Classical Mechanics

Lecture notes

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Preface to the EAP Series

Essential Advanced Physics

Essential Advanced Physics (EAP) is a series of lecture notes and problems with solutions, consisting of the following four parts[1]:

- *Part CM*: *Classical Mechanics* (a one-semester course),
- *Part EM*: *Classical Electrodynamics* (two semesters),
- *Part QM*: *Quantum Mechanics* (two semesters), and
- *Part SM*: *Statistical Mechanics* (one semester).

Each part includes two volumes: *Lecture Notes* and *Problems with Solutions*, and an additional file *Test Problems with Solutions*.

Distinguishing features of this series—in brief

- condensed lecture notes (~250 pp per semester)—much shorter than most textbooks
- emphasis on simple explanations of the main notions and phenomena of physics
- a focus on problem solution; extensive sets of problems with detailed model solutions
- additional files with test problems, freely available to qualified university instructors
- extensive cross-referencing between all parts of the series, which share style and notation

Level and precursors

The goal of this series is to bring the reader to a general physics knowledge level necessary for professional work in the field, regardless on whether the work is theoretical or experimental, fundamental or applied. From the formal point of view, this level (augmented by a few special topic courses in a particular field of concentration, and of course by an extensive thesis research experience) satisfies the typical PhD degree requirements. Selected parts of the series may be also valuable for graduate students and researchers of other disciplines, including astronomy, chemistry, mechanical engineering, electrical, computer and electronic engineering, and material science.

The entry level is a notch lower than that expected from a physics graduate from an average US college. In addition to physics, the series assumes the reader's familiarity with basic calculus and vector algebra, to such an extent that the meaning of the formulas listed in appendix A, 'Selected mathematical formulas' (reproduced at the end of each volume), is absolutely clear.

[1] Note that the (very ambiguous) term *mechanics* is used in these titles in its broadest sense. The acronym *EM* stems from another popular name for classical electrodynamics courses: *Electricity and Magnetism*.

Origins and motivation

The series is a by-product of the so-called 'core physics courses' I taught at Stony Brook University from 1991 to 2013. My main effort was to assist the development of students' problem-solving skills, rather than their idle memorization of formulas. (With a certain exaggeration, my lectures were not much more than introductions to problem solution.) The focus on this main objective, under the rigid time restrictions imposed by the SBU curriculum, had some negatives. First, the list of covered theoretical methods had to be limited to those necessary for the solution of the problems I had time to discuss. Second, I had no time to cover some core fields of physics—most painfully general relativity[2] and quantum field theory, beyond a few quantum electrodynamics elements at the end of *Part QM*.

The main motivation for putting my lecture notes and problems on paper, and their distribution to students, was my desperation to find textbooks and problem collections I could use, with a clear conscience, for my purposes. The available graduate textbooks, including the famous *Theoretical Physics* series by Landau and Lifshitz, did not match the minimalistic goal of my courses, mostly because they are far too long, and using them would mean hopping from one topic to another, picking up a chapter here and a section there, at a high risk of losing the necessary background material and logical connections between the course components—and the students' interest with them. In addition, many textbooks lack even brief discussions of several traditional and modern topics that I believe are necessary parts of every professional physicist's education[3].

On the problem side, most available collections are not based on particular textbooks, and the problem solutions in them either do not refer to any background material at all, or refer to the included short sets of formulas, which can hardly be used for systematic learning. Also, the solutions are frequently too short to be useful, and lack discussions of the results' physics.

Style

In an effort to comply with the Occam's Razor principle[4], and beat Malek's law[5], I have made every effort to make the discussion of each topic as clear as the time/space (and my ability :-) permitted, and as simple as the subject allowed. This effort has resulted in rather succinct lecture notes, which may be thoroughly read by a student during the semester. Despite this briefness, the introduction of every new

[2] For an introduction to this subject, I can recommend either a brief review by S Carroll, *Spacetime and Geometry* (2003, New York: Addison-Wesley) or a longer text by A Zee, *Einstein Gravity in a Nutshell* (2013, Princeton University Press).

[3] To list just a few: the statics and dynamics of elastic and fluid continua, the basics of physical kinetics, turbulence and deterministic chaos, the physics of computation, the energy relaxation and dephasing in open quantum systems, the reduced/RWA equations in classical and quantum mechanics, the physics of electrons and holes in semiconductors, optical fiber electrodynamics, macroscopic quantum effects in Bose–Einstein condensates, Bloch oscillations and Landau–Zener tunneling, cavity quantum electrodynamics, and density functional theory (DFT). All these topics are discussed, if only briefly, in my lecture notes.

[4] *Entia non sunt multiplicanda praeter necessitate*—Latin for 'Do not use more entities than necessary'.

[5] 'Any simple idea will be worded in the most complicated way'.

physical notion/effect and of every novel theoretical approach is always accompanied by an application example or two.

The additional exercises/problems listed at the end of each chapter were carefully selected[6], so that their solutions could better illustrate and enhance the lecture material. In formal classes, these problems may be used for homework, while individual learners are strongly encouraged to solve as many of them as practically possible. The few problems that require either longer calculations, or more creative approaches (or both), are marked by asterisks.

In contrast with the lecture notes, the model solutions of the problems (published in a separate volume for each part of the series) are more detailed than in most collections. In some instances they describe several alternative approaches to the problem, and frequently include discussions of the results' physics, thus augmenting the lecture notes. Additional files with sets of shorter problems (also with model solutions) more suitable for tests/exams, are available for qualified university instructors from the publisher, free of charge.

Disclaimer and encouragement

The prospective reader/instructor has to recognize the limited scope of this series (hence the qualifier *Essential* in its title), and in particular the lack of discussion of several techniques used in current theoretical physics research. On the other hand, I believe that the series gives a reasonable introduction to the *hard core* of physics—which many other sciences lack. With this hard core knowledge, today's student will always feel at home in physics, even in the often-unavoidable situations when research topics have to be changed at a career midpoint (when learning from scratch is terribly difficult—believe me :-). In addition, I have made every attempt to reveal the remarkable logic with which the basic notions and ideas of physics subfields merge into a wonderful single construct.

Most students I taught liked using my materials, so I fancy they may be useful to others as well—hence this publication, for which all texts have been carefully reviewed.

[6] Many of the problems are original, but it would be silly to avoid some old good problem ideas, with long-lost authorship, which wander from one textbook/collection to another one without references. The assignments and model solutions of all such problems have been re-worked carefully to fit my lecture material and style.

Preface to *Classical Mechanics: Lecture Notes*

A fairer title for this part of the series could be *Classical Mechanics and Dynamics*, because the notions of non-relativistic[7] mechanics and dynamics, though much intertwined, are still somewhat different. The term *mechanics*, in its narrow sense, means the derivation of the equations of motion of particles and their systems (including continua such as solids and fluids), followed by the solution of these equations, and interpretation of the results. *Dynamics* is a more ambiguous term; it may mean, in particular:

(i) the part of mechanics that deals with motion (in contrast to *statics*);
(ii) the part of mechanics that deals with reasons for motion (in contrast to *kinematics*); or
(iii) the part of mechanics that focuses on its two last tasks, i.e. the solution of the equations of motion and discussion of the results.

The last definition invites a question. It may look like mechanics and dynamics are just two sequential steps of a single analytical process; why should they be considered separate disciplines[8]? The main reason is that the many differential equations of motion, obtained in classical mechanics, also describe processes in different systems, so their analysis may reveal important features of these systems as well.

To summarize, the term *dynamics* is so ambiguous that, after some hesitation, I have opted to use for this course the traditional name *Classical Mechanics*, implying a broader meaning of the term *mechanics* (also very ambiguous) which includes many results important to non-mechanical systems as well. Note that in other parts of the series (*Quantum Mechanics* and *Statistical Mechanics*) this term is understood in a similarly broad sense.

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[7] Following tradition, an introduction to special relativity is included in *Part EM* of these notes. As a preliminary, the relativistic effects are small if the speed of every particle of the system is much lower than the speed of light, $c \approx 3.00 \times 10^8$ m s^{-1}, and the system size is much larger than the system's *Schwarzschild radius*, $r_s \equiv 2Gm/c^2$, where $G \approx 6.67 \times 10^{-11}$ SI units (m^3 kg s) is the Newtonian gravity constant, and m is the system's mass. (More exact values of c, G, and some other physical constants may be found in appendix B.)

[8] Another important issue concerning definition (iii) of dynamics is that it is suspiciously close to the part of mathematics devoted to differential equation analysis; what is the difference? To answer, we have to dip, for just a second, into the philosophy of physics. This discipline may be described as the art (and a bit of science :-) of the description of Mother Nature by mathematical means; hence in many cases the approaches of a mathematician and a physicist to a problem are very similar. The main difference is that physicists try to express the results of their analysis in less formal terms of a *system's motion* rather than *function properties*. As a result they develop some sort of intuition ('gut feeling') about how other, apparently similar, systems may behave, even if their exact equations of motion are somewhat different—or not known at all. The intuition thus developed has an enormous heuristic power, and most discoveries in physics have been made through gut-feeling-based insights rather than just by plugging one formula into another one.

Perhaps the most distinguishing feature of this course is the substantial attention on the mechanics of continua, including discussions of 1D waves in chapter 6, deformations and elasticity (including 3D waves) in chapter 7, and fluid dynamics in chapter 8. In most classical mechanics courses, these topics are either neglected completely or covered only as an introduction to field theory, without discussion of even basic real-world problems. However, I believe that familiarity with these topics is a necessary part of any professional physicist's education. The discussions of deformation and elasticity in the current chapter 7, and the fluid dynamics in chapter 8, are close to the introductory parts of the corresponding volumes (6 and 7) of the Landau and Lifshitz *Theoretical Physics* series.

Another not-quite-standard feature of this course is the introduction to analytical mechanics, starting with the Lagrangian formalism in chapter 2, from the experiment-based Newton's laws rather from some general ideas, such as the Hamilton principle, with this principle discussed only at the end of the course (section 10.3). I feel that this route emphasizes better the experimental roots of physics, and the secondary nature of any general principles—regardless of their important aesthetic and heuristic values.

Acknowledgments

I am extremely grateful to my faculty colleagues and other readers of the preliminary (circa 2013) version of this series, who provided feedback on certain sections; here they are listed in alphabetical order[9]: A Abanov, P Allen, D Averin, S Berkovich, P-T de Boer, M Fernandez-Serra, R F Hernandez, A Korotkov, V Semenov, F Sheldon, and X Wang. (Obviously, these kind people are not responsible for any remaining deficiencies.)

A large part of my scientific background and experience, reflected in these materials, came from my education, and then research, in the Department of Physics of Moscow State University from 1960 to 1990. The Department of Physics and Astronomy of Stony Brook University provided a comfortable and friendly environment for my work during the following 25+ years.

Last but not least, I would like to thank my wife Lioudmila for all her love, care, and patience—without these, this writing project would have been impossible.

I know very well that my materials are still far from perfection. In particular, my choice of covered topics (always very subjective) may certainly be questioned. Also, it is almost certain that despite all my efforts, not all typos have been weeded out. This is why all remarks (however candid) and suggestions from readers will be greatly appreciated. All significant contributions will be gratefully acknowledged in future editions.

Konstantin K Likharev
Stony Brook, NY

[9] I am very sorry for not keeping proper records from the beginning of my lectures at Stony Brook, so I cannot list all the numerous students and TAs who have kindly attracted my attention to typos in earlier versions of these notes. Needless to say, I am very grateful to all of them as well.

Notation

Abbreviations	Fonts	Symbols
c.c. complex conjugate	$F, \mathscr{F}$ scalar variables[10]	$\dot{}$ time differentiation operator (d/dt)
h.c. Hermitian conjugate	$\mathbf{F}, \boldsymbol{\mathscr{F}}$ vector variables	∇ spatial differentiation vector (*del*)
	$\hat{F}, \hat{\mathscr{F}}$ scalar operators	$\approx$ approximately equal to
	$\hat{\mathbf{F}}, \hat{\boldsymbol{\mathscr{F}}}$ vector operators	$\sim$ of the same order as
	F matrix	$\propto$ proportional to
	$F_{jj'}$ matrix element	$\equiv$ equal to by definition (or evidently)
		$\cdot$ scalar ('dot-') product
		$\times$ vector ('cross-') product
		$\overline{}$ time averaging
		$\langle\ \rangle$ statistical averaging
		$[\ ,\ ]$ commutator
		$\{\ ,\ \}$ anticommutator

Prime signs

The prime signs ($'$, $''$, etc) are used to distinguish similar variables or indices (such as j and j' in the matrix element above), rather than to denote derivatives.

Parts of the series

Part CM: Classical Mechanics *Part EM: Classical Electrodynamics*
Part QM: Quantum Mechanics *Part SM: Statistical Mechanics*

Appendices

Appendix A: Selected mathematical formulas
Appendix B: Selected physical constants

Formulas

The abbreviation Eq. may mean any displayed formula: either the equality, or inequality, or equation, etc.

[10] The same letter, typeset in different fonts, typically denotes different variables.

IOP Publishing

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Lecture notes

Konstantin K Likharev

Chapter 1

Review of fundamentals

This short introductory chapter reviews the basic notions and facts of non-relativistic classical mechanics that should be known to the readers from their undergraduate studies[1]*. For this reason, the discussion is very brief.*

1.1 Kinematics: basic notions

The basic notions of kinematics may be defined in various ways, and some mathematicians pay a lot of attention to analyzing such systems of axioms and the relations between them. In physics, we typically stick to less rigorous methods (in order to proceed faster to particular problems), and end up debating any definition as soon as 'everybody in the room' agrees that we are all speaking about the same thing—at least in the context in which they are discussed. Let me hope that the following notions used in classical mechanics do satisfy this criterion in our 'room':

(i) All the *Euclidean geometry* notions, including the *point*, the straight line, the plane, etc.

(ii) *Reference frames*: platforms for observation and mathematical description of physical phenomena. A reference frame includes a *coordinate system* used for measuring the point's position (namely, its *radius-vector*, **r**, which connects the coordinate origin to the point, see figure 1.1) and a clock measuring *time*, t. A coordinate system may be understood as a certain method of expressing the radius-vector **r** of a point as a set of its *scalar*

[1] The reader is advised to perform (perhaps after reading this chapter as a reminder) a self-check by solving a few problems from the dozen listed in section 1.6. If the results are not satisfactory, it may make sense to start from some remedial reading. For that, I can recommend, for example [1–3].

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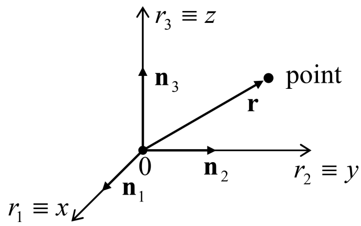

Figure 1.1. Cartesian coordinates of a point.

coordinates. The most important of such systems (but by no means the only one[2]) is the *Cartesian* (orthogonal, linear) *coordinates*[3], r_j, of a point, in which its radius-vector may be represented as the following sum:

$$\mathbf{r} = \sum_{j=1}^{3} \mathbf{n}_j r_j, \tag{1.1}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are unit vectors directed along the coordinate axis—see figure 1.1.

(iii) The *absolute* ('Newtonian') *space/time*, which does not depend on the matter distribution. The space is assumed to have the *Euclidean metric*, which may be expressed as the following relation between the length r of any radius-vector $\mathbf{r}$ and its Cartesian coordinates:

$$r \equiv |\,\mathbf{r}\,| = \left(\sum_{j=1}^{3} r_j^2\right)^{1/2}, \tag{1.2}$$

while time t is assumed to run similarly in all reference frames.

(iv) The (instant) *velocity* of the point,

$$\mathbf{v}(t) \equiv \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}}, \tag{1.3}$$

and its *acceleration*:

$$\mathbf{a}(t) \equiv \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}}. \tag{1.4}$$

(v) *Transfer between reference frames*. The above definitions of vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ depend on the chosen reference frame (they are 'reference-frame-specific'), but we frequently need to relate these vectors as observed in different frames. Within Euclidean geometry, the relation between the radius-vectors in

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[2] For example, the cylindrical/polar and spherical coordinates (see, e.g. appendix A, section A.10), which will be extensively used in this series, are not Cartesian; they are locally orthogonal, but not linear.

[3] In these notes, the Cartesian coordinates are denoted as either $\{r_1, r_2, r_3\}$ or $\{x, y, z\}$, depending on convenience in the particular case. Note that axis numbering is important for operations such as the vector ('cross') product; the 'correct' (meaning generally accepted) numbering order is such that the rotation $\mathbf{n}_1 \to \mathbf{n}_2 \to \mathbf{n}_3 \to \mathbf{n}_1 \ldots$ looks counter-clockwise if watched from a point with all $r_i > 0$—see figure 1.1.

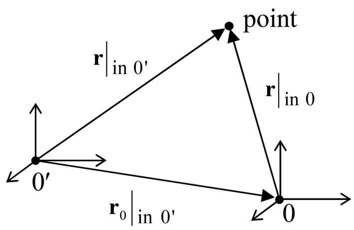


Figure 1.2. Transfer between two reference frames.

two frames with the corresponding axes parallel in the moment of interest (figure 1.2) is very simple:

$$\mathbf{r}|_{\text{in }0'} = \mathbf{r}|_{\text{in }0} + \mathbf{r}_0|_{\text{in }0'}. \tag{1.5}$$

If the frames move versus each other by *translation* only (no mutual rotation!), similar relations are valid for the velocity and the acceleration as well:

$$\mathbf{v}|_{\text{in }0'} = \mathbf{v}|_{\text{in }0} + \mathbf{v}_0|_{\text{in }0'}, \tag{1.6}$$

$$\mathbf{a}|_{\text{in }0'} = \mathbf{a}|_{\text{in }0} + \mathbf{a}_0|_{\text{in }0'}. \tag{1.7}$$

Note that in the case of mutual rotation of the reference frames, notions such as $\mathbf{v}_0|_{\text{in }0'}$ are not well defined, because different points of a rigid body connected to frame 0 may have different velocities when observed in frame $0'$. As a result, generally the transfer laws for velocities and accelerations are more complex than those given by Eqs. (1.6) and (1.7). It will be more natural for me to discuss them at the end of chapter 4, which is devoted to rigid-body motion.

(vi) A *particle* (or 'point particle'): a localized physical object whose size is negligible, and whose shape is irrelevant *for the given problem*. Note that the last qualification is extremely important. For example, the size and shape of the Space Shuttle are not too important for the discussion of its orbital motion, but are paramount when its landing procedures are being developed. Since classical mechanics neglects quantum-mechanical uncertainties[4], in classical mechanics the position of a particle, at any particular instant t, may be identified with a single point, i.e. with a single radius-vector $\mathbf{r}(t)$. The formal final goal of classical mechanics is finding the *laws of motion* $\mathbf{r}(t)$ of all particles participating in a given problem.

1.2 Dynamics: Newton's laws

Generally, classical dynamics is fully described (in addition to the kinematic relations discussed above) by the three *Newton laws*[5]. In contrast to the impression

[4] This approximation is legitimate, crudely, when the product of the coordinate and momentum scales of the particle motion is much larger than the Planck's constant $\hbar \approx 1.054 \times 10^{-34}$ J s. More detailed conditions of the applicability of classical mechanics depend on a particular system—see, e.g. *Part QM* of this lecture note series.

[5] Due to the genius of Sir Isaac Newton, these laws were formulated as early as in the 1680s, far ahead of the science of that time.

that some textbooks on theoretical physics try to create, these laws are experimental in nature, and cannot be derived from *purely* theoretical arguments.

I am confident that the reader of these notes is already familiar with Newton's laws, in one or another formulation. Let me note only that in some formulations *Newton's first law* looks just like a particular case of the second law—for the case of zero net force acting on a particle. In order to avoid this duplication, the first law may be formulated as the following postulate:

> There exists at least one reference frame, called *inertial*, in which any *free particle* (i.e. a particle isolated from the rest of the Universe) moves with $\mathbf{v} =$ const, i.e. with $\mathbf{a} = 0$.

Note that according to Eq. (1.7), this postulate immediately means that there is also an infinite number of inertial frames, because all frames 0′ moving without rotation or acceleration relative to the postulated inertial frame 0 (i.e. having $\mathbf{a}_0|_{\text{in } 0'} = 0$) are also inertial.

On the other hand, the Newton's second and third laws may be postulated *together* in the following elegant way. Each particle, say number k, may be characterized by a scalar constant (called *mass*, m_k), such that at any interaction of N particles (isolated from the rest of the Universe), in any inertial system,

$$\mathbf{P} \equiv \sum_{k=1}^{N} \mathbf{p}_k \equiv \sum_{k=1}^{N} m_k \mathbf{v}_k = \text{const.} \tag{1.8}$$

(Each component of this sum,

$$\mathbf{p}_k \equiv m_k \mathbf{v}_k, \tag{1.9}$$

is called the *mechanical momentum*[6] of the corresponding particle, and the whole sum $\mathbf{P}$, the *total momentum* of the system.)

Let us apply this postulate to just two interacting particles. Differentiating Eq. (1.8), written for this case over time, we obtain

$$\dot{\mathbf{p}}_1 = -\dot{\mathbf{p}}_2. \tag{1.10}$$

Let us give the derivative $\dot{\mathbf{p}}_1$ (which is a vector) the name of *force* exerted on particle 1. In our current case, when the only possible source of force is particle 2, the force may be denoted as $\mathbf{F}_{12}$: $\dot{\mathbf{p}}_1 = \mathbf{F}_{12}$. Similarly, $\mathbf{F}_{21} \equiv \dot{\mathbf{p}}_2$, so that Eq. (1.10) becomes the *Newton's third law*:

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{1.11}$$

Plugging Eq. (1.9) into these force definitions, and differentiating the products $m_k\mathbf{v}_k$, taking into account that the particles' masses are constants[7], we obtain that for k and k' taking any of values 1, 2,

$$m_k \dot{\mathbf{v}}_k \equiv m_k \mathbf{a}_k = \mathbf{F}_{kk'}, \quad \text{where } k' \neq k. \tag{1.12}$$

[6] The more extended term *linear momentum* is typically used in cases when there is a chance of its confusion with the *angular momentum* of the same particle/system—see below.

[7] Note that this may not be true for composite bodies of varying total mass M (e.g. rockets emitting jets, see problem 1.11); in these cases the momentum's derivative may differ from $M\mathbf{a}$.

Now, returning to the general case of several interacting particles, and making an additional (but very natural) assumption that all partial forces $\mathbf{F}_{kk'}$ acting on particle k add up as vectors, we may generalize Eq. (1.12) into *Newton's second law*:

$$m_k \mathbf{a}_k \equiv \dot{\mathbf{p}}_k = \sum_{k' \neq k} \mathbf{F}_{kk'} \equiv \mathbf{F}_k, \tag{1.13}$$

which allows a clear interpretation of the mass as a measure of the particle's *inertia*.

As a matter of principle, if the dependence of all pair forces $\mathbf{F}_{kk'}$ of particle positions (and generally of time as well) is known, Eq. (1.13), augmented with the kinematic relations (1.2) and (1.3), allows the calculation of the laws of motion $\mathbf{r}_k(t)$ of all particles of the system. For example, for one particle the second law (1.13) gives an ordinary differential equation of the second order,

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t), \tag{1.14}$$

which may be integrated—either analytically or numerically.

For certain cases, this is very simple. As an elementary example, Newton's gravitational force

$$\mathbf{F} = -G\frac{mm'}{R^3}\mathbf{R} \tag{1.15}$$

(where $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$ is the distance between particles of masses m and m')[8], is virtually uniform and may be approximated as

$$\mathbf{F} = m\mathbf{g}, \tag{1.16}$$

with the vector $\mathbf{g} \equiv (Gm'/r'^3)\mathbf{r}'$ being constant, for local motions with $r \ll r'$ [9]. As a result, m in Eq. (1.13) cancels, it is reduced to just $\ddot{\mathbf{r}} = \mathbf{g} = \text{const}$, and may be easily integrated twice:

$$\begin{aligned} \dot{\mathbf{r}}(t) \equiv \mathbf{v}(t) &= \int_0^t \mathbf{g} \ dt' + \mathbf{v}(0) = \mathbf{g}t + \mathbf{v}(0), \\ \mathbf{r}(t) &= \int_0^t \mathbf{v}(t')dt' + \mathbf{r}(0) = \mathbf{g}\frac{t^2}{2} + \mathbf{v}(0)t + \mathbf{r}(0), \end{aligned} \tag{1.17}$$

[8] The fact that the masses participating in Eqs. (1.14) and Eq. (1.16) are equal, the so-called *weak equivalence principle*, is actually highly non-trivial, but has been verified experimentally to the relative accuracy of at least 10^{-13}. Due to the conceptual significance of the principle, new experiments, such as MISCROSCOPE (http://smsc.cnes.fr/MICROSCOPE/), are being planned for a substantial, nearly 100-fold accuracy improvement.

[9] Of course, the most important particular case of Eq. (1.16) is the motion of objects near the Earth's surface. In this case, using the fact that Eq. (1.15) remains valid for the gravity field created by a heavy sphere, we obtain $g = GM_E/R_E^2$, where M_E and R_E are the Earth's mass and radius. Plugging in their values, $M_E \approx 5.92 \times 10^{24}$ kg, $R_E \approx 6.37 \times 10^6$ m, we obtain $g \approx 9.74$ m s^{-2}. The experimental value of g varies from 9.78 to 9.83 m s^{-2} at various locations on the Earth's surface (due to the deviations of Earth's shape from a sphere, and the location-dependent effect of the centrifugal 'inertial force'—see section 4.6 below), with an average value of $g \approx 9.807$ m s^{-2}.

thus giving the generic solution for all those undergraduate problems on projectile motion, which should be so familiar to the reader.

All this looks (and indeed is) very simple, but in most other cases Eq. (1.13) leads to more complex calculations. As an example, let us think about how we would solve another simple problem: a bead of mass m sliding, without friction, along a round ring of radius R in a gravity field obeying Eq. (1.16)—see figure 1.3. (This system is of course equivalent to the usual *point pendulum*, i.e. a point mass suspended from point 0 on a light rod or string, and in addition constrained to move in one vertical plane.)

Suppose we are only interested in bead's velocity, v, in the lowest point, after it has been dropped from rest at the rightmost position. If we want to solve this problem using only Newton's laws, we have to take the following steps:

(i) consider the bead in an arbitrary intermediate position on a ring, described, for example, by the angle θ shown in figure 1.3;

(ii) draw all the forces acting on the particle—in our current case, the gravity force $m\mathbf{g}$ and the reaction force $\mathbf{N}$ exerted by the ring—see figure 1.3;

(iii) write the Cartesian components of Newton's second law (1.14) for the bead acceleration: $ma_x = N_x$, $ma_y = N_y - mg$,

(iv) recognize that in the absence of friction, the force $\mathbf{N}$ should be normal to the ring, so that we can use two additional equations, $N_x = -N\sin\theta$ and $N_y = N\cos\theta$;

(v) eliminate unknown variables N, N_x, and N_y from the resulting system of four equations, thus obtaining a single second-order differential equation for one variable, for example θ:

$$mR\ddot{\theta} = -mg \sin\theta; \tag{1.18}$$

(vi) use the mathematical identity $\ddot{\theta} \equiv (d\dot{\theta}/d\theta)\,\dot{\theta}$ to integrate this equation over θ once to obtain an expression relating the velocity $\dot{\theta}$ and the angle θ; and finally

(vii) using our specific initial condition ($\dot{\theta} = 0$ at $\theta = \pi/2$), find the final velocity as $v = R\dot{\theta}$ at $\theta = 0$.

All this is very much doable, but please agree that the procedure it too cumbersome for such a simple problem. Moreover, in many other cases even writing equations of motion along relevant coordinates is very complex, and any help the general theory may provide is highly valuable. In many cases, such help is given by *conservation laws*; let us review the most general of them.

1.3 Conservation laws

Energy conservation is arguably the most general law of physics, but in mechanics it takes the humbler form of *mechanical energy conservation*, which has limited applicability. To derive it, we first have to define the *kinetic energy* of a particle as

$$T \equiv \frac{m}{2}v^2, \tag{1.19}$$

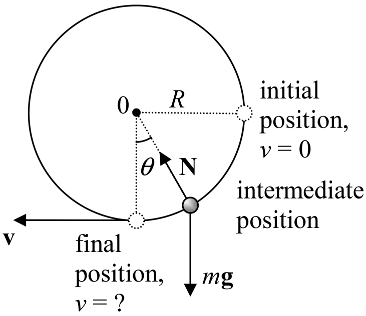


Figure 1.3. A bead sliding along a vertical ring.

and then recast its differential as[10]

$$dT = d\left(\frac{m}{2}v^2\right) \equiv d\left(\frac{m}{2}\mathbf{v}\cdot\mathbf{v}\right) = m\mathbf{v}\cdot d\mathbf{v} = m\frac{d\mathbf{v}\cdot d\mathbf{r}}{dt} = \frac{d\mathbf{p}}{dt}\cdot d\mathbf{r}. \qquad (1.20)$$

Now plugging in the momentum's derivative from Newton's second law, $d\mathbf{p}/dt = \mathbf{F}$, where $\mathbf{F}$ is the full force acting on the particle, we obtain relation $dT = \mathbf{F}\cdot d\mathbf{r}$. Its integration along the particle's trajectory between some points A and B gives the formula that is sometimes called the *work–energy principle*:

$$\Delta T \equiv T(\mathbf{r}_B) - T(\mathbf{r}_A) = \int_A^B \mathbf{F}\cdot d\mathbf{r}, \qquad (1.21)$$

where the integral on the right-hand is called the *work* of the force $\mathbf{F}$ on the path from A to B.

A further step may be taken only for *potential* (also called 'conservative') force that may be represented as a (minus) gradient of some scalar function $U(\mathbf{r})$, called the *potential energy*[11]. The vector operator ∇ (called either 'del' or 'nabla') of spatial differentiation[12] allows a very compact expression of this fact:

$$\mathbf{F} = -\nabla U. \qquad (1.22)$$

For example, for the uniform gravity field (1.16),

$$U = mgh + \text{ const}, \qquad (1.23)$$

where h is the vertical coordinate directed 'up'—opposite to the direction of the vector $\mathbf{g}$.

Integrating the tangential component F_τ of the vector $\mathbf{F}$, given by Eq. (1.22), along an arbitrary path connecting points A and B, we obtain

$$\int_A^B F_\tau dr \equiv \int_A^B \mathbf{F}\cdot d\mathbf{r} = U(\mathbf{r}_A) - U(\mathbf{r}_B), \qquad (1.24)$$

[10] Symbol $\mathbf{a}\cdot\mathbf{b}$ denotes the scalar (or 'dot-') product of vectors $\mathbf{a}$ and $\mathbf{b}$—see, e.g. Eq. (A.43).
[11] Note that because of its definition via the gradient, the potential energy is only defined to an arbitrary additive constant.
[12] Its basic properties are listed in appendix A, section A.8.

i.e. the work of potential forces may be represented as the difference of values of the function $U(\mathbf{r})$ in the initial and final point of the path. (Note that according to Eq. (1.24), work of a potential force on any closed path, with $\mathbf{r}_A = \mathbf{r}_B$, is zero.)

Now returning to Eq. (1.21) and comparing it to Eq. (1.24), we see that

$$T(\mathbf{r}_B) - T(\mathbf{r}_A) = U(\mathbf{r}_A) - U(\mathbf{r}_B), \tag{1.25}$$

so that the *total mechanical energy E*, defined as

$$E \equiv T + U, \tag{1.26}$$

is indeed conserved,

$$E(\mathbf{r}_A) \equiv E(\mathbf{r}_B), \tag{1.27}$$

but for conservative forces only. (Non-conservative forces may change E by either transferring energy from its mechanical form into another form, e.g. heat in the case of friction, or by pumping the energy into the system under consideration from another, 'external' system.)

The mechanical energy conservation allows us to return for a second to the problem shown in figure 1.3 and solve it in one shot by writing Eq. (1.27) for the initial and final points[13]:

$$0 + mgR = \frac{m}{2}v^2 + 0. \tag{1.28}$$

Solving Eq. (1.28) for v immediately gives us the desired answer. Let me hope that the reader agrees that this method of problem solving is much simpler, and I have secured his or her attention to discuss other conservation laws—which may be equally effective.

Momentum. Actually, the conservation of the full linear momentum of any system of particles isolated from the rest of the world has already been discussed and may serve as the basic postulate of classical dynamics—see Eq. (1.8). In the case of one free particle the law is reduced to a trivial result $\mathbf{p}$ = const, i.e. $\mathbf{v}$ = const. If a system of N particles *is* affected by external forces $\mathbf{F}^{(\text{ext})}$, we may write

$$\mathbf{F}_k = \mathbf{F}_k^{(\text{ext})} + \sum_{k=1}^{N} \mathbf{F}_{kk'}. \tag{1.29}$$

If we sum up the resulting Eqs. (1.13) for all particles of the system then, due to Newton's third law (1.11), valid for any indices $k \neq k'$, the contributions of all internal forces $\mathbf{F}_{kk'}$ to the resulting double sum in the right-hand part cancel, and we obtain the equation

$$\dot{\mathbf{P}} = \mathbf{F}^{(\text{ext})}, \quad \text{where } \mathbf{F}^{(\text{ext})} \equiv \sum_{k=1}^{N} \mathbf{F}_k^{(\text{ext})}, \tag{1.30}$$

[13] Here the arbitrary constant in Eq. (1.23) is chosen so that the potential energy is zero in the finite point.

which tells us that the translational motion of the system as the whole is similar to that of a single particle, under the effect of the *net external force* $\mathbf{F}^{(\text{ext})}$. As a simple sanity check, if the external forces have a zero sum, we return to postulate Eq. (1.8). Just one reminder: Eq. (1.30), just as its precursor Eq. (1.13), is only valid in an inertial reference frame.

I hope that the reader knows numerous examples of the application of the linear momentum conservation law, including the problems on car collisions, where the large collision forces are typically not known, so that the direct application of Eq. (1.13) to each car is impracticable.

The *angular momentum* of a particle[14] is defined as the following vector:

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}, \tag{1.31}$$

where $\mathbf{a} \times \mathbf{b}$ means the vector (or 'cross-') product of the vector operands[15]. Differentiating Eq. (1.31) over time, we obtain

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}. \tag{1.32}$$

In the first product, $\dot{\mathbf{r}}$ is just the velocity vector $\mathbf{v}$, parallel to the particle momentum $\mathbf{p} = m\mathbf{v}$, so that this product vanishes, since the vector product of any two parallel vectors equals zero. In the second product, $\dot{\mathbf{p}}$ is equal to the full force $\mathbf{F}$ acting on the particle, so that Eq. (1.32) is reduced to

$$\dot{\mathbf{L}} = \boldsymbol{\tau}, \tag{1.33}$$

where vector

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}, \tag{1.34}$$

is called the *torque* exerted by force $\mathbf{F}$[16]. (Note that the torque is reference-frame-specific—and again, the frame has to be inertial for Eq. (1.33) to be valid, because we have used Eq. (1.13) for its derivation.) For an important particular case of a *central* force $\mathbf{F}$ that is parallel to the radius-vector $\mathbf{r}$ of a particle, the torque vanishes, so that (in that particular reference frame only!) the angular momentum is conserved:

$$\mathbf{L} = \text{const}. \tag{1.35}$$

For a system of N particles, the total angular momentum is naturally defined as

$$\mathbf{L} \equiv \sum_{k=1}^{N} \mathbf{L}_k. \tag{1.36}$$

[14] Here we imply that the internal motions of the particle, including its rotation about its own axis, are negligible. (Otherwise it could not be represented by a point, as was postulated in section 1.1.)

[15] See, e.g. Eq. (A.45).

[16] Alternatively, especially in mechanical engineering, the torque is called the *force moment*.

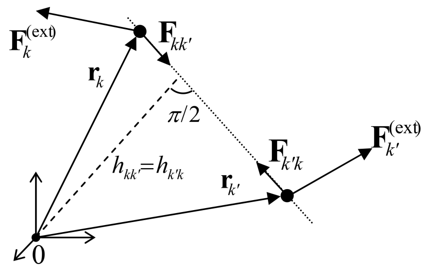


Figure 1.4. Internal and external forces, and the internal torque cancellation in a system of two particles.

Differentiating this equation over time, using Eq. (1.33) for each $\dot{\mathbf{L}}_k$, and again partitioning each force in a accordance with Eq. (1.29), we obtain

$$\dot{\mathbf{L}} = \sum_{\substack{k,k'=1 \\ k'\neq k}}^{N} \mathbf{r}_k \times \mathbf{F}_{kk'} + \boldsymbol{\tau}^{(\text{ext})}, \quad \text{where } \boldsymbol{\tau}^{(\text{ext})} \equiv \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^{(\text{ext})}. \tag{1.37}$$

The first (double) sum may always be divided into pairs of the type ($\mathbf{r}_k \times \mathbf{F}_{kk'} + \mathbf{r}_{k'} \times \mathbf{F}_{k'k}$). With a natural assumption of the central forces, $\mathbf{F}_{kk'} \parallel (\mathbf{r}_k - \mathbf{r}_{k'})$, each of these pairs equals zero. Indeed, in this case both components of the pair are vectors perpendicular to the plane passing through the positions of both particles and the reference-frame origin, i.e. to the plane of the drawing in figure 1.4.

Also, due to Newton's third law (1.11) the two forces are equal and opposite, and the magnitude of each term in the sum may be represented as $|F_{kk'}|h_{kk'}$, equal to the 'lever arms' $h_{kk'} = h_{k'k}$. As a result, each sum ($\mathbf{r}_k \times \mathbf{F}_{kk'} + \mathbf{r}_{k'} \times \mathbf{F}_{k'\text{k}}$), and hence the whole double sum in Eq. (1.37) vanishes, and it is reduced to a very simple result,

$$\dot{\mathbf{L}} = \boldsymbol{\tau}^{(\text{ext})}, \tag{1.38}$$

which is similar to Eq. (1.33) for a single particle, and is the angular analog of Eq. (1.30). In particular, Eq. (1.38) shows that if the full external torque $\boldsymbol{\tau}^{(\text{ext})}$ vanishes for some reason (e.g. if the system of particles is isolated from the rest of the Universe), the conservation law (1.35) is valid for the full angular momentum $\mathbf{L}$, even if its individual components $\mathbf{L}_k$ are not conserved due to inter-particle interactions.

Note again that since the conservation laws may be derived from Newton's laws, they do not introduce anything new to the dynamics of any system. Indeed, from the mathematical point of view, the conservation laws discussed above are just *the first integrals of motion*, which may liberate us from the necessity to integrate the second-order differential equations of motion, following from Newton's laws, twice.

1.4 Potential energy and equilibrium

Another important role of the potential energy U, in particular for dissipative systems whose total mechanical energy E is *not* conserved because it may be drained to the environment, is finding the positions of equilibrium (sometimes called the *fixed points* of the system under analysis) and analyzing their stability with respect to small perturbations. For a single particle, this is very simple: the force (1.22) vanishes at each extremum (minimum or maximum) of the potential energy[17]. Of those fixed points, only the minimums of $U(\mathbf{r})$ are stable—see section 3.2 below for a discussion of this point.

A slightly subtler case is a particle with potential energy $U(\mathbf{r})$, subjected to an *additional* external force $\mathbf{F}^{(\text{ext})}(\mathbf{r})$. In this case, the stable equilibrium is reached at the minimum of not the function $U(\mathbf{r})$, but of what is sometimes called the *Gibbs potential energy*[18]

$$U_G(\mathbf{r}) \equiv U(\mathbf{r}) - \int^{\mathbf{r}} \mathbf{F}^{(\text{ext})}(\mathbf{r}') \cdot d\mathbf{r}', \tag{1.39}$$

which is defined, just as $U(\mathbf{r})$ is, to an arbitrary constant. The proof of Eq. (1.39) is very simple: in an extremum of this function, the total force acting on the particle,

$$\mathbf{F}^{(\text{tot})} = \mathbf{F} + \mathbf{F}^{(\text{ext})} \equiv -\nabla U + \nabla \int^{\mathbf{r}} \mathbf{F}^{(\text{ext})}(\mathbf{r}') \cdot d\mathbf{r}' = -\nabla U_G, \tag{1.40}$$

vanishes, as it should.

Physically, the difference $U_G - U$ specified by Eq. (1.39) is the $\mathbf{r}$-dependent part of the potential energy $U^{(\text{ext})}$ of the external system responsible for the force $\mathbf{F}^{(\text{ext})}$, so that U_G is just the total potential energy $U + U^{(\text{ext})}$, excluding the part that does not depend on $\mathbf{r}$ and hence is irrelevant for the analysis. According to Newton's third law, the force exerted by the particle on the external system equals $(-\mathbf{F}^{(\text{ext})})$, so that its work (and hence the change of $U^{(\text{ext})}$ due to the change of $\mathbf{r}$) is given by the second term on the right-hand side of Eq. (1.39). Thus the condition of equilibrium, $-\nabla U_G = 0$, is just the condition of an extremum of the total potential energy, $U + U^{(\text{ext})}$, of the two interacting systems.

For the simplest (and very frequent) case when the applied force is independent of the particle's position, the Gibbs potential energy is just[19]

$$U_G(\mathbf{r}) \equiv U(\mathbf{r}) - \mathbf{F}^{(\text{ext})} \cdot \mathbf{r} + \text{const.} \tag{1.41}$$

As the simplest example, consider a 1D deformation of the usual elastic spring providing the returning force $(-\kappa x)$, where x is the deviation from its equilibrium.

[17] Assuming that the additional, non-conservative forces (such as viscosity), responsible for the mechanical energy drain vanish at equilibrium—as they typically do. (The static friction is one counter-example.)

[18] Unfortunately, in most textbooks the association of the (unavoidably used) notion of U_G with the glorious name of Josiah Willard Gibbs is postponed until a course on statistical mechanics and/or thermodynamics, where U_G is a part of the *Gibbs free energy*, in contrast to U, which is a part of the *Helmholtz free energy*—see, e.g. *Part SM* section 1.4. In contrast, I use this name throughout my series, because the difference between U_G and U, and hence that between the Gibbs and Helmholtz free energies, has nothing to do with statistics or thermal motion, and belongs to the whole of physics, including not only mechanics, but also electrodynamics and quantum mechanics.

[19] Note that Eq. (1.41) is a particular case of what mathematicians call the *Legendre transformations*.

As follows from Eq. (1.22), its potential energy is $U = \kappa x^2/2 + \text{const}$, so that its minimum corresponds to $x = 0$. Now let us apply an additional external force F, say independent of x. Then the equilibrium deformation of the spring, $x_0 = F/\kappa$, corresponds to the minimum not of U, but rather of the Gibbs potential energy (1.41), taking the form

$$U_G \equiv U - Fx = \frac{\kappa x^2}{2} - Fx. \tag{1.42}$$

1.5 OK, we've got it—can we go home now?

Not yet. In many cases the conservation laws discussed above provide little help, even in systems without dissipation. Consider, for example, a generalization of the bead-on-the-ring problem shown in figure 1.3, in which the ring is rotated by external forces, with a constant angular velocity ω, about its vertical diameter[20]. In this problem (to which I will repeatedly return below, using it as an analytical mechanics 'testbed'), none of the three conservation laws listed in the last section holds. In particular, the bead's energy,

$$E = \frac{m}{2}v^2 + mgh, \tag{1.43}$$

is *not* constant, because the external forces rotating the ring may change it. Of course, we can still solve the problem using Newton's laws, but this is even more complex than for the above case of the ring at rest, in particular because the force **N** exerted on the bead by the ring now may have three rather than two Cartesian components, which are not simply related. On the other hand, it is clear that the bead still has just one degree of freedom (angle θ), so that its dynamics should not be too complicated.

This fact gives a clue as to how situations such as this one could be simplified: if we could only exclude the so-called *reaction forces* such as **N** (that take into account the *external constraints* imposed on the particle motion) in advance, it would help a lot. Such a constraint exclusion may be provided by analytical mechanics, in particular its Lagrangian formulation, to we will now proceed.

Of course, the value of the Lagrangian approach goes far beyond simple systems such as the bead on a ring. Indeed, this system has just two externally imposed constrains: the fixed distance of the bead from the center of the ring, and the instant angle of rotation of the ring about its vertical diameter. Now let us consider the motion of a rigid body. It is essentially a system of a very large number, $N \gg 1$, of particles ($\sim 10^{23}$ of them if we think about atoms in a 1 cm scale body). If the only way to analyze its motion would be to write Newton's laws for each of the particles, the situation would be completely hopeless. Fortunately, the number of constraints imposed

[20] This is essentially a simplified model of the famous mechanical control device called the *centrifugal* (or 'flyball' or 'centrifugal flyball') *governor*—see, e.g. http://en.wikipedia.org/wiki/Centrifugal_governor. Sometimes the device is called the 'Watt's governor', after the famous engineer J Watts who used it in 1788 in one of his first steam engines, but it had been used in European windmills since at least the early 1600s.

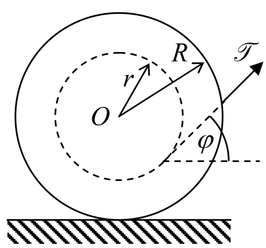
on its motion is almost similarly huge. (At negligible deformations of the body, the distances between each pair of its particles should be constant.) As a result, the number of actual degrees of freedom of such a body (at negligible deformations, just 6—see section 4.1), so that with kind help from analytical mechanics, the motion of the body may be, in many important cases, analyzed even without numerical calculations.

One more important motivation for analytical mechanics is given by the dynamics of 'non-mechanical' systems, for example, of the electromagnetic field—possibly interacting with charged particles, conducting bodies, etc. In many such systems, the easiest (and sometimes the only practicable) way to find the equations of motion is to derive then from either the Lagrangian or Hamiltonian function of the system. Moreover, the Hamiltonian formulation of the analytical mechanics (to be discussed in chapter 10) offers a direct pathway to deriving quantum-mechanical Hamiltonian operators of various systems, which are necessary for the analysis of their quantum properties.

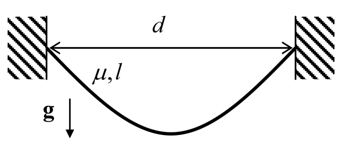
1.6 Problems

Problem 1.1. A bicycle, ridden with velocity v on a wet pavement, has no mudguards on its wheels. How far behind should the following biker ride to avoid being splashed over? Neglect the effects of air resistance.

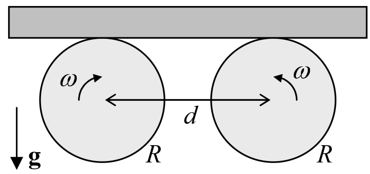
Problem 1.2. Two round disks of radius R are firmly connected with a coaxial cylinder of a smaller radius r, and a thread is wound on the resulting spool. The spool is placed on a horizontal surface, and the thread's end is pooled out at angle ϕ—see the figure below. Assuming that the spool does not slip on the surface, what direction would it roll?



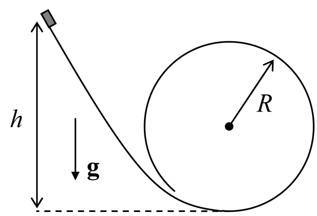
Problem 1.3.* Calculate the equilibrium shape of a flexible, heavy rope of length l, with a constant mass μ per unit length, if it is hung in a uniform gravity field between two points separated by a horizontal distance d—see the figure below.



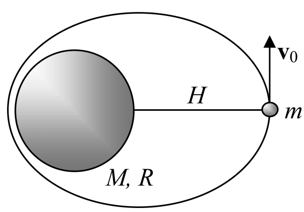
Problem 1.4. A uniform, long, thin bar is placed horizontally on two similar round cylinders rotating toward each other with the same angular velocity ω and displaced by distance d—see the figure below. Calculate the laws of relatively slow horizontal motions of the bar within the plane of the drawing for both possible directions of cylinder rotation, assuming that the friction force between the slipping surfaces of the bar and each cylinder obeys the simple *Coulomb approximation*[21] $|F| = \mu N$, where N is the normal pressure force between them, and μ is a constant (velocity-independent) coefficient. Formulate the condition of validity of your result.



Problem 1.5. A small block slides, without friction, down a smooth slide that ends with a round loop of radius R—see the figure below. What smallest initial height h allows the block to make its way around the loop without dropping from the slide, if it is launched with negligible initial velocity?



Problem 1.6. A satellite of mass m is being launched from height H over the surface of a spherical planet with radius R and mass $M \gg m$—see the figure below. Find the range of initial velocities $\mathbf{v}_0$ (normal to the radius) providing closed orbits above the planet's surface.

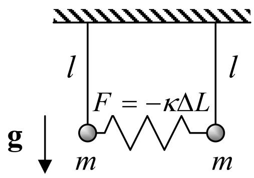


Problem 1.7. Prove that the thin-uniform-disk model of a galaxy describes small harmonic oscillations of stars inside it along the direction normal to the disk,

[21] This approximation was suggested in 1785 by the same C-A de Coulomb who discovered the famous *Coulomb law* of electrostatics, and hence pioneered the whole qualitative science of electricity.

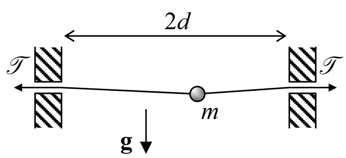
and calculate the frequency of these oscillations in terms of Newton's gravitational constant G and the average density ρ of the star/dust matter of the galaxy.

Problem 1.8. Derive the differential equations of motion for small oscillations of two similar pendula coupled with a spring (see the figure below), within the vertical plane. Assume that at the vertical position of both pendula, the spring is not stretched ($\Delta L = 0$).



Problem 1.9. One popular futuristic concept of travel is digging a straight railway tunnel through the Earth and letting a train go through it without initial velocity—driven only by gravity. Calculate the train's travel time through such a tunnel, assuming that the Earth's density ρ is constant, and neglecting the effects of friction and planetary rotation.

Problem 1.10. A small bead of mass m may slide, without friction, along a light string, stretched with a force $\mathscr{T} \gg mg$ between two points separated by a horizontal distance $2d$—see the figure below. Calculate the frequency of horizontal oscillations of the bead about its equilibrium position.



Problem 1.11. For a rocket accelerating due to a working jet motor (and hence spending its fuel), calculate the relation between its velocity and the remaining mass. *Hint*: For the sake of simplicity, consider 1D motion.

Problem 1.12. Prove the following *virial theorem*[22]. For a set of N particles performing a periodic motion,

$$\overline{T} = -\frac{1}{2}\sum_{k=1}^{N}\overline{\mathbf{F}_k \cdot \mathbf{r}_k},$$

where the top bar means time averaging, in this case over the motion period. What does the virial theorem say about:

[22] It was first stated by R Clausius in 1870.

(i) the 1D motion of a particle in a confining potential $U(x) = ax^{2s}$, with $a > 0$ and $s > 0$, and
(ii) the orbital motion of a particle moving in a central potential $U(r) = -C/r$?

Hint: Explore the time derivative of the following scalar function of time:

$$G(t) \equiv \sum_{k=1}^{N} \mathbf{p}_k \cdot \mathbf{r}_k.$$

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