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Classical Mechanics
Lecture notes
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Chapter 5

Oscillations

In this course, oscillations and waves are discussed in detail because of their key importance for fundamental and applied physics. This chapter starts with discussion of the so-called 'linear' (or 'harmonic') oscillator whose differential equation of motion is linear and hence allows the full analytical solution, and then proceeds to 'nonlinear' and parametric systems whose dynamics may only be explored by either approximate analytical or numerical methods.

5.1 Free and forced oscillations

In section 3.2 we briefly discussed oscillations in a very important Hamiltonian system—a 1D *harmonic oscillator* described by a simple 1D Lagrangian[1]

$$L \equiv T(\dot{q}) - U(q) = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2, \tag{5.1}$$

whose Lagrange equation of motion[2],

$$m\ddot{q} + \kappa q = 0, \quad \text{i.e.} \quad \ddot{q} + \omega_0^2 q = 0, \quad \text{with} \quad \omega_0^2 \equiv \frac{\kappa}{m} \geqslant 0, \tag{5.2}$$

is a *linear homogeneous* differential equation. Its general solution is given by Eq. (3.16), which is frequently useful to recast into another, amplitude-phase form:

$$q(t) = u\cos\omega_0 t + v\sin\omega_0 t = A\cos(\omega_0 t - \varphi), \tag{5.3a}$$

[1] For simplicity of notation, in this chapter I will drop indices 'ef' in the energy components T and U, and parameters such as m, κ, etc. However, the reader should still remember that T and U do not necessarily coincide with the actual kinetic and potential energies (even if those energies may be uniquely identified)—see section 3.1.

[2] ω_0 is usually called the *own frequency* of the oscillator. In quantum mechanics, the Germanized version of the same term, *eigenfrequency*, is more commonly used. In this series I will use either of the terms, depending on the context.

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where A is the *amplitude* and φ the *phase* of the oscillations, which are determined by the initial conditions. Mathematically, it is frequently easier to work with sinusoidal functions as complex exponents, by rewriting Eq. (5.3*a*) in one more form[3]:

$$q(t) = \mathrm{Re}\,[Ae^{-i(\omega_0 t-\varphi)}] = \mathrm{Re}\,[ae^{-i\omega_0 t}], \tag{5.3b}$$

where a is the *complex amplitude* of the oscillations:

$$a \equiv Ae^{i\varphi}, \quad |\,a\,| = A, \quad \mathrm{Re}\,a = A\cos\varphi = u, \quad \mathrm{Im}\,a = A\sin\varphi = v. \tag{5.4}$$

For an autonomous, Hamiltonian oscillator, Eq. (5.3) gives the full classical description of its dynamics. However, it is important to understand that this *free-oscillation* solution, with a constant amplitude A, is due to the conservation of energy $E = T + U = \kappa A^2/2$ of the oscillator. If its energy changes by any reason, the description needs to be generalized.

First of all, if the energy leaks out of the oscillator to its environment (the effect usually called the *energy dissipation*), the free oscillation decay with time. The simplest model of this effect is represented by an additional *linear drag* (or 'kinematic friction') *force*, proportional to the generalized velocity and directed opposite to it:

$$F_v = -\eta\dot{q}, \tag{5.5}$$

where constant η is called the *drag coefficient*[4]. The inclusion of this force modifies the equation of motion (5.2) to become

$$m\ddot{q} + \eta\dot{q} + \kappa q = 0. \tag{5.6a}$$

This equation is frequently rewritten in the form

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0, \quad \text{with } \delta \equiv \frac{\eta}{2m}, \tag{5.6b}$$

where parameter δ is called the *damping coefficient* (or just 'damping'). Note that Eq. (5.6) is still a linear homogeneous second-order differential equation, and its general solution still has the form of the sum (3.13) of two exponents of the type $\exp\{\lambda t\}$, with arbitrary pre-exponential coefficients. Plugging such an exponent into Eq. (5.6), we obtain the following algebraic characteristic equation for λ:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0. \tag{5.7}$$

[3] Note that this is the so-called *physics convention*. Most engineering texts use the opposite sign in the imaginary exponent, $\exp\{-i\omega t\} \to \exp\{i\omega t\}$, with the corresponding sign implications for intermediate formulas, but (of course) similar final results for real variables.

[4] Here Eq. (5.5) is treated as a phenomenological model, but in statistical mechanics such a dissipative term may be *derived* as an average force exerted on the oscillator by its environment, at very general assumptions. As will be discussed in detail later in this series (*Part SM* chapter 5 and *Part QM* chapter 7), due to the ts numerous degrees of freedom of a typical environmental (think about the molecules of air surrounding the usual mechanical pendulum), its force also has a random component; as a result, the *dissipation* is fundamentally related to *fluctuations*. The latter effects may be neglected (as they are in this course) only if E is much higher than the energy scale of the random fluctuations—in the thermal equilibrium at temperature T, the larger of $k_B T$ and $\hbar\omega_0/2$.

Solving this quadratic equation, we get

$$\lambda_{\pm} = -\delta \pm i\omega_0', \quad \text{where } \omega_0' \equiv (\omega_0^2 - \delta^2)^{1/2}, \tag{5.8}$$

so that for not-very-high damping ($\delta < \omega_0$)[5] we obtain the following generalization of Eq. (5.3):

$$\begin{aligned} q_{\text{free}}(t) &= c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t} \\ &= (u_0 \cos \omega_0' t + v_0 \sin \omega_0' t)\, e^{-\delta t} \\ &= A_0 e^{-\delta t} \cos(\omega_0' t - \varphi_0). \end{aligned} \tag{5.9}$$

The result shows that, besides a certain correction to the free-oscillation frequency (which is very small in the most interesting *low damping limit*, $\delta \ll \omega_0$), the energy dissipation leads to an exponential decay of oscillation amplitude with time constant $\tau = 1/\delta$:

$$A = A_0 e^{-t/\tau}, \quad \text{where } \tau \equiv \frac{1}{\delta} = \frac{2m}{\eta}. \tag{5.10}$$

A very popular, dimensionless measure of damping is the so-called *quality factor* Q (or just the *Q-factor*) that is defined as $\omega_0/2\delta$, and may be rewritten in several other useful forms:

$$Q \equiv \frac{\omega_0}{2\delta} = \frac{m\omega_0}{\eta} = \frac{(m\kappa)^{1/2}}{\eta} = \pi \frac{\tau}{\mathcal{T}} = \frac{\omega_0 \tau}{2}, \tag{5.11}$$

where $\mathcal{T} = 2\pi/\omega_0$ is the oscillation period in the absence of damping—see Eq. (3.29). Since the oscillation energy E is proportional to A^2, i.e. decays as $\exp\{-2t/\tau\}$, with the time constant $\tau/2$, the last form of Eq. (5.11) may be used to rewrite the Q-factor in one more form:

$$Q = \omega_0 \frac{E}{(-\dot{E})} \equiv \omega_0 \frac{E}{\mathscr{P}}, \tag{5.12}$$

where $\mathscr{P}$ is the dissipation power. (Two other practical ways to measure Q will be discussed below.) The range of Q-factors of important oscillators is very broad, all the way from $Q \sim 10$ for a human leg (with relaxed muscles), to $Q \sim 10^4$ for the quartz crystals used in 'electronic' clocks and watches, all the way up to $Q \sim 10^{12}$ for carefully designed microwave cavities with superconducting walls.

In contrast to the decaying free oscillations, the *forced oscillations*, induced by an external force $F(t)$, may maintain their amplitude (and hence energy) infinitely, even at non-zero damping. This process may be described using a still linear but now *inhomogeneous* differential equation

$$m\ddot{q} + \eta\, \dot{q} + \kappa q = F(t), \tag{5.13a}$$

[5] Systems with very high damping ($\delta > \omega_0$) can hardly be called oscillators, and although they are used in engineering and physics experiments (e.g. for the shock, vibration, and sound isolation), for their discussion I have to refer the interested reader to special literature—see, e.g. [1]. Let me only note that at very high damping, $\delta \gg \omega_0$, the system may be adequately described by just one parameter: the relaxation time $1/\lambda_+ \approx 2\delta/\omega_0^2 \equiv \eta/m \gg \omega_0$.

or, more conveniently for analysis, the following generalization of Eq. (5.6*b*):

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = f(t), \quad \text{where } f(t) \equiv F(t)/m. \tag{5.13b}$$

For a mechanical linear, dissipative 1D oscillator (5.6), under the effect of an additional external force $F(t)$, Eq. (5.13*a*) is just an expression of Newton's second law. However, according to Eq. (1.41), Eq. (5.13) is valid for any dissipative, *linear*[6] 1D system whose Gibbs potential energy (1.39) has the form $U_G(q, t) = \kappa q^2/2 - F(t)q$.

The forced-oscillation solutions may be analyzed by two mathematically equivalent methods whose relative convenience depends on the character of function $f(t)$

(i) *Frequency domain.* Let us present function $f(t)$ as a Fourier sum of sinusoidal harmonics[7]:

$$f(t) = \sum_{\omega} f_\omega e^{-i\omega t}. \tag{5.14}$$

Then, due to the linearity of Eq. (5.13), its general solution may be represented as a sum of the decaying free oscillations (5.9) with the frequency ω_0', independent of the function $F(t)$, and forced oscillations due to each of the Fourier components of the force[8]:

$$q(t) = q_{\text{free}}(t) + q_{\text{forced}}(t), \quad q_{\text{forced}}(t) = \sum_{\omega} a_\omega e^{-i\omega t}. \tag{5.15}$$

Plugging Eq. (5.15) into Eq. (5.13), and requiring that the factors before each $e^{-i\omega t}$ in both parts be equal, we obtain

$$a_\omega = f_\omega \chi(\omega), \tag{5.16}$$

where the complex function $\chi(\omega)$, which in our particular case equals

$$\chi(\omega) = \frac{1}{(\omega_0^2 - \omega^2) - 2i\omega\delta}, \tag{5.17}$$

is called either the *response function* or (particularly for non-mechanical oscillators) the *generalized susceptibility*. From here, and Eq. (5.4), the amplitude of the oscillations under the effect of a sinusoidal force is

$$A_\omega \equiv |a_\omega| = |f_\omega|\,|\chi(\omega)|, \quad \text{with } |\chi(\omega)| = \frac{1}{\left[(\omega_0^2 - \omega^2)^2 + (2\omega\delta)^2\right]^{1/2}}. \tag{5.18}$$

[6] This is a very unfortunate, but common jargon, meaning 'the system described by linear equations of motion'.

[7] Here, in contrast to Eq. (5.3*b*), we may drop the operator Re, assuming that $f_{-\omega} = f_\omega^*$, so that the imaginary components of the sum compensate each other.

[8] In physics, this mathematical property of linear equations is frequently called the *linear superposition principle*.

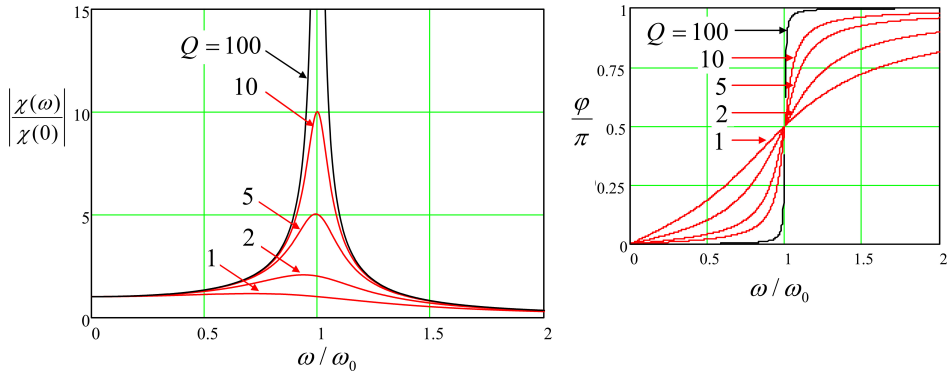


Figure 5.1. Resonance in the linear oscillator, for several values of Q.

This formula describes, in particular, an increase of the oscillation amplitude A_ω at $\omega \to \omega_0$—see the left panel in figure 5.1. At the exact resonance,

$$|\chi(\omega)|_{\omega=\omega_0} = \frac{1}{2\omega_0\delta}, \tag{5.19}$$

so that, according to Eq. (5.11), the ratio of the response magnitudes at $\omega = \omega_0$ and $\omega = 0$ ($|\chi(\omega)|_{\omega=0} = 1/{\omega_0}^2$) is exactly equal to the Q-factor of the oscillator. Thus, the response increase is particularly strong in the low damping limit ($\delta \ll \omega_0$, i.e. $Q \gg 1$); moreover at $Q \to \infty$ and $\omega \to \omega_0$ the response diverges. (This fact is very useful for the approximate methods to be discussed later in this chapter.) This is of course the classical description of the famous phenomenon of *resonance*, so ubiquitous in physics.

Due to the increase of the resonance peak height, its width is inversely proportional to Q.[9] Quantitatively, in the most interesting low-damping limit, i.e. at $Q \gg 1$, the reciprocal Q-factor gives the normalized value of the so-called *full-width at half-maximum* (FWHM) of the resonance curve:

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}. \tag{5.20}$$

Indeed, this $\Delta\omega$ is defined as the difference $(\omega_+ - \omega_-)$ between the two values of ω at which the square of the oscillator response function, $|\chi(\omega)|^2$ (proportional to the oscillation energy), equals half of its resonance value (5.19). In the low damping limit, both these points are very close to ω_0, so that in the first (linear) approximation in $(\omega - \omega_0) \ll \omega_0$, we can take $({\omega_0}^2 - \omega^2) \equiv -(\omega + \omega_0)(\omega - \omega_0) \approx (-2\omega\xi) \approx (-2\omega_0\xi)$, where

$$\xi \equiv \omega - \omega_0 \tag{5.21}$$

[9] Note that the phase shift $\varphi = \arg[\chi(\omega)]$ between the oscillations and the external force (see the right panel in figure 1) makes its steepest change, by $\pi/2$, exactly within the same frequency interval $\Delta\omega$.

is a very convenient parameter called *detuning*, which will be repeatedly used later in this chapter. In this approximation, the second of Eqs. (5.18) is reduced to

$$|\chi(\omega)|^2 = \frac{1}{4\omega^2(\delta^2 + \xi^2)}. \tag{5.22}$$

As a result, the points $\omega_\pm$ correspond to $\xi^2 = \delta^2$, i.e. $\omega_\pm = \omega_0 \pm \delta = \omega_0(1 \pm 1/2Q)$, so that $\Delta\omega \equiv \omega_+ - \omega_- = \omega_0/Q$, thus proving Eq. (5.20).

(ii) *Time domain.* Returning to the general problem of linear oscillations, one may argue that Eqs. (5.9) and (5.15)–(5.17) provide a full solution of the forced oscillation problem. This is formally correct, but this solution may be very inconvenient if the external force is far from a sinusoidal function of time, in particular if it is not periodic at all. In this case, we should first calculate the complex amplitudes f_ω participating in the Fourier sum (5.14). In the case of a non-periodic $f(t)$, this is actually the Fourier integral[10],

$$f(t) = \int_{-\infty}^{+\infty} f_\omega e^{-i\omega t} dt, \tag{5.23}$$

so that f_ω should be calculated using the reciprocal Fourier transform,

$$f_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t) e^{i\omega t'} dt. \tag{5.24}$$

Now we can use Eq. (5.16) for each Fourier component of the resulting forced oscillations, and rewrite the last of Eqs. (5.15) as

$$\begin{aligned} q_{\text{forced}}(t) &= \int_{-\infty}^{+\infty} a_\omega e^{-i\omega t} d\omega \\ &= \int_{-\infty}^{+\infty} \chi(\omega) f_\omega e^{-i\omega t} d\omega \\ &= \int_{-\infty}^{+\infty} d\omega\, \chi(\omega) \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, f(t) e^{i\omega(t'-t)} \\ &= \int_{-\infty}^{+\infty} dt\, f(t) \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \chi(\omega) e^{i\omega(t'-t)}\right], \end{aligned} \tag{5.25}$$

with the response function $\chi(\omega)$ given, in our case, by Eq. (5.17). Apart from requiring two integrations, the relation (5.25) is conceptually uncomforting: it seems to indicate that the oscillator's coordinate at time t depends not only on the external force exerted at earlier times $t' < t$, but also at future times. This would contradict one of the most fundamental principles of physics (and indeed, science as a whole), the *causality*: no effect may precede its cause.

[10] Let me hope that the reader knows that Eq. (5.23) may be used even for periodic functions; in such a case, f_ω is a set of equidistant delta-functions. (A reminder of the basic properties of the δ-function may be found, for example, in appendix A, section A.14.)

Fortunately, a straightforward calculation (left for a reader's exercise) shows that the response function (5.17) satisfies the following rule[11]:

$$\int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega = 0, \quad \text{for } \tau < 0. \tag{5.26}$$

This fact allows the last form of Eq. (5.25) to be rewritten in either of the following equivalent forms:

$$q_{\text{forced}}(t) = \int_{-\infty}^{t} f(t) G(t - t) dt = \int_{0}^{\infty} f(t - \tau)\, G(\tau) d\tau, \tag{5.27}$$

where $G(\tau)$, defined as the Fourier transform of the response function,

$$G(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega, \tag{5.28}$$

is called the (*temporal*) *Green's function* of the system. According to Eq. (5.26), $G(\tau) = 0$ for all $\tau < 0$.

While the second form of Eq. (5.27) is frequently more convenient for calculations, its first form is more suitable for understanding the physical sense of the Green's function. Indeed, let us consider the particular case, when the force is a delta-function

$$f(t) = \delta(t - t'), \qquad \text{with } t' < t, \qquad \text{i.e. } \tau \equiv t - t' > 0, \tag{5.29}$$

representing an ultimately short pulse at the moment t', with a unit 'area' $\int f(t')dt'$. Substituting Eq. (5.29) into Eq. (5.27)[12], we obtain

$$q(t) = G(t - t'). \tag{5.30}$$

Thus the Green's function $G(t - t')$ is just the oscillator's response to a short pulse of force, of a unit 'area', measured at time t. Hence Eq. (5.27) expresses the linear superposition principle in the time domain: the full effect of the force $f(t)$ on an oscillator (actually, any linear system) is a sum of the effects of short pulses of duration dt' and magnitude $f(t')$, each with its own 'weight' $G(t-t')$—see figure 5.2.

This picture may be used for the calculation of the Green's function for a particular system. Indeed, Eqs. (5.29)–(5.30) mean that $G(\tau)$ is just the solution of the differential equation of motion of the system, in our case of Eq. (5.13), with the replacement $t \to \tau$ and a δ-functional right-hand side:

$$\frac{d^2 G(\tau)}{d\tau^2} + 2\delta \frac{dG(\tau)}{d\tau} + \omega_0^2 G(\tau) = \delta(\tau). \tag{5.31}$$

[11] Eq. (5.26) remains true for any linear physical systems in which $f(t)$ represents a cause, and $q(t)$ its effect. Following tradition, I discuss the frequency-domain expression of this causality relation (called the *Kramers–Kronig relations*) in the *Classical Electrodynamics* part of this lecture series—see section 7.3.

[12] Technically, for this integration, t' in Eq. (5.27) should be temporarily replaced with another letter, say t''.

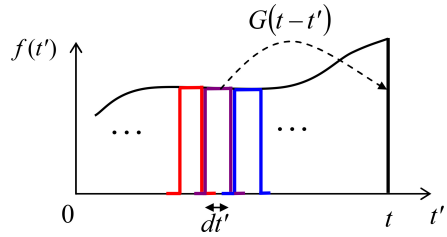


Figure 5.2. A schematic, finite-interval representation of a force $f(t)$ as a sum of short pulses, and their effects on the linear system's response at time t, given by (5.27).

Since Eq. (5.27) describes only the second term in Eq. (5.15), i.e. only the forced rather than free oscillations, we have to exclude the latter by solving Eq. (5.31) with zero initial conditions:

$$G(-0) = \frac{dG}{d\tau}(-0) = 0, \tag{5.32}$$

where $t = -0$ means the instant immediately preceding $t = 0$.

This calculation may be simplified even further. Let us integrate both sides of Eq. (5.31) over an infinitesimal interval including the origin, e.g. $[-d\tau/2, +d\tau/2]$, and then follow the limit $d\tau \to 0$. Since the Green's function has to be continuous because of its physical sense as the (generalized) coordinate, all terms on the left-hand side but the first one vanish, while the first term yields $dG/d\tau|_{+0} - dG/d\tau|_{-0}$. Due to the second of the conditions (5.32), the last of these two derivatives equals zero, while the right-hand side of Eq. (5.31) yields 1 upon the integration. Thus, $G(\tau)$ may be calculated for $\tau > 0$ (i.e. for all times when it is different from zero) by solving the *homogeneous* version of system's equation of motion for $\tau > 0$, with the following special initial conditions:

$$G(0) = 0, \quad \frac{dG}{d\tau}(0) = 1. \tag{5.33}$$

This approach gives us a convenient way for calculation of Green's functions of linear systems. In particular for the oscillator with not-very-low damping ($\delta > \omega_0$, i.e. $Q > 1/2$), imposing the boundary conditions (5.33) on the homogeneous equation's solution (5.9), we immediately obtain

$$G(\tau) = \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau. \tag{5.34}$$

(The same result may be obtained directly from Eq. (5.28) with the response function $\chi(\omega)$ given by Eq. (5.19). This way is, however, a little bit more cumbersome, and is left for a reader exercise.)

Relations (5.27) and (5.34) provide a very convenient recipe for solving most forced oscillation problems. As a very simple example, let us calculate the transient

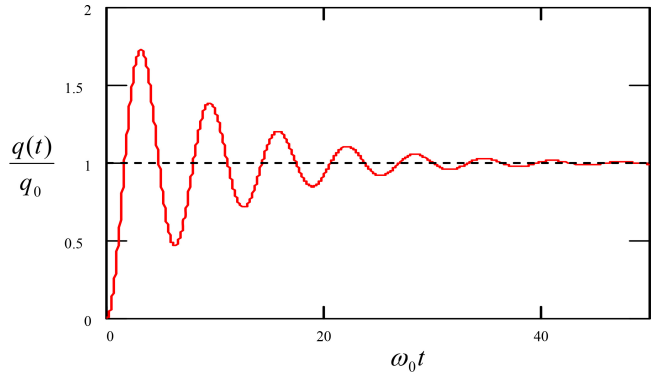


Figure 5.3. The transient process in a linear oscillator, induced by a step-like force $f(t)$, for the particular case $\delta/\omega_0 = 0.1$ (i.e. $Q = 5$).

process in an oscillator under the effect of a constant force being turned on at $t = 0$, i.e. proportional to the theta-function of time:

$$f(t) = f_0\theta(t) \equiv \begin{cases} 0, & t < 0, \\ f_0, & t > 0, \end{cases} \tag{5.35}$$

provided that at $t < 0$ the oscillator was at rest, so that in Eq. (5.15), $q_{\text{free}}(t) \equiv 0$. Then the second form of Eq. (5.27), and Eq. (5.34), yield

$$q(t) = \int_0^\infty f(t-\tau)G(\tau)d\tau = f_0\int_0^t \frac{1}{\omega_0'}e^{-\delta\tau}\sin\omega_0'\tau\, d\tau. \tag{5.36}$$

The simplest way to work out such integrals is to represent the sine function under it as the imaginary part of $\exp\{i\omega_0't\}$, and merge the two exponents, obtaining

$$q(t) = f_0\frac{1}{\omega_0'}\text{Im}\left[\frac{1}{\delta + i\omega_0'}e^{-\delta\tau - i\omega_0'\tau}\right]_0^t = \frac{F_0}{k}\left[1 - e^{-\delta t}\left(\cos\omega_0't + \frac{\delta}{\omega_0'}\sin\omega_0't\right)\right]. \tag{5.37}$$

This result, plotted in figure 5.3, is rather natural: it describes nothing more than the transient from the initial equilibrium position $q = 0$ to the new equilibrium position $q_0 = f_0/\omega_0^2 = F_0/\kappa$, accompanied by decaying oscillations. For this particular simple function $f(t)$, the same result might be also obtained by introducing a new variable $\tilde{q}(t) \equiv q(t) - q_0$ and solving the resulting *homogeneous* equation for $\tilde{q}$ (with the appropriate initial condition $\tilde{q}(0) = -q_0$). However, for more complicated functions $f(t)$ the Green's function approach is irreplaceable.

Note that for any particular linear system, its Green's function should be calculated only once, and then may be repeatedly used in Eq. (5.27) to calculate the system response to various external forces—either analytically or numerically. This property makes the Green's function approach very popular in many other fields of physics—with the corresponding generalization or re-definition of the function[13].

[13] See, e.g. *Part EM* section 2.7 and *Part QM* section 2.2.

5.2 Weakly nonlinear oscillations

In comparison with systems discussed in the last section, which are described by linear differential equations with constant coefficients and thus allow a complete and exact analytical solution, oscillations in nonlinear systems generally present a complex and, generally, analytically intractable problem. However, much insight on possible processes[14] in such systems may be gained from the discussion of the important case of *weakly nonlinear* systems that may be explored analytically. An example of such a system is given by an *anharmonic oscillator*—a 1D system whose higher terms in the potential expansion (3.10) cannot be neglected, but are small and may be accounted for approximately. If, in addition, damping is low (or negligible), and the external harmonic force exerted on the system is not too large (if any), the equation of motion is a slightly modified version of Eq. (5.13):

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}, \ldots), \tag{5.38}$$

where $\omega \approx \omega_0$ is the anticipated frequency of oscillations (whose choice is to a certain extent arbitrary—see below), and the right-hand side f is small (say, scales as some small dimensionless parameter $\varepsilon \ll 1$), and may be considered as a *perturbation*.

Since at $\varepsilon = 0$, this equation has the sinusoidal solution given by Eq. (5.3), one might naïvely think that at a non-vanishing but small ε, the approximate solution to Eq. (5.38) should be sought in the form

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)} + \cdots, \quad \text{where } q^{(n)} \propto \varepsilon^n, \tag{5.39}$$

with $q^{(0)} = A\cos(\omega_0 t - \varphi) \propto \varepsilon^0$. This is a good example of an apparently impeccable mathematical reasoning that would lead to a very inefficient procedure. Indeed, let us apply it to the problem we already know the exact solution for, namely the free oscillations in a linear but damped oscillator, for this occasion assuming the damping to be very low, $\delta/\omega_0 \sim \varepsilon \ll 1$. The corresponding equation of motion (5.6) may be represented in the form (5.38) if we take $\omega = \omega_0$ and

$$f = -2\delta\dot{q}, \quad \delta \propto \varepsilon. \tag{5.40}$$

The naïve approach described above would allow us to find *small* corrections, of the order of δ, to the free, non-decaying oscillations $A\cos(\omega_0 t - \varphi)$. However, we already know from Eq. (5.9) that the main effect of damping is a gradual decrease of the free-oscillation amplitude to zero, i.e. a very *large* change of the amplitude, though at low damping, $\delta \ll \omega_0$, this decay takes large time $t \sim \tau \gg 1/\omega_0$. Hence, if we want our approximate method to be productive (i.e. to work at all time scales, in particular for forced oscillations with established, constant amplitude and phase), we need to account for the fact that the *small* right-hand side of Eq. (5.38) may eventually lead to *essential* changes of the oscillation amplitude A (and sometimes,

[14] They are frequently called *nonlinear oscillations*. This is, again, a very unfortunate but generally accepted slang term for oscillations in systems described by nonlinear equations of motion.

as we will see below, also of the oscillation phase φ) at large times, because of the *slowly accumulating* effects of the small perturbation[15].

This goal may be achieved[16] by the account of these slow changes already in the '0th approximation', i.e. the basic part of the solution in the expansion (5.39):

$$q^{(0)} = A(t)\cos[\omega t - \varphi(t)], \quad \text{with } \dot{A}, \dot{\varphi} \to 0 \quad \text{at } \varepsilon \to 0. \tag{5.41}$$

(It is evident that the solution (5.9) fits this form.) Let me discuss this approach using a particular simple but representative example of a dissipative (but high-Q) pendulum driven by a weak sinusoidal external force with a nearly resonant frequency:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 \sin q = f_0 \cos \omega t, \tag{5.42}$$

with $|\omega - \omega_0|$, $\delta \ll \omega_0$, and the force amplitude f_0 so small that $|q| \ll 1$ at all times. From what we know about the forced oscillations from section 5.1, it is natural to identify ω in the left-hand side of Eq. (5.38) with the force frequency. Expanding $\sin q$ into the Taylor series in small q, keeping only the first two terms of this expansion, and moving all the small terms to the right-hand side, we can rewrite Eq. (5.42) in the canonical form (5.38)[17]:

$$\ddot{q} + \omega^2 q = -2\delta\dot{q} + 2\xi\omega q + \alpha q^3 + f_0 \cos \omega t \equiv f(t, q, \dot{q}). \tag{5.43}$$

Here $\alpha = \omega_0^2/6$ in the case of the pendulum (although the calculations below will be valid for any α), and the second term on the right-hand side was obtained using the approximation already employed in section 5.1: $(\omega^2 - \omega_0^2)q \approx 2\omega(\omega - \omega_0)q = 2\omega\xi q$, where $\xi \equiv \omega - \omega_0$ is the detuning parameter that was already used earlier—see Eq. (5.21).

Now, following the general recipe expressed by Eqs. (5.39) and (5.41), in the first approximation in $f \propto \varepsilon$, we may look for the solution to Eq. (5.43) in the form[18]

$$q(t) = A \cos \Psi + q^{(1)}(t), \quad \text{where } \Psi \equiv \omega t - \varphi, \qquad q^{(1)} \sim \varepsilon. \tag{5.44}$$

Let us plug this solution into both parts of Eq. (5.43), keeping only the terms of the first order in ε. Thanks to our (smart:-) choice of ω on the left-hand side of that equation, the two zero-order terms in that part cancel each other. Moreover, since

[15] The same flexible approach is necessary to approximations used in quantum mechanics. The method discussed here is closer in spirit (but not identical) to the *WKB approximation* (see, e.g. *Part QM* section 2.4) rather to the *perturbation theory* varieties (*Part QM* chapter 6).

[16] The basic idea of this approach was suggested in 1920 by B van der Pol, and in some textbooks its first approximation (on which I will focus) is called the *van der Pol method*. However, in optics and quantum mechanics, it is commonly called the *rotating wave approximation* (RWA). In mathematics-oriented texts, this approach, in particular its extensions to higher approximations, is usually called either the *small parameter method* or the *asymptotic method*. The list of other scientists credited for the development of this method, its variations and extensions includes, most notably, J H Poincaré, N Krylov, N Bogolyubov, and Yu Mitroplolsky.

[17] This equation is frequently called the *Duffing equation* (or the equation of the *Duffing oscillator*), after G Duffing who was the first to carry out its (rather incomplete) analysis in 1918.

[18] For a mathematically rigorous treatment of higher approximations, see, e.g. [2]. A more layman (and somewhat verbose) discussion of various oscillatory phenomena may be found in the classical text [3].

each term on the right-hand side of Eq. (5.43) is already of the order of ε, we may drop $q^{(1)} \propto \varepsilon$ from the substitution into that side at all, because this would give us only terms $O(\varepsilon^2)$ or higher. As a result, we get the following approximate equation:

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} = f^{(0)} \equiv -2\delta \frac{d}{dt}(A\cos\Psi) + 2\xi\omega A\cos\Psi + \alpha(A\cos\Psi)^3 + f_0\cos\omega t. \tag{5.45}$$

According to Eq. (5.41), generally A and φ should be considered as (slow) functions of time. However, let us leave the analyses of the transient process and system's stability until the next section, and use Eq. (5.45) to find stationary oscillations in the system that are established after an initial transient. For that limited task, we may take $A = \text{const}$, $\varphi = \text{const}$, so that $q^{(0)}$ presents sinusoidal oscillations of frequency ω. Sorting the terms on the right-hand side according to their time dependence[19], we see that it has terms with frequencies ω and 3ω:

$$f^{(0)} = \left(2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi\right)\cos\Psi + \left(2\delta\omega A - f_0\sin\varphi\right)\sin\Psi + \frac{1}{4}\alpha A^3\cos 3\Psi. \tag{5.46}$$

Now comes the main trick of the van der Pol approach: mathematically, Eq. (5.45) may be viewed as the equation of oscillations in a linear, *dissipation-free* harmonic oscillator of frequency ω (not ω_0!) under the action of an external force $f(t)$ represented by the right-hand side of the equation. In our particular case, it has three terms: two 'quadrature' components at that very frequency ω, and the third one at frequency 3ω. As we know from our analysis of this problem in section 5.1, if any of the first two components is non-vanishing, $q^{(1)}$ grows to infinity—see Eq. (5.19) with $\delta = 0$. At the same time, by the very structure of the rotating-wave approximation, $q^{(1)}$ has to be finite—moreover, small! The only way out of this contradiction is to require that the amplitudes of both quadrature components of $f^{(0)}$ with frequency ω are equal to zero:

$$2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi = 0, \qquad 2\delta\omega A - f_0\sin\varphi = 0. \tag{5.47}$$

These two *harmonic balance equations* enable us to find both parameters of the forced oscillations: their amplitude A and phase φ. In particular, the phase may be readily eliminated from this system (most easily, by expressing $\sin\varphi$ and $\cos\varphi$ from Eqs. (5.47), and then requiring the sum $\sin^2\varphi + \cos^2\varphi$ to equal 1), and the solution for the amplitude A recast in the following implicit but convenient form:

$$A^2 = \frac{f_0^2}{4\omega^2}\frac{1}{\xi^2(A) + \delta^2}, \quad \text{where } \xi(A) \equiv \xi + \frac{3}{8}\frac{\alpha A^2}{\omega} = \omega - \left(\omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}\right). \tag{5.48}$$

[19] Using the second of Eqs. (5.44), $\cos\omega t$ may be rewritten as $\cos(\Psi + \varphi) \equiv \cos\Psi\cos\varphi - \sin\Psi\sin\varphi$. Then using the identity given, e.g. by Eq. (A.19): $\cos^3\Psi = (3/4)\cos\Psi + (1/4)\cos 3\Psi$, we obtain Eq. (5.46).

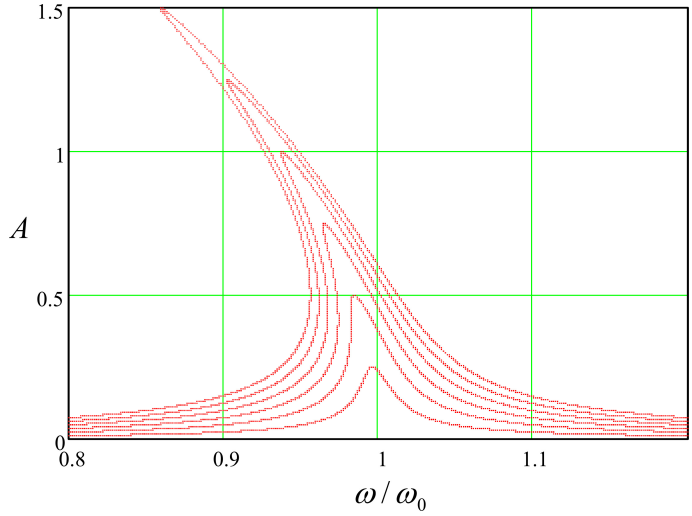


Figure 5.4. The nonlinear resonance in the Duffing oscillator, as described by Eq. (5.48), for the particular case $\alpha = \omega_0^2/6$, $\delta/\omega = 0.01$ (i.e. $Q = 50$), and several values of the parameter f_0/ω_0^2, increased by equal steps from 0.005 to 0.03.

This expression differs from Eq. (5.22) for the linear resonance in the low-damping limit only by the replacement of the detuning ξ with its effective amplitude-dependent value $\xi(A)$—or, equivalently, the replacement of the frequency ω_0 of the oscillator with its effective, amplitude-dependent value

$$\omega_0(A) = \omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}. \tag{5.49}$$

The physical meaning of $\omega_0(A)$ is simple: this is just the frequency of free oscillations of amplitude A in a similar nonlinear system, but with zero damping[20]. Indeed, for $\delta = 0$ and $f_0 = 0$ we could repeat our calculations, assuming that ω is an amplitude-dependent eigenfrequency $\omega_0(A)$. Then the second of Eqs. (5.47) is trivially satisfied, while the second gives Eq. (5.49). Eq. (5.48) enables us to draw the curves of this *nonlinear resonance* by just bending the linear resonance plots (figure 5.1) according to the so-called *skeleton curve* expressed by Eq. (5.49). Figure 5.4 shows the result of this procedure. Note that at small amplitude, $\omega(A) \to \omega_0$, and we return to the usual, 'linear' resonance Eq. (5.22).

To bring our solution to its logical completion, we should still find the first perturbation $q^{(1)}(t)$ from what is left of Eq. (5.45). Since the structure of this equation

[20] This effect of the pendulum's frequency dependence on its oscillation amplitude was described as early as 1673 by C Huygens, who had earlier invented the pendulum clock, increasing time-keeping accuracy by ~3 orders of magnitude.

is similar to Eq. (5.13) with the force of frequency 3ω and zero damping, we may use Eqs. (5.16) and (5.17) to obtain

$$q^{(1)}(t) = -\frac{1}{32\omega^2}\alpha A^3 \cos 3(\omega t - \varphi). \tag{5.50}$$

Adding this perturbation (note the negative sign!) to the sinusoidal oscillation (5.41), we see that as the amplitude A of oscillations in a system with $\alpha > 0$ (e.g. a pendulum) grows, their waveform becomes a bit more 'blunt' near the maximum deviations from the equilibrium.

The same Eq. (5.50) also allows an estimate of the range of validity of our first approximation: since it has been based on the assumption $|q^{(1)}| \ll |q^{(0)}| \leqslant A$, for this particular problem we have to require $\alpha A^2/32\omega^2 \ll 1$. For a pendulum (with $\alpha = {\omega_0}^2/6$), this condition becomes $A^2 \ll 192$. Although numerical coefficients in such strong inequalities should be taken with a grain of salt, the very large magnitude of this particular coefficient gives a good hint that the method should give very good results even for relatively large oscillations with $A \sim 1$. In section 5.7 below, we will see that this is indeed the case.

From the mathematical viewpoint, the next step would be to write the next approximation as

$$q(t) = A \cos \Psi + q^{(1)}(t) + q^{(2)}(t), \qquad q^{(2)} \sim \varepsilon^2, \tag{5.51}$$

and plug it into the Duffing equation (5.43), which (thanks to our special choice of $q^{(0)}$ and $q^{(1)}$) would retain only $\ddot{q}^{(2)} + \omega^2 q^{(2)}$ on its left-hand side. Again, requiring that amplitudes of two quadrature components of frequency ω on the right-hand side to be zero, we may obtain the second-order corrections to A and φ. Then we may use the remaining part of the equation to calculate $q^{(2)}$, and then go after the third-order terms, etc. However, for most purposes the sum $q^{(0)} + q^{(1)}$, and sometimes even just the crudest approximation $q^{(0)}$ alone, are completely sufficient. For example, according to Eq. (5.50), for a simple pendulum ($\alpha = {\omega_0}^2/6$) swinging as much as between the opposite horizontal positions ($A = \pi/2$), the first-order correction $q^{(1)}$ is of the order of 0.5%. (Soon beyond this value, completely new dynamic phenomena begin—see section 5.7 below, but they cannot be described by these successive approximations at all.) For such reasons, higher approximations are rarely pursued for particular systems.

5.3 Reduced equations

A much more important issue is the stability of the solutions described by Eq. (5.48). Indeed, figure 5.4 shows that within a certain range of parameters, these equations give three different values for the oscillation amplitude (and phase), and it is important to understand which of these solutions are stable. Since these solutions are not the fixed points in the sense discussed in section 3.2 (each point in figure 5.4 represents a nearly sinusoidal oscillation), their stability analysis needs a more general approach that would be valid for oscillations with amplitude and phase slowly evolving in time. This approach will also enable the analysis of

non-stationary (in particular the initial transient) processes, which are of key importance for some dynamic systems.

First of all, let us formalize the way the harmonic balance equations, such as Eq. (5.47), are obtained for the general case (5.38)—rather than for the particular Eq. (5.43) considered in the last section. After plugging in the 0th approximation Eq. (5.41) into the right-hand side of Eq. (5.38) we have to require the amplitudes of its both quadrature components of frequency ω to be zero. From the standard Fourier analysis we know that these requirements may be represented as

$$\overline{f^{(0)} \sin \Psi} = 0, \quad \overline{f^{(0)} \cos \Psi} = 0, \tag{5.52}$$

where the top bar means time averaging—in our current case, over the period $2\pi/\omega$ of the right-hand side of Eq. (5.52), with the arguments calculated in the 0th approximation:

$$\begin{aligned} &f^{(0)} \equiv f(t, q^{(0)}, \dot{q}^{(0)}, \cdots) \equiv f(t, A \cos \Psi, -A\omega \sin \Psi, \ldots), \\ &\text{with} \quad \Psi = \omega t - \varphi. \end{aligned} \tag{5.53}$$

Now, for a transient process the contribution of $q^{(0)}$ on the left-hand side of Eq. (5.38) is not zero any longer, because both amplitude and phase may be slow functions of time—see Eq. (5.41). Let us calculate this contribution. The exact result would be

$$\begin{aligned} \ddot{q}^{(0)} + \omega^2 q^{(0)} &\equiv \left(\frac{d^2}{dt^2} + \omega^2 \right) A \cos(\omega t - \varphi) \\ &= (\ddot{A} + 2\dot{\varphi}\omega A - \dot{\varphi}^2 A) \cos(\omega t - \varphi) - 2\dot{A}(\omega - \dot{\varphi}) \sin(\omega t - \varphi). \end{aligned} \tag{5.54}$$

However, in the first approximation in ε, we may neglect the second derivative of A, and also the squares and products of the first derivatives of A and φ (which are all of the second order in ε), so that Eq. (5.54) is reduced to

$$\ddot{q}^{(0)} + \omega^2 q^{(0)} \approx 2A\dot{\varphi}\omega \cos(\omega t - \varphi) - 2\dot{A}\omega \sin(\omega t - \varphi). \tag{5.55}$$

On the right-hand side of Eq. (5.53), we can neglect the time derivatives of the amplitude and phase, because this part is already proportional to the small parameter. Hence, in the first order in ε, Eq. (5.38) becomes

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} = f_{\text{ef}}^{(0)} \equiv f^{(0)} - (2A\dot{\varphi}\omega \cos \Psi - 2\dot{A}\omega \sin \Psi). \tag{5.56}$$

Now, applying Eq. (5.52) to the function $f_{\text{ef}}^{(0)}$, and taking into account that the time averages of $\sin^2 \Psi$ and $\cos^2 \Psi$ are both equal to 1/2, while the time average of the product $\sin\Psi \cos\Psi$ vanishes, we get a pair of so-called *reduced equations* (alternatively called the 'RWA equations' or the 'van der Pol equations') for the time evolution of the amplitude and the phase:

$$\dot{A} = -\frac{1}{\omega} \overline{f^{(0)} \sin \Psi}, \qquad \dot{\varphi} = \frac{1}{\omega A} \overline{f^{(0)} \cos \Psi}. \tag{5.57a}$$

Extending the definition Eq. (5.4) of the complex amplitude of oscillations to their slow evolution in time, $a(t) \equiv A(t) \exp\{i\varphi(t)\}$, and differentiating this relation, the two equations (5.57*a*) may be also rewritten in the form of either one equation for a:

$$\dot{a} = \frac{i}{\omega}\overline{f^{(0)}e^{i(\Psi+\varphi)}} \equiv \frac{i}{\omega}\overline{f^{(0)}e^{i\omega t}}, \tag{5.57b}$$

or two equations for the real and imaginary parts of $a(t) = u(t) + iv(t)$:

$$\dot{u} = -\frac{1}{\omega}\overline{f^{(0)}\sin\omega t}, \quad \dot{v} = \frac{1}{\omega}\overline{f^{(0)}\cos\omega t}. \tag{5.57c}$$

The first-order harmonic balance Eq. (5.52) are evidently just the particular case of the reduced equations (5.57) for stationary oscillations ($\dot{A} = \dot{\varphi} = 0$)[21].

Superficially, the system (5.57*a*) of two coupled, first-order differential equations may look more complex than the initial, second-order differential Eq. (5.38), but actually it is usually much simpler. For example, let us spell them out for the easy case of free oscillations a linear oscillator with damping. For that, we may reuse the ready Eq. (5.46) by taking $\alpha = f_0 = 0$, and thus turning Eq. (5.57*a*) into

$$\dot{A} = -\frac{1}{\omega}\overline{f^{(0)}\sin\Psi} \equiv -\frac{1}{\omega}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\sin\Psi} \equiv -\delta A, \tag{5.58a}$$

$$\dot{\varphi} = \frac{1}{\omega A}\overline{f^{(0)}\cos\Psi} \equiv \frac{1}{\omega A}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\cos\Psi} \equiv \xi. \tag{5.58b}$$

The solution of Eq. (5.58*a*) gives us the same 'envelope' law $A(t) = A(0)e^{-\delta t}$ as the exact solution (5.10) of the initial differential equation, while the elementary integration of Eq. (5.58*b*) yields $\varphi(t) = \xi t + \varphi(0) = \omega t - \omega_0 t + \varphi(0)$. This means that our approximate solution,

$$q^{(0)}(t) = A(t)\cos[\omega t - \varphi(t)] = A(0)e^{-\delta t}\cos[\omega_0 t - \varphi(0)], \tag{5.59}$$

agrees with the exact Eq. (5.9), and misses only the correction (5.8) of the oscillation frequency. (This correction is of the second order in δ, i.e. of the order of ε^2, and hence beyond the accuracy of our first approximation.) It is remarkable how nicely the reduced equations recover the proper frequency of free oscillations in this autonomous system—in which the very notion of ω is ambiguous.

The result is different at forced oscillations. For example, for the (generally nonlinear) Duffing oscillator described by Eq. (5.43) with $f_0 \neq 0$, Eqs. (5.57*a*) yield the reduced equations

$$\dot{A} = -\delta A + \frac{f_0}{2\omega}\sin\varphi, \quad A\dot{\varphi} = \xi(A)A + \frac{f_0}{2\omega}\cos\varphi, \tag{5.60}$$

which are valid for an arbitrary function $\xi(A)$, provided that the nonlinear detuning remains much smaller than the oscillation frequency. Here (after a transient), the

[21] One may ask why we cannot stick to just the one, most compact, complex-amplitude form (5.57*b*) of the reduced equations. The main reason is that when the function $f(q, \dot{q}, t)$ is nonlinear, we cannot replace its real arguments, such as $q = A\cos(\omega t - \varphi)$, with their complex-function representations such as $a\exp\{-i\omega t\}$ (as could be done in the linear problems considered in section 4.1), and we need to use real variables, such as either $\{A, \varphi\}$ or $\{u, v\}$, anyway.

amplitude and phase tend to the stationary states described by Eq. (5.47). This means that φ becomes a constant, so that $q^{(0)} \to A\cos(\omega t - \text{const})$, i.e. the reduced equations again automatically recover the correct frequency of the solution, in this case equal to that of the external force.

Note that each stationary oscillation regime, with a certain amplitude and phase, corresponds to a fixed point of the reduced equations, so that the stability of those fixed points determines that of the oscillations. In what follows, we will carry out such an analysis for several simple systems of key importance for physics and engineering.

5.4 Self-oscillations and phase locking

The motivation for B van der Pol to develop his method was the analysis of one more type of oscillatory motion: *self-oscillations*. Several systems, e.g. electronic radio-frequency (RF) amplifiers with positive feedback, and optical media with quantum level population inversion, provide convenient means for the compensation, and even over-compensation, of the intrinsic energy losses in oscillators. Phenomenologically, this effect may be described as the change of sign of the damping coefficient δ from positive to negative. Since for small oscillations the equation of motion is still linear, we may use Eq. (5.9) to describe its general solution. This equation shows that at $\delta < 0$, even infinitesimal deviations from equilibrium (say, due to unavoidable fluctuations) lead to oscillations with exponentially growing amplitude. Of course, in any real system such a growth cannot persist infinitely, and shall be limited by this or that effect—e.g. in the above examples, respectively, by amplifier's saturation and quantum level population's exhaustion.

In many cases, the amplitude limitation may be described reasonably well by making the following replacement:

$$2\delta\dot{q} \to 2\delta\dot{q} + \beta\dot{q}^3, \tag{5.61}$$

with $\beta > 0$. Let us analyze the effects of such *nonlinear damping*, applying the van der Pol's approach to the corresponding homogeneous differential equation (which also carries his name):

$$\ddot{q} + 2\delta\dot{q} + \beta\dot{q}^3 + \omega_0^2 q = 0. \tag{5.62}$$

Carrying out the dissipative and detuning terms to the right-hand side, and taking them for f in the canonical Eq. (5.38), we can easily calculate the right-hand sides of the reduced equations (5.57*a*), obtaining[22]

$$\dot{A} = -\delta(A)\,A, \qquad \text{where } \delta(A) \equiv \delta + \frac{3}{8}\beta\omega^2 A^2, \tag{5.63a}$$

$$A\dot{\varphi} = \xi\,A. \tag{5.63b}$$

[22] For that, one needs to use the trigonometric identity $\sin^3\Psi = (3/4)\sin\Psi - (1/4)\sin 3\Psi$—see, e.g. Eq. (A.19).

The second of these equations has exactly the same form as Eq. (5.58*b*) for the case of decaying oscillations and hence shows that the self-oscillations (if they happen, i.e. if $A \neq 0$) have the own frequency ω_0 of the oscillator—see Eq. (5.59). Equation (5.63*a*) is more substantive. If the initial damping δ is positive, it has only the trivial fixed point, $A_0 = 0$ (that describes the oscillator at rest), but if δ is negative, there is also another fixed point,

$$A_1 = \left(\frac{8 \mid \delta \mid}{3\beta\omega^2} \right)^{1/2}, \qquad \text{for } \delta < 0, \tag{5.64}$$

which describes steady self-oscillations with a non-zero amplitude A_1.

Let us apply the general approach discussed in section 3.2, the linearization of equations of motion, to this reduced equation. For the trivial fixed point $A_0 = 0$, the linearization of Eq. (5.63*a*) is reduced to discarding the nonlinear term in the definition of the amplitude-dependent damping $\delta(A)$. The resulting linear equation evidently shows that the system's equilibrium point, $A = A_0 = 0$, is stable at $\delta > 0$ and unstable at $\delta < 0$. (We have already discussed this *self-excitation condition* above.) The linearization near the non-trivial fixed point A_1 requires a bit more math: in the first order in $\tilde{A} \equiv A - A_1 \to 0$, we obtain

$$\begin{aligned} \dot{\tilde{A}} \equiv \dot{A} &= -\delta(A_1 + \tilde{A}) - \frac{3}{8}\beta\omega^2(A_1 + \tilde{A})^3 \approx -\delta\,\tilde{A} - \frac{3}{8}\beta\omega^2 3A_1^2\,\tilde{A} \\ &= (-\delta + 3\delta)\,\tilde{A} = 2\delta\tilde{A}, \end{aligned} \tag{5.65}$$

where Eq. (5.64) has been used to eliminate A_1. We see that the fixed point A_1 (and hence the whole process) is stable as soon as it exists ($\delta < 0$)—similar to the situation in our 'testbed problem' (figure 2.1), except that in our current, dissipative system, the stability is 'actual' rather than 'orbital'—see section 5.6 for more.

Now let us consider another important problem: the effect of an external sinusoidal force on a self-excited oscillator. If the force is sufficiently small, its effects on the self-excitation condition and the oscillation amplitude are negligible. However, if the frequency ω of such a weak force is close to the own frequency ω_0 of the oscillator, it may lead to a very important effect of *phase locking*[23]—also called the 'synchronization', although the latter term also has a much broader meaning. At this effect, the oscillator's frequency deviates from ω_0, and becomes exactly equal to the external force's frequency ω, within a certain range

$$-\Delta \leqslant \omega - \omega_0 < +\Delta. \tag{5.66}$$

In order to prove this fact, and also to calculate the phase-locking range width 2Δ, we may repeat the calculation of the right-hand sides of the reduced equations (5.57*a*), adding term $f_0 \cos \omega t$ to the right-hand side of Eq. (5.62)—see Eqs. (5.42)–(5.43). This addition modifies Eqs. (5.63) as follows[24]:

[23] Apparently, the mutual phase-locking of two pendulum clocks was first noticed by the same C Huygens.
[24] Actually, this result should be evident, even without calculations, from the comparison of Eqs. (5.60) and (5.63).

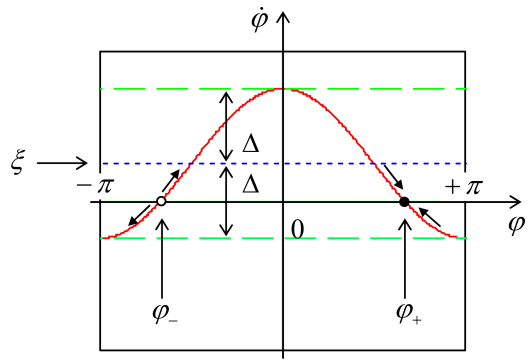


Figure 5.5. The phase plane of a phase-locked oscillator, for the particular case $\xi = \Delta/2$, $f_0 > 0$.

$$\dot{A} = -\delta(A)\, A + \frac{f_0}{2\omega} \sin\varphi, \tag{5.67a}$$

$$A\dot{\varphi} = \xi\, A + \frac{f_0}{2\omega} \cos\varphi. \tag{5.67b}$$

If the system is self-excited, and the external force is weak, its effect on the oscillation amplitude is small, and in the first approximation in f_0 we can take A to be constant and equal to the value A_1 given by Eq. (5.64). Plugging this approximation into Eq. (5.67*b*), we obtain a very simple equation[25]

$$\dot{\varphi} = \xi + \Delta \cos\varphi, \tag{5.68}$$

where in our current case

$$\Delta \equiv \frac{f_0}{2\omega A_1}. \tag{5.69}$$

Within the range $-|\Delta| < \xi < +|\Delta|$, Eq. (5.68) has two fixed points on each 2π-segment of the variable φ:

$$\varphi_\pm = \pm\cos^{-1}\left(-\frac{\xi}{\Delta}\right) + 2\pi n. \tag{5.70}$$

It is easy to linearize Eq. (5.68) near each point to analyze their stability in our usual way; however, let me use this case to demonstrate another convenient way to achieve this in 1D systems, using the so-called *phase plane*—the plot of the right-hand side of Eq. (5.68) as a function of φ (see figure 5.5).

Since the positive values of this function correspond to the growth of φ in time, and vice versa, we may draw the arrows showing the direction of phase evolution. From these graphics, it is clear that one of these fixed points (for $f_0 > 0$, φ_+) is stable,

[25] This equation is ubiquitous in phase-locking systems, including even some digital electronic circuits used for that purpose.

while its counterpart (in this case, φ_-) is unstable. Hence the magnitude of Δ given by Eq. (5.69) is indeed the phase-locking range (or rather it half) that we wanted to find. Note that the range is proportional to the amplitude of the phase-locking signal—perhaps the most important feature of this effect.

In order to complete our simple analysis, based on the assumption of fixed oscillation amplitude, we need to find the condition of its validity. For that, we may linearize Eq. (5.67*a*), for the stationary case, near the value A_1, just as we have done in Eq. (5.65) for the transient process. The stationary result,

$$\tilde{A} \equiv A - A_1 = \frac{1}{2\,|\,\delta\,|}\frac{f_0}{2\omega}\sin\varphi_\pm \approx A_1 \left|\frac{\Delta}{2\delta}\right| \sin\varphi_\pm, \tag{5.71}$$

shows that our assumption, $|\tilde{A}| \ll A_1$, and hence the final result (5.69), are valid if the phase-locking range, 2Δ, is much smaller than $4\,|\delta\,|$.

5.5 Parametric excitation

In both problems solved in the last section, the stability analysis was easy because it could be carried out for just one slow variable, *either* amplitude *or* phase. More generally, such analysis of the reduced equations involves both these variables. The classical example of such a situation is provided by one important physical phenomenon—the *parametric excitation* of oscillations. An elementary example of such excitation is given by a pendulum with a variable parameter, for example the suspension length $l(t)$—see figure 5.6. Experiments (including those with playground swings) and numerical simulations show that if the length is changed (*modulated*) periodically, with a frequency 2ω that is close to $2\omega_0$ and a sufficiently large swing Δl, the equilibrium position of the pendulum becomes unstable and it starts swinging with frequency ω equal to *exactly* half of the modulation frequency (and hence only *approximately* equal to the average eigenfrequency ω_0 of the oscillator).

For an elementary analysis of this effect we may consider the simplest case when the oscillations are small. At the lowest point ($\theta = 0$), where the pendulum moves with the highest velocity $v_{\max}$, the string's tension $\mathcal{T}$ is *higher* than mg by the centripetal force: $\mathcal{T}_{\max} = mg + mv_{\max}^2/l$. In contrast, at the maximum deviation of the pendulum from the equilibrium, the force is *lower* than mg, because of

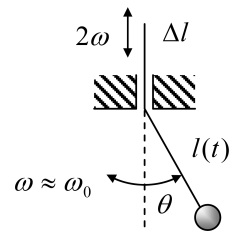


Figure 5.6. Parametric excitation of a pendulum.

the suspension line's tilt: $\mathcal{T}_{\min} = mg\cos\theta_{\max}$. Using the energy conservation, $E = mv_{\max}^2/2 = mgl(1-\cos\theta_{\max})$, we may express these values as $\mathcal{T}_{\max} = mg + 2E/l$ and $\mathcal{T}_{\min} = mg - E/l$. Now, if during each oscillation period the string is pulled up slightly by Δl (with $|\Delta l| \ll l$) at each of its two passages through the lowest point, and is allowed to go down by the same amount at each of two points of the maximum deviation, the net work of the external force per period is positive:

$$\mathcal{W} \approx 2(\mathcal{T}_{\max} - \mathcal{T}_{\min})\Delta l \approx 6\frac{\Delta l}{l}E, \tag{5.72}$$

and hence results in an increase of the oscillator's energy. If the so-called *modulation depth* $\Delta l/2l$ is sufficient, this increase may overcompensate the energy drained out by damping. Quantitatively, Eq. (5.10) shows that low damping ($\delta \ll \omega_0$) leads to the following energy decrease,

$$\Delta E \approx -4\pi\frac{\delta}{\omega_0}E, \tag{5.73}$$

per oscillation period. Comparing Eqs. (5.72) and (5.73), we see that the net energy flow into the oscillations is positive, $\mathcal{W} + \Delta E > 0$, i.e. oscillation amplitude has to grow, if[26]

$$\frac{\Delta l}{l} > \frac{2\pi\delta}{3\omega_0} \equiv \frac{\pi}{3Q}. \tag{5.74}$$

Since this result is independent of the oscillation energy E, the growth of energy and amplitude is exponential (until E becomes so large that some of our assumptions fails), so that Eq. (5.74) is the condition of parametric excitation—in this simple model.

However, this result does not account for a possible difference between the oscillation frequency ω and the eigenfrequency ω_0, and also does not clarify whether the best phase shift between the oscillations and parameter modulation, assumed in the above calculation, may be sustained automatically. In order to address these issues, we may apply the van der Pol approach to a simple but reasonable model:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2(1 + \mu\cos 2\omega t)q = 0, \tag{5.75}$$

describing the parametric excitation in a linear oscillator with a sinusoidal modulation of the parameter $\omega_0^2(t)$. Rewriting this equation in the canonical form (5.38),

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}) \equiv -2\delta\dot{q} + 2\xi\omega q - \mu\omega_0^2 q\cos 2\omega t, \tag{5.76}$$

[26] A modulation of a pendulum's mass (say, by periodic pumping water in and out of a suspended bottle) gives a qualitatively similar result. Note, however, that parametric oscillations cannot be excited by modulating *any* oscillator's parameter—for example, oscillator's damping coefficient (at least if it stays positive at all times), because it does not change the system's energy, just the energy drain rate.

and assuming that the dimensionless ratios δ/ω and $|\xi|/\omega$, and the modulation depth μ are all much less than 1, we may use the general Eqs. (5.57*a*) to obtain the following reduced equations:

$$\begin{aligned} \dot{A} &= -\delta\, A - \frac{\mu\omega}{4} A \sin 2\varphi, \\ A\dot{\varphi} &= A\xi\ - \frac{\mu\omega}{4} A \cos 2\varphi. \end{aligned} \tag{5.77}$$

These equations evidently have a fixed point, with $A_0 = 0$, but its stability analysis (though possible) is not absolutely straightforward, because the phase φ of oscillations is undetermined at that point. In order to avoid this (technical rather than conceptual) difficulty, we may use, instead of the real amplitude and phase of oscillations, either their complex amplitude $a = A \exp\{i\varphi\}$, or its Cartesian components u and v—see Eq. (5.4). Indeed, for our function f, Eq. (5.57*b*) gives

$$\dot{a} = (-\delta + i\xi)a - i\frac{\mu\omega}{4} a^*, \tag{5.78}$$

while Eqs. (5.57*c*) yield

$$\begin{aligned} \dot{u} &= -\delta u - \xi v - \frac{\mu\omega}{4} v, \\ \dot{v} &= -\delta v + \xi u - \frac{\mu\omega}{4} u. \end{aligned} \tag{5.79}$$

We see that in contrast to Eqs. (5.77), in the Cartesian coordinates $\{u, v\}$ the trivial fixed point $a_0 = 0$ (i.e. $u_0 = v_0 = 0$) is absolutely regular. Moreover, Eqs. (5.78) and (5.79) are already linear, so they do not require any additional linearization. Thus we may use the same approach as was already used in sections 3.2 and 5.1, i.e. look for the solution of Eq. (5.79) in the exponential form $\exp\{\lambda t\}$. However, now we are dealing with two variables, and should allow them to have, for each value of λ, a certain ratio u/v. For that, we may take the partial solution in the form

$$u = c_u e^{\lambda t}, \quad v = c_v e^{\lambda t}, \tag{5.80}$$

where constants c_u and c_v are frequently called the *distribution coefficients.* Plugging this solution into Eqs. (5.79), we obtain for them the following system of two linear algebraic equations:

$$\begin{aligned} (-\delta - \lambda)c_u + \left(-\xi - \frac{\mu\omega}{4}\right)c_v &= 0, \\ \left(\xi - \frac{\mu\omega}{4}\right)c_u + (-\delta - \lambda)c_v &= 0. \end{aligned} \tag{5.81}$$

The characteristic equation of this system, i.e. the condition of compatibility of Eqs. (5.81),

$$\begin{vmatrix} -\delta-\lambda & -\xi-\dfrac{\mu\omega}{4} \\ \xi-\dfrac{\mu\omega}{4} & -\delta-\lambda \end{vmatrix} \equiv \lambda^2 + 2\delta\lambda + \delta^2 + \xi^2 - \left(\frac{\mu\omega}{4}\right)^2 = 0, \tag{5.82}$$

has two roots:

$$\lambda_\pm = -\delta \pm \left[\left(\frac{\mu\omega}{4}\right)^2 - \xi^2\right]^{1/2}. \tag{5.83}$$

Requiring the fixed point to be unstable, $\mathrm{Re}\,\lambda_+ > 0$, we get the parametric excitation condition

$$\frac{\mu\omega}{4} > (\delta^2 + \xi^2)^{1/2}. \tag{5.84}$$

Thus the parametric excitation may indeed happen without any artificial phase control: the arising oscillations self-adjust their phase to pick up energy from the external source responsible for the parameter variation.

Our key result (5.84) may be compared with two other calculations. First, in the case of negligible damping ($\delta = 0$), Eq. (5.84) turns into condition $\mu\omega/4 > |\xi|$. This result may be compared with the well-developed theory of the so-called *Mathieu equation*, whose canonical form is

$$\frac{d^2y}{dv^2} + (a - 2b\cos 2v)y = 0. \tag{5.85}$$

With the substitutions $y \to q$, $v \to \omega t$, $a \to (\omega_0/\omega)^2$, $b \to -\mu/2$, this equation is just a particular case of Eq. (5.75) for $\delta = 0$. In terms of Eq. (5.85), the result of our analysis may be re-written just as $b > |a - 1|$, and is supposed to be valid for $b \ll 1$. This condition is shown in figure 5.7 together with the numerically calculated[27] stability boundaries of the Mathieu equation.

One can see that the van der Pol approximation works just fine within its applicability limit (and a bit beyond :-), although it fails to predict some other features of the Mathieu equation, such as the existence of higher, more narrow regions of parametric excitation (at $a \approx n^2$, i.e. $\omega_0 \approx \omega/n$, for all integer n), and some spill-over of the stability region into the lower half-plane $a < 0$.[28] The reason for these failures is the fact that, as can be seen in figure 5.7, these phenomena do not appear in the first approximation in the parameter modulation amplitude $\mu \propto \varepsilon$, that is the realm of applicability of the reduced equation (5.79).

[27] Such calculations may be substantially simplified by the use of the so-called *Floquet theorem*, which is also the mathematical basis for the discussion of wave propagation in periodic media—see the next chapter.

[28] This region describes, for example, the counter-intuitive stability of an inverted pendulum with the periodically modulated length, within a limited range of the modulation depth μ.

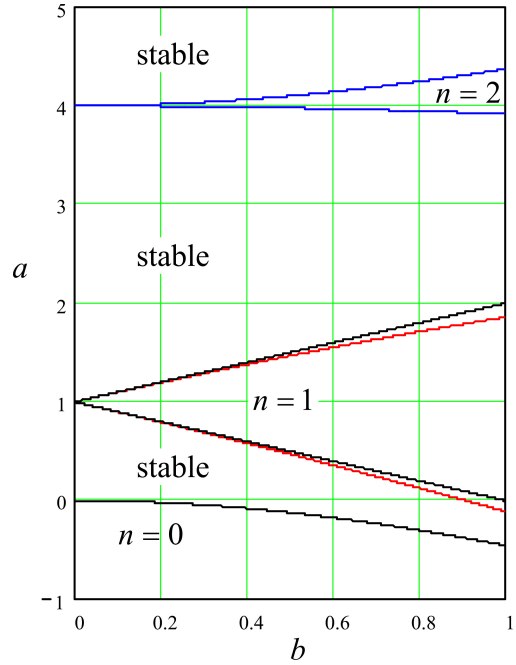


Figure 5.7. Stability boundaries of the Mathieu equation (5.85), as calculated: numerically (curves) and using the reduced equations (5.79) (dashed straight lines). In the regions numbered by various n, the trivial solution $y = 0$ of the equation is unstable, i.e. its general solution $y(v)$ includes an exponentially growing term.

In the opposite case of non-zero damping but exact tuning ($\xi = 0$, $\omega \approx \omega_0$), Eq. (5.84) gives

$$\mu > \frac{4\delta}{\omega_0} \equiv \frac{2}{Q}. \tag{5.86}$$

This condition may be compared with Eq. (5.74), taking $\Delta l/l = 2\mu$. The comparison shows that while the structure of these conditions is similar, the numerical coefficients are different by a factor close to 2. The first reason for this difference is that the instant parameter change at optimal moments of time is more efficient then the smooth, sinusoidal variation described by Eq. (5.75). Even more significantly, the change of the pendulum's length modulates not only its frequency $\omega_0 \equiv (g/l)^{1/2}$ as Eq. (5.75) implies, but also its *mechanical impedance* $Z \equiv (gl)^{1/2}$—the notion to be discussed in detail in the next chapter. (The analysis of the general case of the simultaneous modulation of ω_0 and Z is left for a reader exercise.)

Before moving on, let me summarize the most important differences between the parametric and forced oscillations:

(i) Parametric oscillations completely disappear outside of their excitation range, while the forced oscillations have a non-zero amplitude for any frequency and amplitude of the external force—see Eq. (5.18).

(ii) Parametric excitation may be described by a linear *homogeneous* equation—e.g. Eq. (5.75)—which cannot predict any finite oscillation amplitude within the excitation range, even at finite damping. In order to describe stationary parametric oscillations, some nonlinear effect has to be taken into account. (Again, I am leaving analyses of such effects for a reader exercise.)

One more important feature of the parametric oscillations will be discussed at the end of the next section.

5.6 Fixed-point classification

The reduced equations (5.79) give us a good pretext for a brief discussion of an important general topic of dynamics: fixed points of a system described by two time-independent, first-order differential equations with time-independent coefficients[29]. After their linearization near a fixed point, the equations for deviations can always be expressed in the form similar to Eq. (5.79):

$$\begin{aligned} \dot{\tilde{q}}_1 &= M_{11}\tilde{q}_1 + M_{12}\tilde{q}_2, \\ \dot{\tilde{q}}_2 &= M_{21}\tilde{q}_1 + M_{22}\tilde{q}_2, \end{aligned} \tag{5.87}$$

where $M_{jj'}$ (with $j, j' = 1, 2$) are some real scalars, which may be viewed upon as the elements of a 2×2 matrix M. Looking for an exponential solution of the type (5.80),

$$\tilde{q}_1 = c_1 e^{\lambda t}, \quad \tilde{q}_2 = c_2 e^{\lambda t}, \tag{5.88}$$

we obtain a more general system of two linear equations for the distribution coefficients $c_{1,2}$:

$$\begin{aligned} (M_{11} - \lambda)c_1 + M_{12}c_2 &= 0, \\ M_{21}c_1 + (M_{22} - \lambda)c_2 &= 0. \end{aligned} \tag{5.89}$$

These equations are consistent if

$$\begin{vmatrix} M_{11} - \lambda & M_{12} \\ M_{21} & M_{22} - \lambda \end{vmatrix} = 0, \tag{5.90}$$

giving us a quadratic characteristic equation

$$\lambda^2 - \lambda(M_{11} + M_{22}) + (M_{11}M_{22} - M_{12}M_{21}) = 0. \tag{5.91}$$

Its solution[30],

[29] Autonomous systems described by a single, second-order homogeneous differential equation, say $F(q, \dot{q}, \ddot{q}) = 0$, also belong to this class, because we may always treat the generalized velocity $\dot{q} \equiv v$ as a new variable, and use this definition as one first-order differential equation, and the initial equation, in the form $F(q, v, \dot{v}) = 0$, as the second first-order equation.

[30] In the terms of linear algebra, $\lambda_\pm$ are the *eigenvalues*, and the corresponding sets of the distribution coefficients $[c_1, c_2]_\pm$, the *eigenvectors* of the matrix M with elements $M_{jj'}$.

$$\lambda_{\pm} = \frac{1}{2}(M_{11} + M_{22}) \pm \frac{1}{2}[(M_{11} - M_{22})^2 + 4M_{12}M_{21}]^{1/2}, \tag{5.92}$$

shows that the following situations are possible:

A. The expression under the square root, $(M_{11} - M_{22})^2 + 4M_{12}M_{21}$, is positive. In this case, both characteristic exponents $\lambda_{\pm}$ are real, and we can distinguish three sub-cases:

(i) Both λ_+ and λ_- are negative. As Eq. (5.88) shows, in this case the deviations $\tilde{q}$ tend to zero at $t \to \infty$, i.e. the fixed point is stable. Because of generally different magnitudes of exponents $\lambda_{\pm}$, the process represented on the phase plane $[\tilde{q}_1, \tilde{q}_2]$ (see figure 5.8a, with the solid arrows, for an example) may be

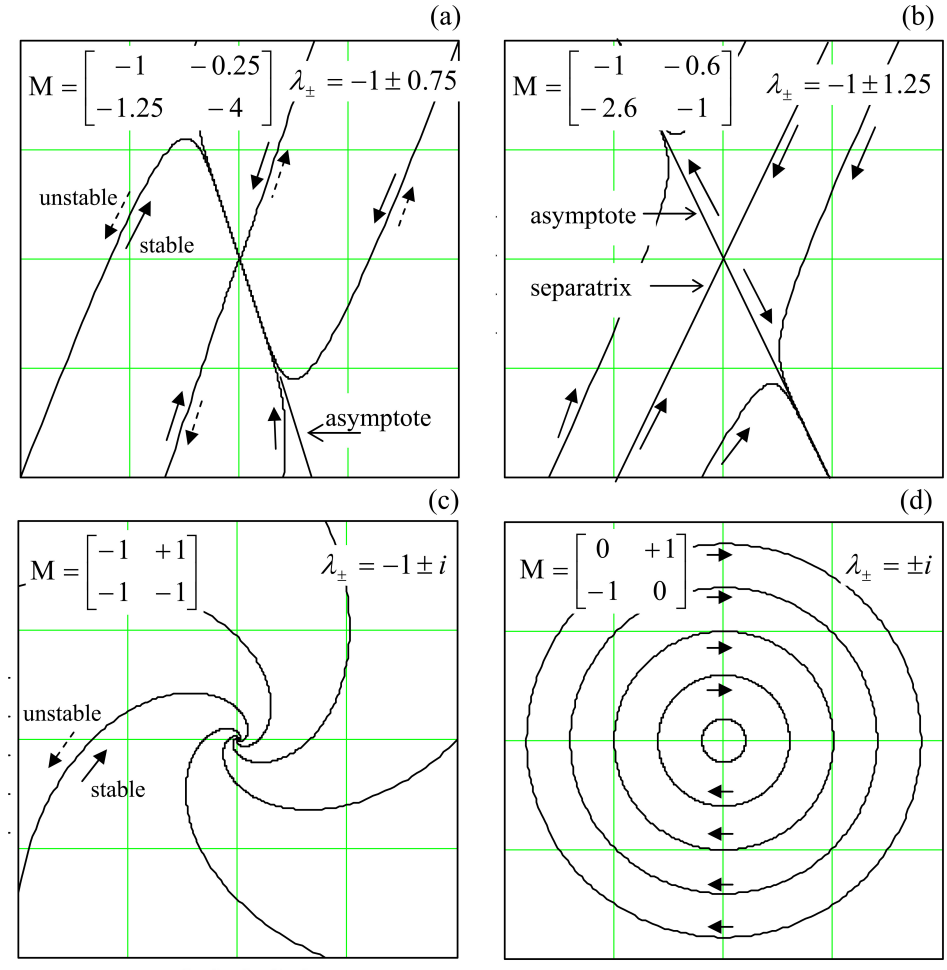


Figure 5.8. Typical trajectories on the phase-plane $[\tilde{q}_1, \tilde{q}_2]$ near fixed points of different types: (a) node, (b) saddle, (c) focus, and (d) center. The particular values of the matrix M, used in the first three panels, correspond to Eq. (5.81) for the parametric excitation, with $\xi = \delta$, and three different values of the parameter $\mu\omega/4\delta$: (a) 1.25, (b) 1.6, and (c) 0.

seen as consisting of two stages: first, a faster (with the rate $|\lambda_-| > |\lambda_+|$) relaxation to a linear *asymptote*[31], and then a slower decline, with the rate $|\lambda_+|$, along this line, i.e. at the virtually fixed ratio of the variables. Such a fixed point is called the *stable node*.

(ii) Both λ_+ and λ_- are positive. This case of an *unstable node* differs from the previous one only by the direction of motion along the phase-plane trajectories—see the dashed arrows in figure 5.8a. Here the variable ratio is also approaching a constant soon, now the one corresponding to $\lambda+ > \lambda_-$.

(iii) Finally, in the case of a *saddle* ($\lambda_+ > 0$, $\lambda_- < 0$) the system dynamics is different (figure 5.8b): after the rate $|\lambda_-|$ relaxation to the asymptote, the perturbation starts to grow, with the rate λ_+, along one of two opposite directions. (The direction is determined on which side of another straight line, called the *separatrix*, the system has been initially.) So the saddle[32] is an unstable fixed point.

B. The expression under the square root in Eq. (5.92), $(M_{11} - M_{22})^2 + 4M_{12}M_{21}$, is negative. In this case the square root is imaginary, making the real parts of both roots equal, $\mathrm{Re}\,\lambda_\pm = (M_{11} + M_{22})/2$, and their imaginary parts equal but opposite. As a result, here there can be just two types of fixed points:

(i) *Stable focus*, at $(M_{11} + M_{22}) < 0$. The phase-plane trajectories are spirals going to the origin (i.e. toward the fixed point)—see figure 5.8c with the solid arrow.

(ii) *Unstable focus*, taking place at $(M_{11} + M_{22}) > 0$, differs from the stable focus only by the direction of motion along the phase trajectories—see the dashed arrow in the same figure 5.8c.

C. Frequently, the border case, $M_{11} + M_{22} = 0$, corresponding to the orbital ('indifferent') stability already discussed in section 3.2, is also distinguished, and the corresponding fixed point is referred to as the *center*—see figure 5.8d. Considering centers as a separate category makes sense because such fixed points are typical for Hamiltonian systems, whose first integral of motion may be frequently represented as the distance of the phase point from a fixed point. For example, introducing new variables $\tilde{q}_1 \equiv \tilde{q}$, $\tilde{q}_2 \equiv m\dot{\tilde{q}}_1$, we may rewrite Eq. (3.12) of a harmonic oscillator without dissipation (again, with indices 'ef' dropped for brevity), as a system of two first-order differential equations:

$$\dot{\tilde{q}}_1 = \frac{1}{m}\tilde{q}_2, \quad \dot{\tilde{q}}_2 = -\kappa\tilde{q}_1, \tag{5.93}$$

i.e. as a particular case of Eq. (5.87), with $M_{11} = M_{22} = 0$, and $M_{12}M_{21} = -\kappa/m \equiv -\omega_0{}^2 < 0$, and hence $(M_{11} - M_{22})^2 + 4M_{12}M_{21} = -4\omega_0{}^2 < 0$, and $M_{11} + M_{22} = 0$. On

[31] The asymptote direction may be found by plugging the value λ_+ back into Eq. (5.89) and finding the corresponding ratio c_1/c_2. Note that the separation of the system's evolution into the two stages is conditional, being most vivid in the case of a large difference between the exponents λ_+ and λ_-.

[32] The term 'saddle' is due to the fact that system's dynamics in this case is qualitatively similar to those of a viscous motion in the 2D potential $U(\tilde{q}_1, \tilde{q}_2)$ having the shape of a horse saddle (or a mountain pass).

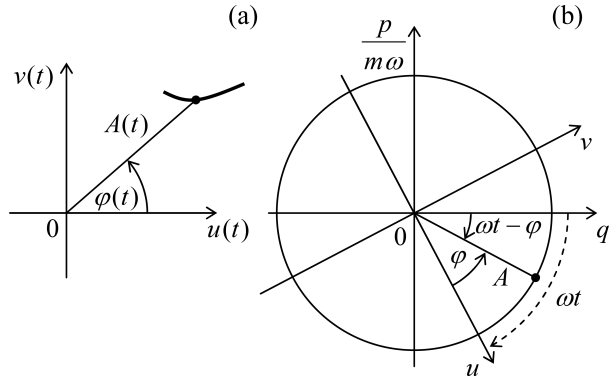


Figure 5.9. (a) Representation of a sinusoidal oscillation (point) and a slow transient process (line) on the Poincaré plane and (b) the relation between 'fast' phase plane to the 'slow' (Poincaré) plane.

the symmetrized phase plane $[\tilde{q}_1,\ \tilde{q}_2/Z]$, where the parameter $Z \equiv (\kappa m)^{1/2} \equiv m\omega_0$ is the oscillator's impedance, the sinusoidal oscillations of amplitude A are represented by a circle of radius A about the center-type fixed point $A = 0$. In the case when $\tilde{q}_1 \equiv \tilde{q}$ is the linear coordinate q of an actual mechanical oscillator, so that $\tilde{q}_2 \equiv m\dot{\tilde{q}}_1$ is its linear momentum $p = m\dot{q}$, such a circular trajectory corresponds to the conservation of the oscillator's energy

$$E \equiv T + U \equiv \frac{p^2}{2m} + \frac{\kappa q^2}{2} \equiv \frac{\kappa}{2}\left[\tilde{q}_1^2 + \left(\frac{\tilde{q}_2}{Z}\right)^2\right] = \frac{\kappa A^2}{2} = \text{const}. \tag{5.94}$$

This is a convenient moment for a brief discussion of the so-called *Poincaré* (or 'slow-variable', or 'stroboscopic') *plane*[33]. From the point of view of the basic Eq. (5.41), the sinusoidal oscillations $q(t) = A\cos(\omega t - \varphi)$, described by a circular trajectory on the actual (symmetrized) phase plane, correspond to a fixed point $\{A, \varphi\}$, which may conveniently be represented by a steady geometric point on a plane with these polar coordinates—see figure 5.9a. (As follows from Eq. (5.4), the Cartesian coordinates of the point on that plane are just the variables $u \equiv A\cos\varphi$ and $v \equiv A\cos\varphi$ used, in particular, in the last section.) The quasi-sinusoidal process (5.41), with slowly changing A and φ, may be represented by a slow motion of that point on this Poincaré plane.

Figure 5.9b shows a convenient way to visualize the relation between the actual phase plane of an oscillator, with the 'fast' symmetrized coordinates q and p/Z, and the Poincaré plane with 'slow' coordinates u and v: the latter plane rotates relative to the former one, about the origin, clockwise, with the angular velocity ω[34]. Another, 'stroboscopic' way to generate the Poincaré plane pattern is to have a fast glance at the 'real' phase-plane just once during the oscillation period $\mathcal{T} = 2\pi/\omega$.

[33] Named after J H Poincaré (1854–1912), who is credited, among many other achievements, with his contributions to special relativity (see, e.g. *Part EM* chapter 9), and the basic idea of unstable trajectories, responsible for the deterministic chaos—to be discussed in chapter 9 of this volume.

[34] This notion of phase-plane rotation is the origin of the term 'rotating wave approximation', RTA. (The word 'wave' is an artifact of this method's wide application in classical and quantum optics.)

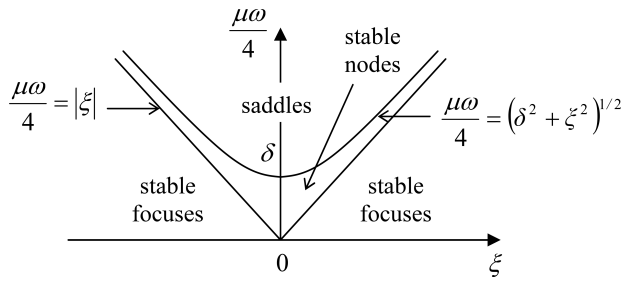


Figure 5.10. The types of the trivial fixed point of a parametric oscillator.

In many cases, the representation on the Poincaré plane is more convenient than that on the 'real' phase-plane. In particular, we have already seen that the reduced equations for such important phenomena as the phase-locking and the parametric oscillations, whose original differential equations include time explicitly, are time-independent—see, e.g. Eqs. (5.75) and (5.79) describing the latter effect. This simplification brings the equations into the category considered in this section, and enables the classification of their fixed points, which may shed additional light on their dynamic properties.

In particular, figure 5.10 shows the classification of the fixed points of a parametric oscillator, which follows from equation (5.83). As the parameter modulation depth μ is increased, the type of the trivial fixed point $A_1 = 0$ on the Poincaré plane changes from a stable focus (pertinent to a simple oscillator with damping) to a stable node and then to a saddle describing the parametric excitation. In the last case, the two directions of the perturbation growth, so prominently featured in figure 5.8b, correspond to the two possible values of the oscillation phase φ, with the phase choice determined by initial conditions.

This double degeneracy of the parametric oscillation's phase could already be noticed from Eq. (5.77), because they are evidently invariant with respect to the replacement $\varphi \to \varphi + \pi$. Moreover, the degeneracy is not an artifact of the reduced equations, because the initial equation (5.75) is already invariant with respect to the corresponding replacement $q(t) \to q(t - \pi/\omega)$. This invariance means that all other characteristics (e.g. the amplitude) of the parametric oscillations excited with either of two phases are *absolutely* similar. At the dawn of the computer age (in the late 1950s and early 1960s), there were substantial attempts, particularly in Japan, to use this property for storage and processing digital information coded in the phase-binary form. Though these attempts have not survived competition with simpler approaches, based on voltage-binary coding, some current trends in the development of prospective reversible and quantum computers may be traced back to this idea.

5.7 Numerical approaches

If the amplitude of oscillations, for whatever reason, becomes so large that the nonlinear terms in the equation describing the oscillator are comparable to its linear terms, numerical methods are virtually the only avenue available for their theoretical study. In Hamiltonian 1D systems, such methods may be applied directly to the

integral (3.26), but dissipative and/or parametric systems typically lack first integrals of motion similar to Eq. (3.24), so that the initial differential equation has to be solved.

Let us discuss the general idea of such methods on the example of what mathematicians call the *Cauchy problem* (finding the solution for all moments of time, starting from the known initial conditions) for first-order differential equation

$$\dot{q} = f(t, q). \tag{5.95}$$

(The generalization to a system of several such equations is straightforward.) Breaking the time axis into small, equal steps h (figure 5.11) we can reduce the equation integration problem to finding the function value at the next time point, $q_{n+1} \equiv q(t_{n+1}) \equiv q(t_n + h)$ from the previously found value $q_n = q(t_n)$—and, if necessary, the values of q at other previous time steps.

In the simplest approach (called the *Euler method*), q_{n+1} is found using the following formula:

$$\begin{aligned} q_{n+1} &= q_n + k, \\ k &\equiv h f(t_n, q_n). \end{aligned} \tag{5.96}$$

This approximation is equivalent to the replacement of the genuine function $q(t)$, on the segment $[t_n, t_{n+1}]$, with the two first terms of its Taylor expansion in point t_n:

$$q(t_n + h) \approx q(t_n) + \dot{q}(t_n)h \equiv q(t_n) + hf(t_n, q_n). \tag{5.97}$$

This approximation has an error proportional to h^2. One could argue that making the step h sufficiently small, the Euler method's error might be made arbitrary small, but even with the number-crunching power of modern computers, the computation time necessary to reach sufficient accuracy may be too high for large problems[35]. In addition, the increase of the number of time steps, which is necessary at $h \to 0$, increases the total rounding errors, and eventually may cause an increase, rather than the reduction of the overall error of the computed result.

A more efficient way is to modify Eq. (5.96) to include the terms of the second order in h. There are several ways to do this, for example using the *second-order Runge–Kutta* method:

$$\begin{aligned} q_{n+1} &= q_n + k_2, \\ k_2 &\equiv hf\left(t_n + \frac{h}{2}, q_n + \frac{k_1}{2}\right), \quad k_1 \equiv hf(t_n, q_n). \end{aligned} \tag{5.98}$$

One can readily check that this method gives the exact result if the function $q(t)$ is a quadratic polynomial, and hence in the general case its errors are of the order of h^3.

[35] In addition, the Euler method is not time-reversible—the handicap which may be essential for integration of Hamiltonian systems described by systems of second-order differential equations. However, this drawback may be readily overcome by the so-called *leapfrogging*—the overlap of time steps h for a generalized coordinate and the corresponding generalized velocity.

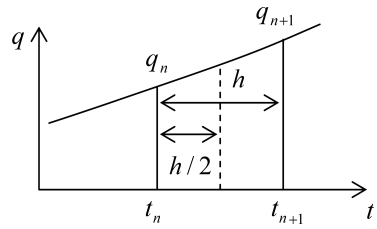


Figure 5.11. The basic notions used at numerical integration of ordinary differential equations.

We see that the main idea here is to first break the segment $[t_n, t_{n+1}]$ in half (see figure 5.11), then evaluate the right-hand side of the differential equation (5.95) at the point intermediate (in both t and q) between the points number n and $(n + 1)$, and then use this information to predict q_{n+1}.

The advantage of the Runge–Kutta approach is that it may be extended to the fourth order, without an additional breaking of the interval $[t_n, t_{n+1}]$:

$$\begin{aligned} q_{n+1} &= q_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ k_4 &\equiv hf\left(t_n + h,\, q_n + k_3\right), \quad k_3 \equiv hf\left(t_n + \frac{h}{2},\, q_n + \frac{k_2}{2}\right), \\ k_2 &\equiv hf\left(t_n + \frac{h}{2},\, q_n + \frac{k_1}{2}\right), \quad k_1 \equiv hf(t_n,\, q_n). \end{aligned} \tag{5.99}$$

This method reaches much lower error, $O(h^5)$, without being not too cumbersome. These features have made the fourth-order Runge–Kutta the default method in most numerical libraries. Its extension to higher orders is possible, but requires more complex formulas, and is justified only for some special cases, e.g. very abrupt functions $q(t)$[36]. The most frequent enhancement of the method is the automatic adjustment of the step h to reach the specified accuracy, but not make more calculations than necessary.

Figure 5.12 shows a typical example of an application of that method to the very simple problem of a damped linear oscillator, for two values of fixed time step h (expressed in terms of the number N of such steps per oscillation period). The black lines connect the points obtained by the fourth-order Runge–Kutta method, while the points connected with the green lines represent the exact analytical solution (5.22). Few-percent errors start to appear only at as few as ~10 time steps per period, so that the method is indeed very efficient.

Let me hope that the discussion in the next section will make the conveniences and the handicaps of the numerical approach to the solution of problems of nonlinear dynamics very clear.

[36] The most popular approaches in such cases are the *Richardson extrapolation*, the *Bulirsch–Stoer algorithm*, and a set of *prediction–correction techniques*, e.g. the *Adams–Bashforth–Moulton method*—see the literature recommended in appendix A, section A.16(iii).

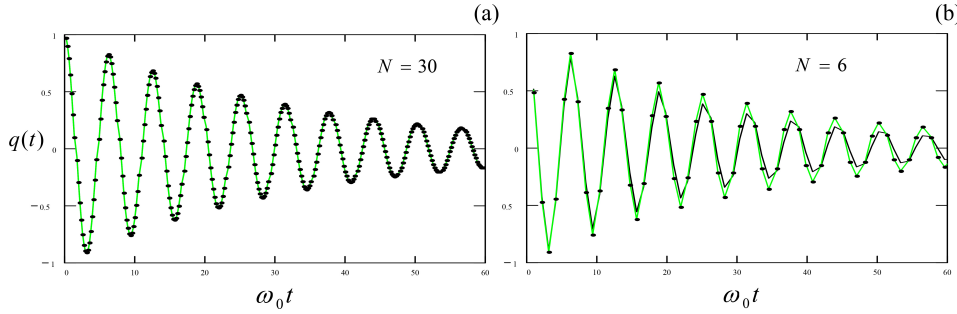


Figure 5.12. Results of the Runge–Kutta solution of Eq. (5.6) (with $\delta/\omega_0 = 0.03$) for: (a) 30 and (b) 6 points per oscillation period. The results are shown by points; the black and green lines are only the guides for the eye.

5.8 Higher harmonic and subharmonic oscillations

Figure 5.13 shows the numerically calculated[37] transient process and stationary oscillations in a linear oscillator and in a very representative nonlinear system, the pendulum described by Eq. (5.42), both with the same resonance frequency ω_0 for small oscillations. Both systems are driven by a sinusoidal external force of the same amplitude and frequency—in this illustration, equal to the small-oscillation own frequency ω_0 of both systems. The plots show that despite a very substantial amplitude of the pendulum oscillations (an angle amplitude of about one radian), their waveform remains almost exactly sinusoidal[38]. On the other hand, the nonlinearity affects the oscillation amplitude very substantially. These results imply that the corresponding reduced equation (5.60), which is based on the assumption (5.41), may work very well far beyond its formal restriction $|q| \ll 1$.

Still, the waveform of oscillations in a nonlinear system always differs from that of the applied force—in our case, from the sine function of frequency ω. This fact is frequently formulated as the generation, by the system, of *higher harmonics*. Indeed, the Fourier theorem tells us that any non-sinusoidal periodic function of time may be represented as a sum of its basic harmonic of frequency ω and higher harmonics with frequencies $n\omega$, with integer $n > 1$.

Note that an effective generation of higher harmonics is only possible with adequate nonlinearity of the system. For example, consider the nonlinear term αq^3 used in the equations explored in sections 5.2 and 5.3. If the waveform $q(t)$ is approximately sinusoidal, such term will have only the basic (first) and the third harmonics—see, e.g. Eq. (5.50). As another example, the 'pendulum nonlinearity' $\sin q$ cannot produce, without a constant component ('bias') in $q(t)$, any even harmonic, including the second one. The most efficient generation of harmonics may be achieved using systems with the sharpest nonlinearities—e.g. semiconductor

[37] All numerical results shown in this section have been obtained by the fourth-order Runge–Kutta method with the automatic step adjustment which guarantees the relative error of the order of 10^{-4}—much smaller than the pixel size in the plots.

[38] In this particular case, the higher harmonic content is about 0.5%, dominated by the third harmonic whose amplitude and phase are in a very good agreement with Eq. (5.50).

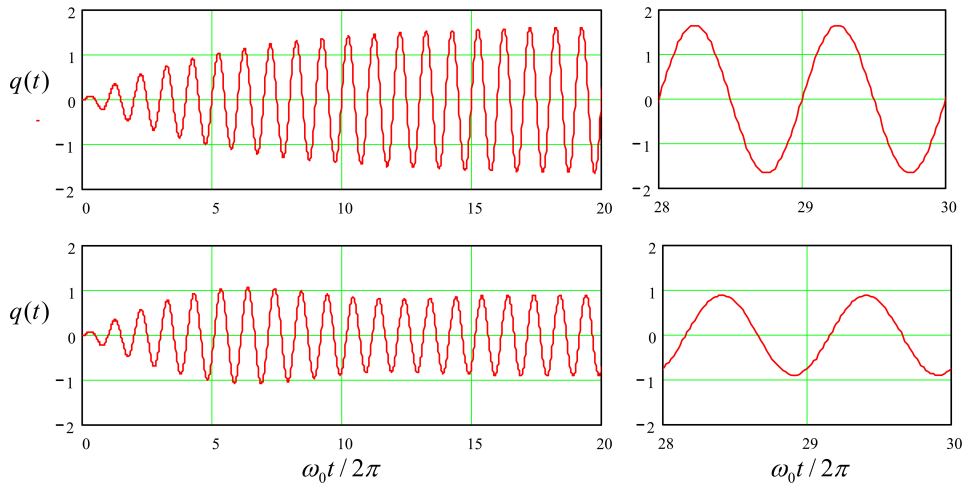


Figure 5.13. The oscillations induced by a similar sinusoidal external force (turned on at $t = 0$) in two systems with the same small-oscillation frequency ω_0 and low damping: a linear oscillator (two top panels) and a pendulum (two bottom panels). In all cases, $\delta/\omega_0 = 0.03$, $f_0 = 0.1$, and $\omega = \omega_0$.

diodes whose current may follow an exponential dependence on the applied voltage through several orders of magnitude[39].

Another way to increase the contents of an nth higher harmonic in a nonlinear oscillator is to reduce the excitation frequency ω to $\sim\omega_0/n$, so that the oscillator resonated at the frequency $n\omega \approx \omega_0$ of the desirable harmonic. For example, figure 5.14a shows the oscillations in a pendulum described by the same Eq. (5.42), but driven at frequency $\omega = \omega_0/3$. One can see that the third harmonic amplitude may be comparable with that of the basic harmonic, in particular if the external frequency is additionally lowered (figure 5.14b) to accommodate for the deviation of the effective frequency $\omega_0(A)$ of own oscillations from its small-oscillation value ω_0—see Eq. (5.49), figure 5.4 and their discussion in section 5.2 above.

However, numerical modeling of nonlinear oscillators, as well as experiments with their physical implementations, bring more surprises. For example, the bottom panel of figure 5.15 shows the oscillations in a pendulum under the effect of a strong sinusoidal force with a frequency ω close to $3\omega_0$. One can see that at some parameter values and initial conditions, the system's oscillation spectrum is heavily contributed (almost dominated) by the third *sub*harmonic, i.e. the Fourier component of frequency $\omega/3 \approx \omega_0$.

This counter-intuitive phenomenon of such *subharmonic generation* may be explained as follows. Let us assume that subharmonic oscillations of frequency $\omega/3 \approx \omega_0$ have somehow appeared, and coexist with the forced oscillations of frequency 3ω:

[39] This method is used in practice, for example, for the generation of electromagnetic waves with frequencies in the terahertz range (10^{12}–10^{13} Hz), which still lacks efficient electronic self-oscillators.

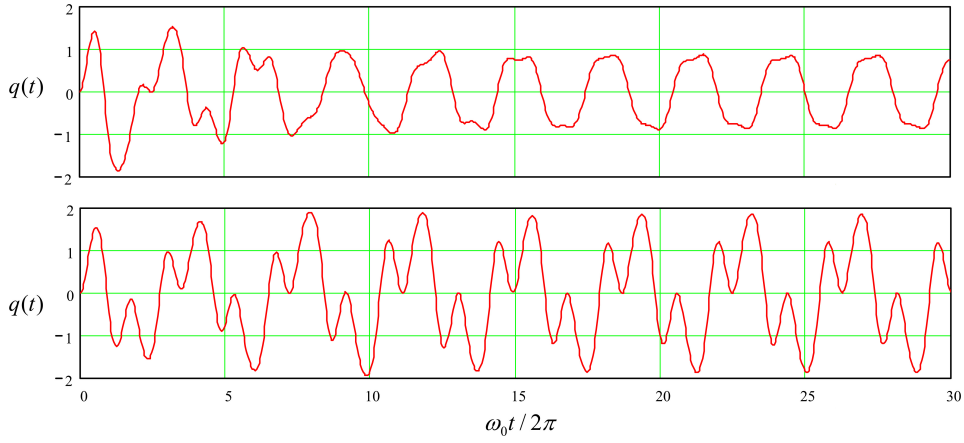


Figure 5.14. The oscillations induced in a pendulum, with damping $\delta/\omega_0 = 0.03$, driven by a sinusoidal external force of amplitude $f_0 = 0.75$, and frequency $\omega_0/3$ (top panel) and $0.8\omega_0/3$ (bottom panel).

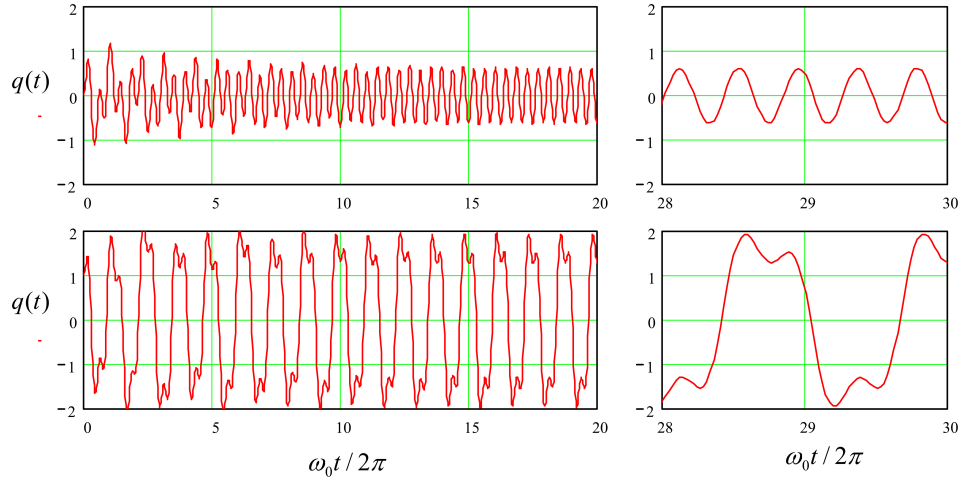


Figure 5.15. The oscillations sustained in a pendulum with $\delta/\omega_0 = 0.03$, by a sinusoidal external force of amplitude $f_0 = 3$ and frequency $3\omega_0 \times 0.8$, at initial conditions $q(0) = 0$ (the top row) and $q(0) = 1$ (the bottom row).

$$q(t) \approx A\cos\Psi + A_{\rm sub}\cos\Psi_{\rm sub}, \qquad \text{where } \Psi \equiv \omega t - \varphi, \qquad \Psi_{\rm sub} \equiv \frac{\omega t}{3} - \varphi_{\rm sub}. \quad (5.100)$$

Then the leading nonlinear term, αq^3, of the Taylor expansion of the pendulum's nonlinearity $\sin q$, is proportional to

$$\begin{aligned} q^3 &= (A\cos\Psi + A_{\rm sub}\cos\Psi_{\rm sub})^3 \\ &= A^3\cos^3\Psi + 3A^2A_{\rm sub}\cos^2\Psi\cos\Psi_{\rm sub} \\ &\quad + 3AA_{\rm sub}^2\cos\Psi\cos^2\Psi_{\rm sub} + A_{\rm sub}^3\cos^3\Psi_{\rm sub}. \end{aligned} \quad (5.101)$$

While the first and the last terms of this expression depend only of amplitudes of the individual components of oscillations, the two middle terms are more interesting because they produce so-called *combinational frequencies* of the two components. For our case, the third term,

$$3AA_{\rm sub}^2 \cos\Psi \cos^2\Psi_{\rm sub} = \frac{3}{4} AA_{\rm sub}^2 \cos(\Psi - 2\Psi_{\rm sub}) + \ldots, \tag{5.102}$$

is of special importance, because it produces, in addition to other combinational frequencies, the subharmonic component with the total phase

$$\Psi - 2\Psi_{\rm sub} = \frac{\omega t}{3} - \varphi + 2\varphi_{\rm sub}. \tag{5.103}$$

Thus, within a certain range of the mutual phase shift between the Fourier components, this nonlinear contribution is synchronous with the subharmonic oscillations, and describes the interaction that can deliver to it the energy from the external force, so that the oscillations may be sustained. Note, however, that the amplitude of the term (5.102) describing this energy exchange is proportional to the square of $A_{\rm sub}$, and vanishes at the linearization of the equations of motion near the trivial fixed point. This means that the point is always stable, i.e. the third subharmonic cannot be self-excited and always needs an initial 'kick-off'—compare the two panels of figure 5.15. The same is evidently true for higher subharmonics.

Only the second subharmonic is a special case. Indeed, let us make a calculation similar to Eq. (5.101), by replacing Eq. (5.100) with

$$q(t) \approx A\cos\Psi + A_{\rm sub}\cos\Psi_{\rm sub}, \quad \text{where } \Psi \equiv \omega t - \varphi, \quad \Psi_{\rm sub} \equiv \frac{\omega t}{2} - \varphi_{\rm sub}, \tag{5.104}$$

for a nonlinear term proportional to q^2:

$$\begin{aligned} q^2 = (A\cos\Psi + A_{\rm sub}\cos\Psi_{\rm sub})^2 &= A^2\cos^2\Psi \\ &+ 2AA_{\rm sub}\cos\Psi\cos\Psi_{\rm sub} + A_{\rm sub}^2\cos^2\Psi_{\rm sub}. \end{aligned} \tag{5.105}$$

Here the combinational-frequency term capable of supporting the second subharmonic,

$$\begin{aligned} 2AA_{\rm sub}\cos\Psi\cos\Psi_{\rm sub} &= AA_{\rm sub}\cos(\Psi - \Psi_{\rm sub}) \\ &= AA_{\rm sub}\cos(\omega t - \varphi + \varphi_{\rm sub}) + \ldots, \end{aligned} \tag{5.106}$$

is linear in the subharmonic amplitude, i.e. survives the equation linearization near the trivial fixed point. This mean that the second subharmonic may arise spontaneously, from infinitesimal fluctuations.

Moreover, such excitation of the second subharmonic is very similar to the parametric excitation that was discussed in detail in section 5.5, and this similarity is not coincidental. Indeed, let us redo the expansion (5.105) at a somewhat different assumption—that the oscillations are a sum of the forced oscillations at the external force frequency ω, and an *arbitrary but weak* perturbation:

$$q(t) = A\cos(\omega t - \varphi) + \tilde{q}(t), \qquad \text{with} \quad |\tilde{q}| \ll A. \tag{5.107}$$

Then, neglecting the small term proportional to $\tilde{q}^2$, we obtain

$$q^2 \approx A^2 \cos^2(\omega t - \varphi) + 2\tilde{q}(t)A \cos(\omega t - \varphi). \tag{5.108}$$

Besides the inconsequential phase φ, the second term in the last formula is *exactly* similar to the term describing the parametric effects in Eq. (5.75). This fact means that for a weak perturbation, a system with a quadratic nonlinearity in the presence of a strong 'pumping' signal of frequency ω is equivalent to a system with parameters changing in time with frequency ω. This fact is broadly used for the parametric excitation at high (e.g. optical) frequencies where the mechanical means of parameter modulation (see, e.g. figure 5.5) are not practicable. The necessary quadratic nonlinearity at optical frequencies may be provided by a *non-centrosymmetric nonlinear crystal*, e.g. the β-phase barium borate (BaB_2O_4).

Before finishing this chapter, let me elaborate a bit on a general topic: the relation between the numerical and analytical approaches to problems of dynamics (and physics as a whole). We have just seen that sometimes numerical solutions, such as those shown in figure 5.15b, may give vital clues for previously unanticipated phenomena such as the excitation of subharmonics. (The phenomenon of deterministic chaos, which will be discussed in chapter 9, presents another example of such 'numerical discoveries'.) One might also argue that in the absence of exact analytical solutions, numerical simulations may be the main theoretical tool for the study of such phenomena. These hopes are, however, muted by the general problem that is frequently called the *curse of dimensionality*[40], in which the last word refers to the number of input parameters of the problem to be solved[41].

Indeed, let us have another look at figure 5.15. We have been lucky to find a new phenomenon, the third subharmonic generation, for a particular set of parameters—in this case, five of them: $\delta/\omega_0 = 0.03$, $\omega/\omega_0 = 2.4$, $f_0 = 3$, $q(0) = 1$, and $dq/dt\,(0) = 0$. Could we tell anything about how common this effect is? Are subharmonics with different n possible in the system? The only way to address these questions computationally is to carry out similar numerical simulations at many points of the d-dimensional (in this case, $d = 5$) space of parameters. Say we have decided that breaking the reasonable range of each parameter to $N = 100$ points is sufficient. (For many problems, even more points are necessary—see, e.g. section 9.1.) Then the total number of numerical experiments to carry out is $N^d = (10^2)^5 = 10^{10}$—not a simple task even for powerful modern computing facilities. (In addition to the pure number of required CPU cycles, consider the storage and analysis of the results.) For many important problems of nonlinear dynamics, e.g. turbulence, the parameter dimensionality d is substantially larger, and the computer resources necessary for even one numerical experiment are much greater.

[40] This term was coined in 1957 by R Bellman in the context of optimal control theory (where the dimensionality means the number of parameters affecting the system under control), but gradually has spread throughout quantitative sciences using numerical methods.

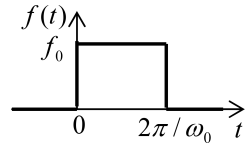
[41] In *Part EM* section 1.2, I discuss the curse's implications for a different case, when both analytical and numerical solutions to the same problem are possible.

In the view of the curse of dimensionality, approximate analytical considerations, such as those outlined above for the subharmonic excitation, are invaluable. More generally, physics used to stand on two legs, experiment and (analytical) theory. The enormous progress of computer performance during the last few decades has provided it with one more point of support (a tail?)—numerical simulation. This does not mean we can afford to cut and throw away any of the legs we are standing on.

5.9 Problems

Problem 5.1. For a system with the response function given by Eq. (5.17), prove Eq. (5.26), and use an approach different to the one used in section 5.1 to derive Eq. (5.34).
Hint: You may like to use the *Cauchy integral theorem* and the Cauchy integral formula for analytical functions of complex variables[42].

Problem 5.2. A square-wave pulse of force (see the figure below) is exerted on a linear oscillator with eigenfrequency ω_0 (with no damping), initially at rest. Calculate the law of motion $q(t)$, sketch it, and interpret the result.



Problem 5.3. At $t = 0$, a sinusoidal external force $F(t) = F_0 \cos \omega t$, is applied to a linear oscillator with eigenfrequency ω_0 and damping δ, which was at rest at $t \leqslant 0$.

(i) Derive the general expression for the time evolution of the oscillator's coordinate, and interpret the result.
(ii) Spell out the result for the case of resonance ($\omega = \omega_0$) in a system with low damping ($\delta \ll \omega_0$), and, in particular, explore the limit $\delta \to 0$.

Problem 5.4. A pulse of external force $F(t)$, with a finite duration $\mathcal{T}$, is exerted on a linear oscillator, initially at rest in its equilibrium position. Neglecting dissipation, calculate the change of oscillator's energy, using two different methods, and compare the results.

Problem 5.5. For a system with the following Lagrangian function:

$$L = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2 + \frac{\varepsilon}{2}\dot{q}^2 q^2,$$

calculate the frequency of free oscillations as a function of their amplitude A, at $A \to 0$, using two different approaches.

[42] See, e.g. Eq. (A.91).

Problem 5.6. For a system with the Lagrangian function

$$L = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2 + \varepsilon\dot{q}^4,$$

with small parameter ε, use the van der Pol method to find the frequency of free oscillations as a function of their amplitude.

Problem 5.7. On the plane $[a_1, a_2]$ of two real, time-independent parameters a_1 and a_2, find the regions in which the fixed point of the following system of equations,

$$\dot{q}_1 = a_1(q_2 - q_1),$$
$$\dot{q}_2 = a_2 q_1 - q_2,$$

is unstable, and sketch the regions of each fixed-point type—stable and unstable nodes, focuses, etc.

Problem 5.8. Solve problem 5.3(ii) using the reduced equations, and compare the result with the exact solution.

Problem 5.9. Use the reduced equations to analyze the forced oscillations in an oscillator with weak nonlinear damping, described by equation

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \beta\dot{q}^3 = f_0 \cos\omega t,$$

with $\omega \approx \omega_0$; $\beta, \delta > 0$; and $\beta\omega A^2 \ll 1$. In particular, find the stationary amplitude of forced oscillations and analyze their stability. Discuss the effect(s) of the nonlinear term on the resonance.

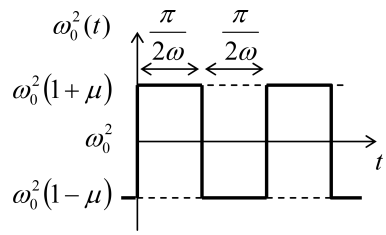
Problem 5.10. Within the approach discussed in section 5.4, calculate the average frequency of a self-oscillator outside the range of its phase-locking by an external sinusoidal force.

Problem 5.11.* Use the reduced equations to analyze stability of the forced nonlinear oscillations described by the Duffing equation (5.43). Relate the result to the slope of resonance curves (figure 5.4).

Problem 5.12. Use the van der Pol method to find the condition of parametric excitation of the oscillator described by the following equation:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2(t)q = 0,$$

where $\omega_0^2(t)$ is the square-wave function shown in the figure to the right, with $\omega \approx \omega_0$.



Problem 5.13. Use the van der Pol method to analyze parametric excitation of an oscillator with a weak nonlinear damping, described by equation

$$\ddot{q} + 2\delta\dot{q} + \beta\dot{q}^3 + \omega_0^2(1 + \mu\cos 2\omega t)q = 0,$$

with $\omega \approx \omega_0$; β, $\delta > 0$; and μ, $\beta\omega A^2 \ll 1$. In particular, find the amplitude of stationary oscillations and analyze their stability.

Problem 5.14.* Adding nonlinear term αq^3 to the left-hand side of Eq. (5.76),

(i) find the corresponding addition to the reduced equations,
(ii) calculate the stationary amplitude A of the parametric oscillations,
(iii) find the type and stability of each fixed point of the reduced equations, and
(iv) sketch the Poincaré phase-plane of the system in major parameter regions.

Problem 5.15. Use the van der Pol method to find the condition of parametric excitation of an oscillator with weak modulation of both the effective mass $m(t) = m_0(1 + \mu_m \cos 2\omega t)$ and the spring constant $\kappa(t) = \kappa_0[1 + \mu_\kappa \cos(2\omega t - \psi)]$, with the same frequency $2\omega \approx 2\omega_0$, at arbitrary modulation depths ratio μ_m/μ_k and phase shift ψ. Interpret the result in terms of modulation of the instantaneous frequency $\omega(t) \equiv [\kappa(t)/m(t)]^{1/2}$ and the mechanical impedance $Z(t) \equiv [\kappa(t)m(t)]^{1/2}$ of the oscillator.

Problem 5.16.* Find the condition of parametric excitation of a nonlinear oscillator described by equation

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \gamma q^2 = f_0 \cos 2\omega t,$$

with sufficiently small δ, γ, f_0, and $\xi \equiv \omega - \omega_0$.

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