor fields.

1.6.3 Spinor fields

We go back to the Dirac equation in the form

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{1.171}$$

This equation is assumed to be covariant under Lorentz transformations and hence one must have the transformed equation

$$(i\hbar\gamma'^\mu\partial'_\mu - mc)\psi' = 0. \tag{1.172}$$

The Dirac γ matrices are assumed to be invariant under Lorentz transformations and thus

$$\gamma'_\mu = \gamma_\mu. \tag{1.173}$$

The spinor ψ will be assumed to transform under Lorentz transformations linearly, namely

$$\psi(x) \longrightarrow \psi'(x') = S(\Lambda)\psi(x). \tag{1.174}$$

Furthermore we have

$$\partial'_\nu = (\Lambda^{-1})^\mu{}_\nu \partial_\mu. \tag{1.175}$$

Thus, equation (1.172) is of the form

$$(i\hbar(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma'^\mu S(\Lambda)\partial_\nu - mc)\psi = 0. \tag{1.176}$$

We can get immediately

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma'^\mu S(\Lambda) = \gamma^\nu. \tag{1.177}$$

Equivalently

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu. \tag{1.178}$$

This is the transformation law of the γ matrices under Lorentz transformations. Thus the γ matrices are invariant under the simultaneous rotations of the vector and spinor indices under Lorentz transformations. This is analogous to the fact that Pauli matrices σ^i are invariant under the simultaneous rotations of the vector and spinor indices under spatial rotations.

The matrix $S(\Lambda)$ form a four-dimensional representation of the Lorentz group which is called the spinor representation. This representation is reducible and it is denoted by $(1/2, 0) \oplus (0, 1/2)$. It remains to find the matrix $S(\Lambda)$. We consider an infinitesimal Lorentz transformation

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}L^{\alpha\beta}, \quad \Lambda^{-1} = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}L^{\alpha\beta}. \tag{1.179}$$

We can write $S(\Lambda)$ as

$$S(\Lambda) = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}, \quad S^{-1}(\Lambda) = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}. \tag{1.180}$$

The infinitesimal form of equation (1.178) is

$$-(L^{\alpha\beta})^\mu{}_\nu \gamma_\mu = [\gamma_\nu, \Gamma^{\alpha\beta}]. \tag{1.181}$$

The fact that the index μ is rotated with $L^{\alpha\beta}$ means that it is a vector index. The spinor indices are the matrix components of the γ matrices which are rotated with the generators $\Gamma^{\alpha\beta}$. A solution is given by

$$\Gamma^{\mu\nu} = \frac{i\hbar}{4}[\gamma^\mu, \gamma^\nu]. \tag{1.182}$$

Explicitly

$$\begin{aligned}
\Gamma^{0i} &= \frac{i\hbar}{4}[\gamma^0, \gamma^i] = -\frac{i\hbar}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\
\Gamma^{ij} &= \frac{i\hbar}{4}[\gamma^i, \gamma^j] = -\frac{i\hbar}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix} = \frac{\hbar}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.
\end{aligned} \tag{1.183}$$

Clearly Γ^{ij} are the generators of rotations. They are the direct sum of two copies of the generators of rotation in three-dimensional space. Thus, we conclude that Γ^{0i} are the generators of boosts.

1.7 Representations of the Lorentz group

1.7.1 The Lorentz group $SO(1, 3)$ and its Lie algebra $so(1, 3)$

We start by recalling that the spacetime points x, the spacetime metric $\eta_{\mu\nu}$ and the spacetime interval ds^2 are given respectively by

$$x \equiv x^\mu = (ct, \vec{x}), \quad x_\mu = \eta_{\mu\nu}x^\nu \tag{1.184}$$

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1) \tag{1.185}$$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - d\vec{x}^2. \tag{1.186}$$

First we note that Lorentz transformations act on x in Minkowski spacetime $\mathbf{M}^4$ in the same way that rotations act on $\vec{x}$ in Euclidean space $\mathbf{R}^3$. Indeed, the interval ds^2 is invariant under the linear Lorentz transformations

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.187}$$

if and only if the transformations Λ satisfy

$$\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \Leftrightarrow \Lambda^T\eta\Lambda = \eta. \tag{1.188}$$

This is the analog of the orthogonality condition $R^TR = 1$ found in the case of the rotation group $SO(3)$ in Euclidean space $\mathbf{R}^3$. Similarly, equation (1.188) defines the Lorentz group, which is denoted by $SO(1, 3)$, in Minkowski spacetime $\mathbf{M}^4$. The condition (1.188) leads immediately to the determinant

$$\det\Lambda = \pm 1. \tag{1.189}$$

Again, this is the analog of $\det R = \pm 1$ in Euclidean space $\mathbf{R}^3$.

The Lorentz group contains (1) rotations, (2) boosts (these are the purely Lorentz transformations), (3) space reflection P and (4) time reflection T.

Furthermore, we note that by setting $\rho = \sigma = 0$ in equation (1.188) we obtain

$$(\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geqslant 1 \Rightarrow |\Lambda^0{}_0| \geqslant 1. \tag{1.190}$$

We can then characterize the various Lorentz transformations as follows:

- The proper orthochronous transformations $L_+^\uparrow$: $\det\Lambda = 1$, $\Lambda^0{}_0 > 0$.
- The improper orthochronous transformations $L_-^\uparrow$: $\det\Lambda = -1$, $\Lambda^0{}_0 > 0$. This involves space reflection P.
- The proper non-orthochronous transformations $L_+^\downarrow$: $\det\Lambda = 1$, $\Lambda^0{}_0 < 0$. This involves time reflection T.

- The improper non-orthochronous transformations $L_-^\downarrow$: $\det\Lambda = -1$, $\Lambda^0{}_0 < 0$. This involves time and space reflections T and P.

The set $L_+^\uparrow$ of all proper orthochronous transformations is the proper Lorentz group which is the basic object. Everything else can be derived from $L_+^\uparrow$ by the action of P ($L_-^\uparrow$), T ($L_+^\downarrow$) or P and T ($L_-^\downarrow$).

The proper Lorentz group contains three basic rotations in the planes 12, 13 and 23 and three basic boosts (rotations with an imaginary angle) along the axes 1, 2 and 3.

The generators of the infinitesimal rotations (generators of the Lie algebra $so(3)$ of the rotation group $SO(3)$ in three dimensions) acting in $\mathbf{R}^3$ were found to be given by the orbital angular momentum $L_i = -iA_i$ given by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\quad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.191}$$

When these generators act in spacetime $\mathbf{M}^4$ they are naturally embedded in the 4×4 matrices (using the same symbols)

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \equiv A^{23},\quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \equiv A^{31}$$
$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv A^{12}. \tag{1.192}$$

The generators L_i were determined to be the orbital angular momentum with standard commutation relations. Equivalently, the generators A^{ij} satisfy the commutation relations

$$[A^{ij}, A^{kl}] = \eta^{ik}A^{jl} - \eta^{il}A^{jk} - \eta^{jk}A^{il} + \eta^{jl}A^{ik}. \tag{1.193}$$

This is a four-dimensional representation of the rotation group since spacetime is four-dimensional. An infinitesimal rotation is then given by (with $\omega_{12} = \theta_3$, $\omega_{31} = \theta_2$ and $\omega_{23} = \theta_1$)

$$\Lambda(\delta\omega) = 1 + \frac{1}{2}\delta\omega_{ij}A^{ij}. \tag{1.194}$$

The finite rotation is obtained by exponentiation (the group is obtained from the Lie algebra by exponentiation), viz

$$\Lambda(\omega) = \exp\left(\frac{1}{2}\omega_{ij}A^{ij}\right). \tag{1.195}$$

This is equivalent to viewing the finite rotation as a succession of infinite number of identical infinitesimal rotations.

Similarly, we have found that the boost along the axis x_1 is given explicitly by

$$\begin{aligned} x^{0'} &= \gamma(x^0 - \beta x^1) \\ x^{1'} &= \gamma(x^1 - \beta x^0) \\ x^{2'} &= x^2 \\ x^{3'} &= x^3. \end{aligned} \tag{1.196}$$

The corresponding Lorentz transformation is then given by

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.197}$$

where $\cosh u = \gamma$. Hence, this boost can be understood as a (non-compact) rotation in the plane 01 with an imaginary angle iu. The Lie algebra is the tangent vector space to the group manifold at the identity. Thus, we need to consider an infinitesimal boost by taking a small velocity v (compared to the speed light c) which corresponds to a small angle u. We get then the generator

$$A^{10} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.198}$$

By the same token the generators corresponding to the boosts along the x_2 and x_3 are found to be given by

$$A^{20} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.199}$$

The boost generators A^{i0} are also four-dimensional. In fact A^{i0} and A^{ij} (written collectively as $A^{\mu\nu}$) provide the four-dimensional representation of the Lie algebra $so(1, 3)$ of the Lorentz group $SO(1, 3)$. The defining algebra is given by a straightforward generalization of equation (1.193) which reads

$$[A^{\mu\nu}, A^{\rho\sigma}] = \eta^{\mu\rho}A^{\nu\sigma} - \eta^{\mu\sigma}A^{\nu\rho} - \eta^{\nu\rho}A^{\mu\sigma} + \eta^{\nu\sigma}A^{\mu\rho}. \tag{1.200}$$

The most general infinitesimal and finite Lorentz transformations in this representation will then be given by

$$\Lambda(\delta\omega) = 1 + \frac{1}{2}\delta\omega_{\mu\nu}A^{\mu\nu}, \quad \Lambda(\omega) = \exp\left(\frac{1}{2}\omega_{\mu\nu}A^{\mu\nu}\right). \tag{1.201}$$

The most general representation of the Lorentz Lie algebra $so(1, 3)$ will be given by some N-dimensional generators $B^{\mu\nu}$ satisfying exactly the algebra

$$[B^{\mu\nu}, B^{\rho\sigma}] = \eta^{\mu\rho}B^{\nu\sigma} - \eta^{\mu\sigma}B^{\nu\rho} - \eta^{\nu\rho}B^{\mu\sigma} + \eta^{\nu\sigma}B^{\mu\rho}. \quad (1.202)$$

The most general infinitesimal and finite Lorentz transformations in this representation will be given by

$$U(\Lambda) = 1 + \frac{1}{2}\delta\omega_{\mu\nu}B^{\mu\nu}, \quad U(\Lambda) = \exp\left(\frac{1}{2}\omega_{\mu\nu}B^{\mu\nu}\right). \quad (1.203)$$

1.7.2 Representations of the Lorentz group

What is the most general solution $B^{\mu\nu}$ of equation (1.202)?

As we have seen, from Shur's lemma, the problem of finding the most general solution of equation (1.202) is equivalent to the problem of finding the most general irreducible representation of the Lorentz group $SO(1, 3)$ and this requires us to find the Casimir operators of the group.

This is easy in this case. We introduce the new generators

$$X_i = -\frac{1}{2}(iM_i + N_i), \quad Y_i = -\frac{1}{2}(iM_i - N_i). \quad (1.204)$$

The M's and N's are defined by

$$M_i = \frac{1}{2}\varepsilon_{ijk}B_{jk}, \quad N_i = B_{0i}. \quad (1.205)$$

They satisfy

$$[M_i, M_j] = -\varepsilon_{ijk}M_k, \quad [N_i, N_j] = \varepsilon_{ijk}M_k, \quad [M_i, N_j] = -\varepsilon_{ijk}N_k. \quad (1.206)$$

We can verify immediately that the commutation relations (1.202) are equivalent to

$$[X_i, X_j] = i\varepsilon_{ijk}X_k, \quad [Y_i, Y_j] = i\varepsilon_{ijk}Y_k, \quad [X_i, Y_j] = 0. \quad (1.207)$$

Thus, the X's and Y's generate two commuting copies of the $so(3)$ Lie algebra. Hence, the Lie algebra $so(1, 3)$ of the Lorentz group is the direct sum of two copies of the Lie algebras $so(3)$ of the rotation group. We can then write the Casimir operators of the Lie algebra $so(1, 3)$ of the Lorentz group. They are

$$\vec{X}^2 = X_1^2 + X_2^2 + X_3^2, \quad \vec{Y}^2 = Y_1^2 + Y_2^2 + Y_3^2. \quad (1.208)$$

The irreducible representations of the Lie algebra $so(1, 3)$ are characterized by two integers j and k which are the spin quantum numbers of the two angular momentum operators $\vec{X}$ and $\vec{Y}$. These representations are $(2j + 1)(2k + 1)$-dimensional given explicitly by

$$\begin{aligned} \vec{X}^2|jm\rangle|kn\rangle &= j(j+1)|jm\rangle|kn\rangle \\ X_3|jm\rangle|kn\rangle &= m|jm\rangle|kn\rangle \\ \vec{Y}^2|jm\rangle|kn\rangle &= k(k+1)|jm\rangle|kn\rangle \\ Y_3|jm\rangle|kn\rangle &= n|jm\rangle|kn\rangle. \end{aligned} \tag{1.209}$$

As in the case of the rotation group we have here tensor representations (for integer values of $j + k$) and spinor representations for half-integer values of $j + k$. Under space or time reflections the representations (j, k) and (k, j) get interchanged. Also, the tensor product of two representations (j_1, k_1) and (j_2, k_2) are given by the quantum mechanical rule

$$\begin{aligned} (j_1, k_1) \otimes (j_2, k_2) = \sum_{\oplus}(j, k), \quad |j_1 - j_2| \leqslant j \leqslant j_1 + j_2, \\ \times |k_1 - k_2| \leqslant k \leqslant k_1 + k_2. \end{aligned} \tag{1.210}$$

Some examples of the irreducible representations (j, k) were given in the previous section. The scalar field corresponds to $(0, 0)$ and $J_{\mu\nu} = 0$. The vector field corresponds to $(1/2, 1/2)$ and $J_{\mu\nu} = \mathcal{J}_{\mu\nu}$. The Dirac spinor field corresponds to the reducible representation $(1/2, 0) \oplus (0, 1/2)$ and $J_{\mu\nu} = \Gamma_{\mu\nu}$. The Weyl spinor fields (left-handed or right-handed Dirac fields) correspond to the irreducible representations $(1/2, 0)$ and $(0, 1/2)$. As a final example we consider the reducible representation given by the direct sum $(1, 0) \oplus (0, 1)$ which corresponds to an antisymmetric tensor field such as the electromagnetic field strength $F_{\mu\nu}$ (the irreducible components correspond to the self-dual and anti-self-dual fields).

1.8 Exercises

Exercise 1:

Show explicitly that the scalar product of two 4-vectors in spacetime is invariant under boosts. Show that the scalar product is then invariant under all Lorentz transformations.

Exercise 2:

- By using Lorentz transformations show that moving clocks cannot be synchronized and derive an explicit formula for the relativity of simultaneity.
- Show that the proper time of a point particle—the proper time is the time measured by an inertial observer flying with the particle—is invariant under Lorentz transformations. We assume that the particle is moving with a velocity $\vec{u}$ with respect to an inertial observer O.
- Define the 4-vector velocity of the particle in spacetime. What is the spatial component.
- Define the energy–momentum 4-vector in spacetime and deduce the relativistic energy.
- Express the energy in terms of the momentum.
- Define the 4-vector force.

Exercise 3:

Derive the velocity addition rule in special relativity.

Exercise 4:

- Show that the Weyl representation of Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1.211}$$

  solves Dirac–Clifford algebra.
- Show that

$$(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0. \tag{1.212}$$

- Show that the Dirac equation can be put in the form of a Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi, \tag{1.213}$$

  with some Hamiltonian H.

Exercise 5:

From the invariance of the interval ds^2 under Poincaré transformations show that the condition which must be satisfied by Lorentz transformations is given by

$$\eta = \Lambda^T\eta\Lambda. \tag{1.214}$$

Show also that

$$\Lambda_\rho{}^\mu = (\Lambda^{-1})^\mu{}_\rho \tag{1.215}$$

$$\partial'_\nu = (\Lambda^{-1})^\mu{}_\nu\partial_\mu \tag{1.216}$$

$$\partial'_\mu\partial'^\mu = \partial_\mu\partial^\mu. \tag{1.217}$$

Exercise 6:

Show that the Klein–Gordon equation is covariant under Lorentz transformations.

Exercise 7:

- Write down the transformation property under ordinary rotations of a vector in three dimensions. What are the generators J^i? What are the dimensions of the irreducible representations and the corresponding quantum numbers?

- The generators of rotation can be alternatively given by

$$J^{ij} = \varepsilon^{ijk} J^k. \tag{1.218}$$

Calculate the commutators $[J^{ij}, J^{kl}]$.

- Write down the generators of the Lorentz group $J^{\mu\nu}$ by simply generalizing J^{ij} and show that

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}). \tag{1.219}$$

- Verify that

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar\left(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu\right), \tag{1.220}$$

is a solution. This is called the vector representation of the Lorentz group.

- Write down a finite Lorentz transformation matrix in the vector representation. Write down an infinitesimal rotation in the xy-plane and an infinitesimal boost along the x-axis.

Exercise 8:

- Introduce $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. Show that

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\mu p^\mu) = m^2c^2. \tag{1.221}$$

- Show that the normalization condition $\bar{u}u = 2mc$ for $u^{(1)}$ and $u^{(2)}$ yields

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}. \tag{1.222}$$

- Show that the normalization condition $\bar{v}v = -2mc$ for $v^{(1)}(p) = u^{(3)}(-p)$ and $v^{(2)}(p) = u^{(4)}(-p)$ yields

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} - p^3}}. \tag{1.223}$$

- Show that we can rewrite the spinors u and v as

$$u^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi^i \\ \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^i \end{pmatrix} \tag{1.224}$$

$$v^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\,\eta^i \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\,\eta^i \end{pmatrix}. \tag{1.225}$$

Determine ξ^i and η^i.

Exercise 9:

Let $u^{(r)}(p)$ and $v^{(r)}(p)$ be the positive-energy and negative-energy solutions of the free Dirac equation. Show that

$$\bar{u}^{(r)}u^{(s)} = 2mc\delta^{rs}, \quad \bar{v}^{(r)}v^{(s)} = -2mc\delta^{rs}, \quad \bar{u}^{(r)}v^{(s)} = 0, \quad \bar{v}^{(r)}u^{(s)} = 0 \tag{1.226}$$

$$u^{(r)+}u^{(s)} = \frac{2E}{c}\delta^{rs}, \quad v^{(r)+}v^{(s)} = \frac{2E}{c}\delta^{rs} \tag{1.227}$$

$$u^{(r)+}(E, \vec{p}\,)v^{(s)}(E, -\vec{p}\,) = 0, \quad v^{(r)+}(E, -\vec{p}\,)u^{(s)}(E, \vec{p}\,) = 0 \tag{1.228}$$

$$\sum_{s=1}^{2} u^{(s)}(E, \vec{p}\,)\bar{u}^{(s)}(E, \vec{p}\,) = \gamma^\mu p_\mu + mc, \quad \sum_{s=1}^{2} v^{(s)}(E, \vec{p}\,)\bar{v}^{(s)}(E, \vec{p}\,) = \gamma^\mu p_\mu - mc. \tag{1.229}$$

Exercise 10:

Determine the transformation property of the spinor ψ under Lorentz transformations in order that the Dirac equation is covariant.

Exercise 11:

Determine the transformation rule under Lorentz transformations of $\bar{\psi}$, $\bar{\psi}\psi$, $\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$ and $\bar{\psi}\Gamma^{\mu\nu}\psi$.

Exercise 12:

- Write down the solution of the Clifford algebra in three Euclidean dimensions. Construct a basis for 2×2 matrices in terms of Pauli matrices.
- Construct a basis for 4×4 matrices in terms of Dirac matrices. Hint: Show that there are 16 antisymmetric combinations of the Dirac gamma matrices in $1 + 3$ dimensions.

Exercise 13:

- We define the gamma five matrix (chirality operator) by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{1.230}$$

Show that

$$\gamma^5 = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{1.231}$$

$$(\gamma^5)^2 = 1 \tag{1.232}$$

$$(\gamma^5)^+ = \gamma^5 \tag{1.233}$$

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{1.234}$$

$$[\gamma^5, \Gamma^{\mu\nu}] = 0. \tag{1.235}$$

- We write the Dirac spinor as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{1.236}$$

By working in the Weyl representation show that the Dirac representation is reducible.

Hint: Compute the eigenvalues of γ^5 and show that they do not mix under Lorentz transformations.

- Rewrite the Dirac equation in terms of ψ_L and ψ_R. What is their physical interpretation?

1.9 Solutions

Exercise 14:

(1) Let us look at the clock found at the origin of the reference frame O'. We set then $x' = 0$ in Lorentz transformations. We get the time dilation effect, viz

$$t' = \frac{t}{\gamma}. \tag{1.237}$$

At time $t = 0$ the clocks in O' read different times depending on their location since

$$t' = -\gamma\frac{v}{c^2}x. \tag{1.238}$$

Hence moving clocks cannot be synchronized.

We consider now two events A and B with coordinates (x_A, t_A) and (x_B, t_B) in O and coordinates (x'_A, t'_A) and (x'_B, t'_B) in O'. We can then compute

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right). \tag{1.239}$$

Thus, if the two events are simultaneous with respect to O, i.e. $\Delta t = 0$ they are not simultaneous with respect to O' since

$$\Delta t' = -\gamma\frac{v}{c^2}\Delta x. \tag{1.240}$$

(2) The trajectory of a particle in spacetime is called a world line. We take two infinitesimally close points on the world line given by (x^0, x^1, x^2, x^3) and $(x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$. Clearly $dx^1 = u^1dt$, $dx^2 = u^2dt$ and $dx^3 = u^3dt$ where $\vec{u}$ is the velocity of the particle measured with respect to the observer O, viz

$$\vec{u} = \frac{d\vec{x}}{dt}. \tag{1.241}$$

The interval with respect to O is given by

$$ds^2 = -c^2dt^2 + d\vec{x}^2 = (-c^2 + u^2)dt^2. \tag{1.242}$$

Let O' be the observer or inertial reference frame moving with respect to O with the velocity $\vec{u}$. We stress here that $\vec{u}$ is thought of as a constant velocity only during the infinitesimal time interval dt. The interval with respect to O' is given by

$$ds^2 = -c^2d\tau^2. \tag{1.243}$$

Hence

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}}\,dt. \tag{1.244}$$

The time interval $d\tau$ measured with respect to O', which is the observer moving with the particle, is the proper time of the particle.

(3) The 4-vector velocity η is naturally defined by the components

$$\eta^\mu = \frac{dx^\mu}{d\tau}. \tag{1.245}$$

The spatial part of η is precisely the proper velocity $\vec{\eta}$ defined by

$$\vec{\eta} = \frac{d\vec{x}}{d\tau} = \frac{1}{\sqrt{1 - u^2/c^2}}\vec{u}. \tag{1.246}$$

The temporal part is

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{c}{\sqrt{1 - u^2/c^2}}. \tag{1.247}$$

(4) The law of conservation of momentum and the principle of relativity put together forces us to define the momentum in relativity as mass times the proper velocity and not the mass time of the ordinary velocity, viz

$$\vec{p} = m\vec{\eta} = m\frac{d\vec{x}}{d\tau} = \frac{m}{\sqrt{1-u^2/c^2}}\vec{u}. \tag{1.248}$$

This is the spatial part of the 4-vector momentum

$$p^\mu = m\eta^\mu = m\frac{dx^\mu}{d\tau}. \tag{1.249}$$

The temporal part is

$$p^0 = m\eta^0 = m\frac{dx^0}{d\tau} = \frac{mc}{\sqrt{1-u^2/c^2}} = \frac{E}{c}. \tag{1.250}$$

The relativistic energy is defined by

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}}. \tag{1.251}$$

The 4-vector p^μ is called the energy–momentum 4-vector.

(5) We note the identity

$$p_\mu p^\mu = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2c^2. \tag{1.252}$$

Thus

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4}. \tag{1.253}$$

The rest mass is m and the rest energy is clearly defined by

$$E_0 = mc^2. \tag{1.254}$$

(6) Newton's first law is automatically satisfied because of the principle of relativity. The second law takes in the theory of special relativity the usual form provided we use the relativistic momentum, viz

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{1.255}$$

Newton's third law does not in general hold in the theory of special relativity. We can define a 4-vector proper force which is called the Minkowski force by the following equation

$$K^\mu = \frac{dp^\mu}{d\tau}. \tag{1.256}$$

The spatial part is

$$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{1}{\sqrt{1 - u^2/c^2}}\vec{F}. \tag{1.257}$$

Exercise 15:

We consider a particle in the reference frame O moving a distance dx in the x-direction during a time interval dt. The velocity with respect to O is

$$u = \frac{dx}{dt}. \tag{1.258}$$

In the reference frame O' the particle moves a distance dx' in a time interval dt' given by

$$dx' = \gamma(dx - vdt) \tag{1.259}$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right). \tag{1.260}$$

The velocity with respect to O' is therefore

$$u' = \frac{dx'}{dt'} = \frac{u - v}{1 - vu/c^2}. \tag{1.261}$$

In general if $\vec{V}$ and $\vec{V}'$ are the velocities of the particle with respect to O and O' respectively and $\vec{v}$ is the velocity of O' with respect to O. Then

$$\vec{V}' = \frac{\vec{V} - \vec{v}}{1 - \vec{V}\vec{v}/c^2}. \tag{1.262}$$

Exercise 16:

The Dirac equation can trivially be put in the form

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{\hbar c}{i}\gamma^0\gamma^i\partial_i + mc^2\gamma^0\right)\psi. \tag{1.263}$$

The Dirac Hamiltonian is

$$H = \frac{\hbar c}{i}\vec{\alpha}\vec{\nabla} + mc^2\beta, \quad \alpha^i = \gamma^0\gamma^i, \quad \beta = \gamma^0. \tag{1.264}$$

This is a Hermitian operator as it should be.

Exercise 17:

A Poincaré transformation combines a translation a with a Lorentz transformation Λ:

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}. \tag{1.265}$$

The invariance of the interval ds^2 under Poincaré transformations means that

$$ds^2 = \eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{1.266}$$

This leads to the condition

$$\eta_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = \eta_{\rho\sigma} \Leftrightarrow \Lambda^T \eta \Lambda = \eta. \tag{1.267}$$

Explicitly we write this as

$$\begin{aligned} \eta^{\mu}_{\nu} &= \Lambda_{\rho}{}^{\mu} \eta^{\rho}_{\beta} \Lambda^{\beta}{}_{\nu} \\ &= \Lambda_{\rho}{}^{\mu} \Lambda^{\rho}{}_{\nu}. \end{aligned} \tag{1.268}$$

But we also have

$$\delta^{\mu}_{\nu} = (\Lambda^{-1})^{\mu}{}_{\rho} \Lambda^{\rho}{}_{\nu}. \tag{1.269}$$

In other words, we have

$$\Lambda_{\rho}{}^{\mu} = (\Lambda^{-1})^{\mu}{}_{\rho}. \tag{1.270}$$

Since $x^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} x'^{\nu}$ we have

$$\frac{\partial x^{\mu}}{\partial x'^{\nu}} = (\Lambda^{-1})^{\mu}{}_{\nu}. \tag{1.271}$$

Hence

$$\partial'_{\nu} = (\Lambda^{-1})^{\mu}{}_{\nu} \partial_{\mu}. \tag{1.272}$$

Thus

$$\begin{aligned} \partial'_{\mu} \partial'^{\mu} &= \eta^{\mu\nu} \partial'_{\mu} \partial'_{\nu} \\ &= \eta^{\mu\nu} (\Lambda^{-1})^{\rho}{}_{\mu} (\Lambda^{-1})^{\lambda}{}_{\nu} \partial_{\rho} \partial_{\lambda} \\ &= \eta^{\mu\nu} \Lambda_{\mu}{}^{\rho} \Lambda_{\nu}{}^{\lambda} \partial_{\rho} \partial_{\lambda} \\ &= (\Lambda^T \eta \Lambda)^{\rho\lambda} \partial_{\rho} \partial_{\lambda} \\ &= \partial_{\mu} \partial^{\mu}. \end{aligned} \tag{1.273}$$

Exercise 18:

(1) We have

$$V'^{i}(x') = R^{ij} V^{j}(x). \tag{1.274}$$

The generators are given by the angular momentum operators J^i which satisfy the commutation relations

$$[J^i, J^j] = i\hbar\varepsilon^{ijk}J^k. \tag{1.275}$$

Thus, a rotation with an angle $|\theta|$ about the axis $\hat{\theta}$ is obtained by exponentiation, viz

$$R = e^{-i\theta^i J^i}. \tag{1.276}$$

The matrices R form an n-dimensional representation with $n = 2j + 1$ where j is the spin quantum number. The quantum numbers are therefore given by j and m.

(2) The angular momentum operators J^i are given by

$$J^i = -i\hbar\varepsilon^{ijk}x^j\partial^k. \tag{1.277}$$

Thus

$$\begin{aligned} J^{ij} &= \varepsilon^{ijk}J^k \\ &= -\, i\hbar(x^i\partial^j - x^j\partial^i). \end{aligned} \tag{1.278}$$

We compute

$$[J^{ij}, J^{kl}] = i\hbar(\eta^{jk}J^{il} - \eta^{ik}J^{jl} - \eta^{jl}J^{ik} + \eta^{il}J^{jk}). \tag{1.279}$$

(3) Generalization to four-dimensional Minkowski space yields

$$J^{\mu\nu} = -i\hbar(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{1.280}$$

Now we compute the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}). \tag{1.281}$$

(4) A solution of is given by the 4×4 matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right). \tag{1.282}$$

Equivalently

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = i\hbar\left(\eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}\right). \tag{1.283}$$

We compute

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta(\mathcal{J}^{\rho\sigma})^\beta{}_\lambda = (i\hbar)^2\left(\eta^{\mu\alpha}\eta^{\rho\nu}\delta^\sigma_\lambda - \eta^{\mu\alpha}\eta^{\sigma\nu}\delta^\rho_\lambda - \eta^{\nu\alpha}\eta^{\rho\mu}\delta^\sigma_\lambda + \eta^{\nu\alpha}\eta^{\sigma\mu}\delta^\rho_\lambda\right) \tag{1.284}$$

$$(\mathcal{J}^{\rho\sigma})^\alpha{}_\beta(\mathcal{J}^{\mu\nu})^\beta{}_\lambda = (i\hbar)^2\left(\eta^{\rho\alpha}\eta^{\mu\sigma}\delta^\nu_\lambda - \eta^{\rho\alpha}\eta^{\sigma\nu}\delta^\mu_\lambda - \eta^{\sigma\alpha}\eta^{\rho\mu}\delta^\nu_\lambda + \eta^{\sigma\alpha}\eta^{\nu\rho}\delta^\mu_\lambda\right). \tag{1.285}$$

Hence

$$\begin{aligned}[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^{\alpha}{}_{\lambda} &= (i\hbar)^2\Big(\eta^{\mu\sigma}\big[\eta^{\nu\alpha}\delta^{\rho}_{\lambda} - \eta^{\rho\alpha}\delta^{\nu}_{\lambda}\big] - \eta^{\nu\sigma}\big[\eta^{\mu\alpha}\delta^{\rho}_{\lambda} - \eta^{\rho\alpha}\delta^{\mu}_{\lambda}\big] \\ &\quad - \eta^{\mu\rho}\big[\eta^{\nu\alpha}\delta^{\sigma}_{\lambda} - \eta^{\sigma\alpha}\delta^{\nu}_{\lambda}\big] + \eta^{\nu\rho}\big[\eta^{\mu\alpha}\delta^{\sigma}_{\lambda} - \eta^{\sigma\alpha}\delta^{\mu}_{\lambda}\big]\Big) \\ &= i\hbar[\eta^{\mu\sigma}(\mathcal{J}^{\nu\rho})^{\alpha}{}_{\lambda} - \eta^{\nu\sigma}(\mathcal{J}^{\mu\rho})^{\alpha}{}_{\lambda} - \eta^{\mu\rho}(\mathcal{J}^{\nu\sigma})^{\alpha}{}_{\lambda} + \eta^{\nu\rho}(\mathcal{J}^{\mu\sigma})^{\alpha}{}_{\lambda}].\end{aligned} \tag{1.286}$$

(5) A finite Lorentz transformation in the vector representation is

$$\Lambda = e^{-\frac{i}{2\hbar}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}. \tag{1.287}$$

$\omega_{\mu\nu}$ is an antisymmetric tensor. An infinitesimal transformation is given by

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}. \tag{1.288}$$

A rotation in the xy-plane corresponds to $\omega_{12} = -\omega_{21} = -\theta$ while the rest of the components are zero, viz

$$\Lambda^{\alpha}{}_{\beta} = \left(1 + \frac{i}{\hbar}\theta\mathcal{J}^{12}\right)^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.289}$$

A boost in the x-direction corresponds to $\omega_{01} = -\omega_{10} = -\beta$ while the rest of the components are zero, viz

$$\Lambda^{\alpha}{}_{\beta} = \left(1 + \frac{i}{\hbar}\beta\mathcal{J}^{01}\right)^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.290}$$

Exercise 19:

(1) We compute

$$\sigma_{\mu}p^{\mu} = \frac{E}{c} - \vec{\sigma}\vec{p} = \begin{pmatrix} \frac{E}{c} - p^3 & -(p^1 - ip^2) \\ -(p^1 + ip^2) & \frac{E}{c} + p^3 \end{pmatrix} \tag{1.291}$$

$$\bar{\sigma}_{\mu}p^{\mu} = \frac{E}{c} + \vec{\sigma}\vec{p} = \begin{pmatrix} \frac{E}{c} + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & \frac{E}{c} - p^3 \end{pmatrix}. \tag{1.292}$$

Thus

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\mu p^\mu) = m^2c^2. \tag{1.293}$$

(2) Recall the four possible solutions:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad u^{(1)} = N^{(1)} \begin{pmatrix} 1 \\ 0 \\ \dfrac{\frac{E}{c} + p^3}{mc} \\ \dfrac{p^1 + ip^2}{mc} \end{pmatrix} \tag{1.294}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad u^{(4)} = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ \dfrac{p^1 - ip^2}{mc} \\ \dfrac{\frac{E}{c} - p^3}{mc} \end{pmatrix} \tag{1.295}$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad u^{(3)} = N^{(3)} \begin{pmatrix} \dfrac{\frac{E}{c} - p^3}{mc} \\ -\dfrac{p^1 + ip^2}{mc} \\ 1 \\ 0 \end{pmatrix} \tag{1.296}$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad u^{(2)} = N^{(2)} \begin{pmatrix} -\dfrac{p^1 - ip^2}{mc} \\ \dfrac{\frac{E}{c} + p^3}{mc} \\ 0 \\ 1 \end{pmatrix}. \tag{1.297}$$

The normalization condition is

$$\bar{u}u = u^+\gamma^0 u = u_A^+ u_B + u_B^+ u_A = 2mc. \tag{1.298}$$

We obtain

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}. \tag{1.299}$$

(3) Recall that

$$v^{(1)}(E, \vec{p}\,) = u^{(3)}(-E, -\vec{p}\,) = N^{(3)}\begin{pmatrix} -\frac{\frac{E}{c} - p^3}{mc} \\ \frac{p^1 + ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}, \tag{1.300}$$

$$v^{(2)}(E, \vec{p}\,) = u^{(4)}(-E, -\vec{p}\,) = N^{(4)}\begin{pmatrix} 0 \\ 1 \\ -\frac{p^1 - ip^2}{mc} \\ -\frac{\frac{E}{c} - p^3}{mc} \end{pmatrix}. \tag{1.301}$$

The normalization condition in this case is

$$\bar{v}v = v^{+}\gamma^0 v = v_A^{+}v_B + v_B^{+}v_A = -2mc. \tag{1.302}$$

We obtain now

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} - p^3}}. \tag{1.303}$$

(4) Let us define

$$\xi_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.304}$$

We have

$$u^{(1)} = N^{(1)}\begin{pmatrix} \xi_0^1 \\ \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc}\xi_0^1 \end{pmatrix} = N^{(1)}\frac{1}{\sqrt{\sigma_\mu p^\mu}}\begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\,\xi_0^1 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\,\xi_0^1 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\,\xi^1 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\,\xi^1 \end{pmatrix} \tag{1.305}$$

$$u^{(2)} = N^{(2)} \begin{pmatrix} \dfrac{\dfrac{E}{c} - \vec{\sigma}\vec{p}}{mc} \xi_0^2 \\ \xi_0^2 \end{pmatrix} = N^{(2)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi_0^2 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi_0^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi^2 \\ \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^2 \end{pmatrix}. \tag{1.306}$$

The spinors ξ^1 and ξ^2 are defined by

$$\xi^1 = N^{(1)} \frac{1}{\sqrt{\sigma_\mu p^\mu}} \xi_0^1 = \sqrt{\frac{\bar{\sigma}_\mu p^\mu}{\dfrac{E}{c} + p^3}}\, \xi_0^1 \tag{1.307}$$

$$\xi^2 = N^{(2)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \xi_0^2 = \sqrt{\frac{\sigma_\mu p^\mu}{\dfrac{E}{c} + p^3}}\, \xi_0^2. \tag{1.308}$$

They satisfy

$$(\xi^r)^+ \xi^s = \delta^{rs}. \tag{1.309}$$

Similarly, let us define

$$\eta_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.310}$$

Then we have

$$v^{(1)} = N^{(3)} \begin{pmatrix} -\dfrac{\dfrac{E}{c} - \vec{\sigma}\vec{p}}{mc} \eta_0^1 \\ \eta_0^1 \end{pmatrix} = -N^{(3)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \eta_0^1 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\, \eta_0^1 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \eta^1 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\, \eta^1 \end{pmatrix} \tag{1.311}$$

$$v^{(2)} = N^{(4)} \begin{pmatrix} \eta_0^2 \\ -\dfrac{\dfrac{E}{c} + \vec{\sigma}\vec{p}}{mc} \eta_0^2 \end{pmatrix} = N^{(4)} \frac{1}{\sqrt{\sigma_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \eta_0^2 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\, \eta_0^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \eta^2 \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\, \eta^2 \end{pmatrix} \tag{1.312}$$

$$\eta^1 = -N^{(3)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \eta_0^1 = -\sqrt{\frac{\sigma_\mu p^\mu}{\dfrac{E}{c} - p^3}}\, \eta_0^1 \tag{1.313}$$

$$\eta^2 = N^{(4)}\frac{1}{\sqrt{\sigma_\mu p^\mu}}\eta_0^2 = \sqrt{\frac{\bar{\sigma}_\mu p^\mu}{\frac{E}{c} - p^3}}\eta_0^2. \tag{1.314}$$

Again they satisfy

$$(\eta^r)^+\eta^s = \delta^{rs}. \tag{1.315}$$

Exercise 20:

(1) We have

$$u^{(r)}(E, \vec{p}\,) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\,\xi^r \\ \sqrt{\bar{\sigma}_\mu p^\mu}\,\xi^r \end{pmatrix}, \quad v^{(r)}(E, \vec{p}\,) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\,\eta^r \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\,\eta^r \end{pmatrix}. \tag{1.316}$$

We compute

$$\bar{u}^{(r)}u^{(s)} = u^{(r)+}\gamma^0 u^{(s)} = 2\xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\,\xi^s = 2mc\xi^{r+}\xi^s = 2mc\delta^{rs} \tag{1.317}$$

$$\bar{v}^{(r)}v^{(s)} = v^{(r)+}\gamma^0 v^{(s)} = -2\eta^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\,\eta^s = -2mc\eta^{r+}\eta^s = -2mc\delta^{rs}. \tag{1.318}$$

We have used

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu) = m^2c^2 \tag{1.319}$$

$$\xi^{r+}\xi^s = \delta^{rs}, \quad \eta^{r+}\eta^s = \delta^{rs}. \tag{1.320}$$

We also compute

$$\bar{u}^{(r)}v^{(s)} = u^{(r)+}\gamma^0 v^{(s)} = -\xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\,\eta^s + \xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\,\eta^s = 0. \tag{1.321}$$

A similar calculation yields

$$\bar{v}^{(r)}u^{(s)} = u^{(r)+}\gamma^0 v^{(s)} = 0. \tag{1.322}$$

(2) Next we compute

$$u^{(r)+}u^{(s)} = \xi^{r+}(\sigma_\mu p^\mu + \bar{\sigma}_\mu p^\mu)\xi^s = \frac{2E}{c}\xi^{r+}\xi^s = \frac{2E}{c}\delta^{rs} \tag{1.323}$$

$$v^{(r)+}v^{(s)} = \eta^{r+}(\sigma_\mu p^\mu + \bar{\sigma}_\mu p^\mu)\eta^s = \frac{2E}{c}\eta^{r+}\eta^s = \frac{2E}{c}\delta^{rs}. \tag{1.324}$$

We have used

$$\sigma^\mu = (1, \sigma^i), \quad \sigma^\mu = (1, -\sigma^i). \tag{1.325}$$

We also compute

$$u^{(r)+}(E, \vec{p})v^{(s)}(E, -\vec{p}) = \xi^{r+}\Big(\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)} - \sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\Big)\xi^s = 0. \tag{1.326}$$

Similarly, we compute that

$$v^{(r)+}(E, -\vec{p})u^{(s)}(E, \vec{p}) = 0. \tag{1.327}$$

In the above two equation we have used the fact that

$$v^{(r)}(E, -\vec{p}) = \begin{pmatrix} \sqrt{\bar{\sigma}_\mu p^\mu}\,\eta^r \\ -\sqrt{\sigma_\mu p^\mu}\,\eta^r \end{pmatrix}. \tag{1.328}$$

(3) Next we compute

$$\begin{aligned}\sum_s u^{(s)}(E, \vec{p})\bar{u}^{(s)}(E, \vec{p}) &= \sum_s u^{(s)}(E, \vec{p})u^{(s)+}(E, \vec{p})\gamma^0 \\ &= \sum_s \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\,\xi^s\xi^{s+}\sqrt{\sigma_\mu p^\mu} & \sqrt{\sigma_\mu p^\mu}\,\xi^s\xi^{s+}\sqrt{\bar{\sigma}_\mu p^\mu} \\ \sqrt{\bar{\sigma}_\mu p^\mu}\,\xi^s\xi^{s+}\sqrt{\sigma_\mu p^\mu} & \sqrt{\bar{\sigma}_\mu p^\mu}\,\xi^s\xi^{s+}\sqrt{\bar{\sigma}_\mu p^\mu} \end{pmatrix} \\ &\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned} \tag{1.329}$$

We use

$$\sum_s \xi^s\xi^{s+}=1. \tag{1.330}$$

We obtain

$$\sum_s u^{(s)}(E, \vec{p})\bar{u}^{(s)}(E, \vec{p}) = \begin{pmatrix} mc & \sigma_\mu p^\mu \\ \bar{\sigma}_\mu p^\mu & mc \end{pmatrix} = \gamma^\mu p_\mu + mc. \tag{1.331}$$

Similarly we use

$$\sum_s \eta^s\eta^{s+} = 1 \tag{1.332}$$

to calculate

$$\sum_s v^{(s)}(E, \vec{p})\bar{v}^{(s)}(E, \vec{p}) = \begin{pmatrix} -mc & \sigma_\mu p^\mu \\ \bar{\sigma}_\mu p^\mu & -mc \end{pmatrix} = \gamma^\mu p_\mu - mc. \tag{1.333}$$

Exercise 21:

Under Lorentz transformations we have the following transformation laws

$$\psi(x) \longrightarrow \psi'(x') = S(\Lambda)\psi(x) \tag{1.334}$$

$$\gamma_\mu \longrightarrow \gamma'_\mu = \gamma_\mu \tag{1.335}$$

$$\partial_\mu \longrightarrow \partial'_\nu = (\Lambda^{-1})^\mu{}_\nu \partial_\mu. \tag{1.336}$$

Thus the Dirac equation $(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$ becomes

$$(i\hbar\gamma'^\mu\partial'_\mu - mc)\psi' = 0, \tag{1.337}$$

or equivalently

$$(i\hbar(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma'^\mu S(\Lambda)\partial_\nu - mc)\psi = 0. \tag{1.338}$$

We must therefore have

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu, \tag{1.339}$$

or equivalently

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu. \tag{1.340}$$

We consider an infinitesimal Lorentz transformation

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}, \quad \Lambda^{-1} = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}. \tag{1.341}$$

The corresponding $S(\Lambda)$ must also be infinitesimal of the form

$$S(\Lambda) = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}, \quad S^{-1}(\Lambda) = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}. \tag{1.342}$$

By substitution we get

$$-(\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma_\mu = [\gamma_\nu, \Gamma^{\alpha\beta}]. \tag{1.343}$$

Explicitly this reads

$$-i\hbar\left(\delta_\nu^\beta\gamma^\alpha - \delta_\nu^\alpha\gamma^\beta\right) = \left[\gamma_\nu, \Gamma^{\alpha\beta}\right], \tag{1.344}$$

or equivalently

$$\begin{aligned}
&[\gamma_0, \Gamma^{0i}] = i\hbar\gamma^i \\
&[\gamma_j, \Gamma^{0i}] = -i\hbar\delta_j^i\gamma^0 \\
&[\gamma_0, \Gamma^{ij}] = 0 \\
&[\gamma_k, \Gamma^{ij}] = -i\hbar\left(\delta_k^j\gamma^i - \delta_k^i\gamma^j\right).
\end{aligned} \tag{1.345}$$

A solution is given by

$$\Gamma^{\mu\nu} = \frac{i\hbar}{4}[\gamma^\mu, \gamma^\nu]. \tag{1.346}$$

Exercise 22:

The Dirac spinor ψ changes under Lorentz transformations as

$$\psi(x) \longrightarrow \psi'(x') = S(\Lambda)\psi(x) \tag{1.347}$$

$$S(\Lambda) = e^{-\frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}}. \tag{1.348}$$

Since $(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0$ we get $(\Gamma^{\mu\nu})^+ = \gamma^0\Gamma^{\mu\nu}\gamma^0$. Therefore

$$S(\Lambda)^+ = \gamma^0 S(\Lambda)^{-1}\gamma^0. \tag{1.349}$$

In other words

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x') = \bar{\psi}(x)S(\Lambda)^{-1}. \tag{1.350}$$

As a consequence

$$\bar{\psi}\psi \longrightarrow \bar{\psi}'\psi' = \bar{\psi}\psi \tag{1.351}$$

$$\bar{\psi}\gamma^5\psi \longrightarrow \bar{\psi}'\gamma^5\psi' = \bar{\psi}\psi \tag{1.352}$$

$$\bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}'\gamma^\mu\psi' = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi \tag{1.353}$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \longrightarrow \bar{\psi}'\gamma^\mu\gamma^5\psi' = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma^5\psi. \tag{1.354}$$

We have used $[\gamma^5, \Gamma^{\mu\nu}] = 0$ and $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$. Finally we compute

$$\begin{aligned} \bar{\psi}\Gamma^{\mu\nu}\psi \longrightarrow \bar{\psi}'\Gamma^{\mu\nu}\psi' &= \bar{\psi}S^{-1}\Gamma^{\mu\nu}S\psi \\ &= \bar{\psi}\frac{i\hbar}{4}[S^{-1}\gamma^\mu S, S^{-1}\gamma^\nu S]\psi \\ &= \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\bar{\psi}\Gamma^{\alpha\beta}\psi. \end{aligned} \tag{1.355}$$

Exercise 23:

(1) The Clifford algebra in three Euclidean dimensions is solved by Pauli matrices, viz

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}, \quad \gamma^i \equiv \sigma^i. \tag{1.356}$$

Any 2×2 matrix can be expanded in terms of the Pauli matrices and the identity. In other words

$$M_{2\times 2} = M_0\mathbf{1} + M_i\sigma_i. \tag{1.357}$$

(2) Any 4×4 matrix can be expanded in terms of a 16 antisymmetric combination of the Dirac gamma matrices. The four-dimensional identity

and the Dirac matrices provide the first five independent 4×4 matrices. The product of two Dirac gamma matrices yield six different matrices which, because of $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, can be encoded in the six matrices $\Gamma^{\mu\nu}$ defined by

$$\Gamma^{\mu\nu} = \frac{i\hbar}{4}[\gamma^\mu, \gamma^\nu]. \tag{1.358}$$

There are four independent 4×4 matrices formed by the product of three Dirac gamma matrices. They are

$$\gamma^0\gamma^1\gamma^2, \quad \gamma^0\gamma^1\gamma^3, \quad \gamma^0\gamma^2\gamma^3, \quad \gamma^1\gamma^2\gamma^3. \tag{1.359}$$

These can be rewritten as

$$i\varepsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma^5. \tag{1.360}$$

The product of four Dirac gamma matrices leads to an extra independent 4×4 matrix which is precisely the gamma five matrix. In total there are $1 + 4 + 6 + 4 + 1 = 16$ antisymmetric combinations of Dirac gamma matrices. Hence, any 4×4 matrix can be expanded as

$$M_{4\times4} = M_0\mathbf{1} + M_\mu\gamma^\mu + M_{\mu\nu}\Gamma^{\mu\nu} + M_{\mu\nu\alpha}i\varepsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma^5 + M_5\gamma^5. \tag{1.361}$$

Exercise 24:

(1) We have

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{1.362}$$

Thus

$$\begin{aligned}
-\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= -\frac{i}{4!}(4)\varepsilon_{0abc}\gamma^0\gamma^a\gamma^b\gamma^c \\
&= -\frac{i}{4!}(4.3)\varepsilon_{0ij3}\gamma^0\gamma^i\gamma^j\gamma^3 \\
&= -\frac{i}{4!}(4.3.2)\varepsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= i\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= \gamma^5.
\end{aligned} \tag{1.363}$$

We have used

$$\varepsilon_{0123} = -\varepsilon^{0123} = -1. \tag{1.364}$$

We also verify

$$\begin{aligned}(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3.\,\gamma^0\gamma^1\gamma^2\gamma^3\\ &= \gamma^1\gamma^2\gamma^3.\,\gamma^1\gamma^2\gamma^3\\ &= -\gamma^2\gamma^3.\,\gamma^2\gamma^3\\ &= 1\end{aligned} \tag{1.365}$$

$$\begin{aligned}(\gamma^5)^+ &= -i(\gamma^3)^+(\gamma^2)^+(\gamma^1)^+(\gamma^0)^+\\ &= i\gamma^3\gamma^2\gamma^1\gamma^0\\ &= -i\gamma^0\gamma^3\gamma^2\gamma^1\\ &= -i\gamma^0\gamma^1\gamma^3\gamma^2\\ &= i\gamma^0\gamma^1\gamma^2\gamma^3\\ &= \gamma^5\end{aligned} \tag{1.366}$$

$$\{\gamma^5, \gamma^0\} = \{\gamma^5, \gamma^1\} = \{\gamma^5, \gamma^2\} = \{\gamma^5, \gamma^3\} = 0. \tag{1.367}$$

From this last property we conclude directly that

$$[\gamma^5, \Gamma^{\mu\nu}] = 0. \tag{1.368}$$

(2) Hence the Dirac representation is reducible. To see this more clearly we work in the Weyl or chiral representation given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{1.369}$$

In this representation we compute

$$\gamma^5 = i\begin{pmatrix} \sigma^1\sigma^2\sigma^3 & 0 \\ 0 & \sigma^1\sigma^2\sigma^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.370}$$

Hence by writing the Dirac spinor as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{1.371}$$

we get

$$\Psi_R = \frac{1+\gamma^5}{2}\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}, \tag{1.372}$$

and

$$\Psi_L = \frac{1-\gamma^5}{2}\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}. \tag{1.373}$$

In other words

$$\gamma^5\Psi_L = -\Psi_L, \quad \gamma^5\Psi_R = \Psi_R. \tag{1.374}$$

The spinors Ψ_L and Ψ_R do not mix under Lorentz transformations since they are eigenspinors of γ^5 which commutes with Γ^{ab}. In other words

$$\Psi_L(x) \longrightarrow \Psi'_L(x') = S(\Lambda)\Psi_L(x) \tag{1.375}$$

$$\Psi_R(x) \longrightarrow \Psi'_R(x') = S(\Lambda)\Psi_R(x). \tag{1.376}$$

(3) The Dirac equation is

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{1.377}$$

In terms of ψ_L and ψ_R this becomes

$$i\hbar(\partial_0 + \sigma^i\partial_i)\psi_R = mc\psi_L, \quad i\hbar(\partial_0 - \sigma^i\partial_i)\psi_L = mc\psi_R. \tag{1.378}$$

For a massless theory we get two fully decoupled equations

$$i\hbar(\partial_0 + \sigma^i\partial_i)\psi_R = 0, \quad i\hbar(\partial_0 - \sigma^i\partial_i)\psi_L = 0. \tag{1.379}$$

These are known as the Weyl equations. They are relevant in describing neutrinos. It is clear that ψ_L describes a left-moving particle and ψ_R describes a right-moving particle.

References

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