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Calculation of Refractive Index Distribution from Interferograms Using the Born and Rytov's Approximation

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Discussion is made of calculational reconstruction of refractive index distribution from "multidirectional interferograms." Two approximate calculation formulae are given for two-dimensional case. One is derived using the Born approximation and corresponds to a generalization of the formula in X-ray structure analysis. The other is obtained using Rytov's approximation and a generalization of straight path approximation.

§1. Introduction

Three-dimensional distribution of refractive index can be measured by "multi-directional interferometry."[1] The procedure is in two steps.[2] The first step is to measure complex amplitude of a scattered or transmitted light wave when a wave is incident on an object with spatially varying refractive index. This measurement can be made by using ordinary or holographic interferometry. Many similar measurements are performed corresponding to many incident waves propagating in different directions. In this paper, details of these measurements will not be discussed; the measurement of complex amplitude is assumed possible. The second step is to calculate refractive index distribution from the measured complex amplitudes. Almost all the proposed calculation procedures are based on the assumption that light travels straight.[1,3–5] This restricts its applicability to large objects having small refractive index variation: *e.g.* in aerodynamic experiments. On the other hand, Wolf has proposed a calculation procedure[2] using the Born approximation for three-dimensional case. Some experiments were performed according to the theory.[6,7] However, when a small object is to be measured, Wolf's calculation procedure is found to contain certain difficulty in practical computation, as will be shown in §2.

In order to find a calculation procedure applicable to the whole range of objects, the discussion should be based on Maxwell's equations. In this paper, however, we neglect vectorial nature or polarization of light and also assume that a total complex amplitude $U^t(\boldsymbol{r})$ of monochromatic light is related to refractive index $n(\boldsymbol{r})$ at a point $\boldsymbol{r}$ by the wave equation

$$\nabla^2 U^t(\boldsymbol{r}) + k_v^2\{n(\boldsymbol{r})\}^2 U^t(\boldsymbol{r}) = 0, \qquad (1)$$

where k_v is the propagation constant in vacuo. The present problem is to find a procedure for calculating the coefficient $n(\boldsymbol{r})$ from the behaviour of $U^t(\boldsymbol{r})$'s outside the object for many different incident waves or under many different boundary conditions with eq. (1) as the basis. This problem may be classified into an inverse problem of a partial differential equation.[8]

This paper presents two approximate solutions of the inverse problem when incident waves are plane. For simplicity two-dimensional problem is discussed, and the inversion formulae are given only for two-dimensional case.* The essential points of the discussions and conclusions, however, will be applicable to three-dimensional problem.

§2. The Born Approximation

2.1 *Approximation of the wave equation*

As an incident wave is plane,

$$U^i(\boldsymbol{s}, \boldsymbol{r}) = \exp(ik_v n_0 \boldsymbol{s}\cdot\boldsymbol{r}), \qquad (2)$$

where $\boldsymbol{s}$ is a unit vector in the direction of propagation and n_0 is the constant refractive index outside the object. The total wave U^t is written as the sum of the incident wave U^i and the scattered one U^s. Under the assumption

$$|U^i| \gg |U^s|, \qquad (3)$$

eq. (1) is reduced to the equation

$$\nabla^2 U^s + k_0^2 U^s = F_1 U^i, \qquad (4)$$

where

$$F_1 = -k_0^2\{(n/n_0)^2 - 1\} \qquad (5)$$

*Derivations of the inversion formulae are not given in this paper; they will be published in the near future.

and $k_0=k_v n_0$. Equation (4) is the first order Born approximation of eq. (1). By using Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ of eq. (4), U^s can be expressed by an integral representation[2)]

$$U^s(s, \boldsymbol{r}) = -\frac{1}{4\pi}\int F_1(\boldsymbol{r}')U^i(s, \boldsymbol{r}')G(\boldsymbol{r}, \boldsymbol{r}')\,\mathrm{d}V', \quad (6)$$

where

$$G(\boldsymbol{r}, \boldsymbol{r}') = \mathrm{i}\pi H_0(k_0|\boldsymbol{r}-\boldsymbol{r}'|) \quad (7)$$

for the two-dimensional problem. In this and the subsequent equations the Hankel function of the first kind and mth order is denoted by H_m. The present problem is reduced to the problem of finding an inversion formula of eq. (6).

To estimate the valid region of the approximation, we introduce a simple model which has a constant refractive index $n_0+\Delta n$ inside a circle with radius a:

$$n(\boldsymbol{r}) = \begin{cases} n_0 & \text{for } |\boldsymbol{r}|>a \\ n_0+\Delta n & \text{for } |\boldsymbol{r}|\leqq a. \end{cases} \quad (8)$$

When we calculate U^s using eq. (6) for this model and compare U^i with U^s at $|\boldsymbol{r}|=0$, eq. (3) is converted to the condition[9)]

$$k_v \Delta n a \ll 1. \quad (9)$$

If we adopt this as a rough criterion of validity, the Born approximation seems effective for a small object unless refractive index variation is very small. For example, we obtain the condition $a \ll 1$ mm for light in visible spectrum when $\Delta n \sim 10^{-4}$.

2.2 *Inversion formula*

Wolf has proposed an inversion formula of eq. (6).[2)] In his proposal U^s is measured on two fixed planes (x=const) shown in Fig. 1 with broken lines and its Fourier transform is calculated. However, when an object having small dimension is considered, the scattered wave is almost cylindrical for an ordinary size of the measuring surface. Phase of the cylindrical wave varies very rapidly on the measuring planes. In order to express U^s properly, the scattered wave has to be sampled at an interval in the order of wavelength, which causes the computation to require extraordinarily long time and large memory.

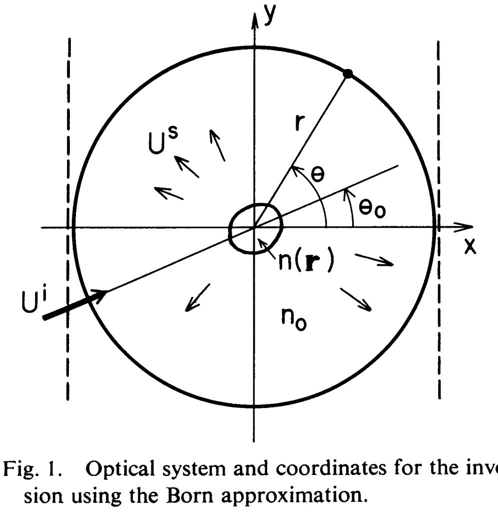

Fig. 1. Optical system and coordinates for the inversion using the Born approximation.

To remove this difficulty we have to measure U^s on a cylindrical surface (r=const) shown in Fig. 1 as a bold circle. The inversion formula for this system is obtained by expressing the position with polar coordinates (r, θ) and (r', θ'), and the propagation direction of the incident waves with θ_0:

$$H_m(k_0 r)\,\mathrm{i}^{m+1}A(m, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} U^s(r, \theta, \theta_0) \times \exp(-\mathrm{i}m\theta)\,\mathrm{d}\theta, \quad (10\text{-a})$$

$$\hat{F}_1(k, \xi) = -\frac{1}{\pi^2}\sum_{m=-\infty}^{\infty} A(m, \theta_0) \times \exp\{\mathrm{i}m(2\xi-\theta_0-\pi)\}, \quad (10\text{-b})$$

where

$$k = 2k_0\cos(\xi-\theta_0-\pi) \quad \frac{\pi}{2} \leqq \xi-\theta_0 \leqq \frac{3}{2}\pi. \quad (11)$$

In these equations $\hat{F}_1$ is the Fourier transform of F_1 and (k, ξ) is the polar coordinate in the Fourier plane. For a single incident wave, $\hat{F}_1$ can be calculated on a circle represented by eq. (11) and shown in Fig. 2. From measurements with various θ_0, we obtain $\hat{F}_1$ inside the circle of radius $2k_0$ and the Fourier inversion can be made with resolution of $\lambda_0/2$, where λ_0 is the wavelength in the surrounding medium. Distribution of refractive index can be calculated with this resolution.

2.3 *Relation to X-ray Structure analysis*

When $k_0 r \to \infty$, U^s in eq. (6) is approximated by the equation

$$U^s(r, \theta, \theta_0) = -\mathrm{i}\pi^2\sqrt{\frac{2}{\pi k_0 r}} \times \exp\left\{\mathrm{i}\left(k_0 r - \frac{\pi}{4}\right)\right\} U_0^s(\theta, \theta_0), \quad (12)$$

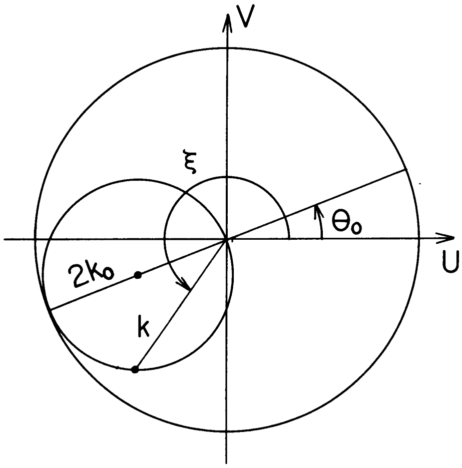

Fig. 2. Fourier plane and the region accessible by the inversion using the Born approximation.

where

$$U_0^s(\theta, \theta_0) = \frac{1}{4\pi^2}\int F_1(r', \theta') \times \exp[ik_0 r'\{\cos(\theta'-\theta_0) - \cos(\theta-\theta')\}] \times r'\,dr'\,d\theta'. \quad (13)$$

Substitution from eq. (12) into eq. (10) leads to

$$\hat{F}_1[2k_0\cos(\xi-\theta_0-\pi), \xi] = U_0^s(2\xi-\theta_0-\pi, \theta_0). \quad (14)$$

This agrees with the basic equation in X-ray structure analysis.[10] It should be noted that U_0^s does not contain the unimportant phase factor $\exp(ik_0 r)$, which causes the difficulty in the Wolf's calculation procedure. Equation (10) corresponds to a generalization of the inversion formula of the X-ray structure analysis, which is valid also when the measuring surface is not in the Fraunhofer region.

§3. Rytov's Approximation

3.1 *Approximation of the wave equation*

We shall introduce logarithmic amplitudes defined by the equations

$$\psi^t = \ln U^t,\ \psi^i = \ln U^i,\ \psi_1 = \ln(U^t/U^i). \quad (15)$$

By using these notations, eq. (10) is rewritten into an equation

$$\nabla^2\psi_1 + (\nabla\psi_1)^2 + 2\nabla\psi^i\nabla\psi_1 + k_v^2(n^2 - n_0^2) = 0. \quad (16)$$

Under the assumption

$$|\nabla\psi^i| \gg |\nabla\psi_1|, \quad (17)$$

this equation is reduced to

$$\nabla^2(\psi_1 U^i) + k_0^2(\psi_1 U^i) = F_1 U^i. \quad (18)$$

As this equation has the same form as eq. (4), its integral representation is given in the same form as eq. (6) in terms of the Green's function defined by eq. (7):

$$\psi_1(s, r)U^i(s, r) = -\frac{1}{4\pi}\int F_1(r')U^i(s, r') \times G(r, r')\,dV'. \quad (19)$$

This approximation is due to Rytov.[11] The inversion of this equation is the present problem.

A rough criterion[12] of the validity of this approximation is obtained from eq. (17) as

$$n_0 \gg |n - n_0| \quad (20)$$

for large-scale distribution of refractive index: i.e. for an object where no significant variation of refractive index exists within a scale much larger than the wavelength of light. For small-scale distribution, the amplitude of the scattered wave must be small compared with the amplitude of the incident wave in addition to eq. (20). This condition is the same as that of the Born approximation.

As far as the above criterion is concerned, Rytov's approximation seems to impose no severe restriction on the size of the object with large-scale distribution of refractive index.

3.2 *Inversion formula*

The inversion formula similar to the Born approximation may be applied to the present problem because both integral representations have a similar form. However, in order to avoid rapid phase variation on the left side of eq. (19), we have to measure ψ_1 on a plane parallell to the wavefront of the incident light. Such a measuring surface (x_0 = const) is shown in Fig. 3. In the figure the coordinate system (x, y) is fixed and the coordinate system (x_0, y_0) rotates according to the direction of the incident wave. In this system U^t may be called as a transmitted wave. The inversion formula is obtained in a similar procedure given by Wolf[2] for the inversion of eq. (6). The result is as follows:

$$\hat{\psi}_1(x_0, v, \theta_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_1(x_0, y_0, \theta_0) \times \exp(-ivy_0)\,dy_0, \quad (21\text{-a})$$

$$\hat{F}_1(U, V) = \frac{iu}{\pi}\exp\{-i(u-k_0)x_0\}\hat{\psi}_1(x_0, v, \theta_0), \quad (21\text{-b})$$

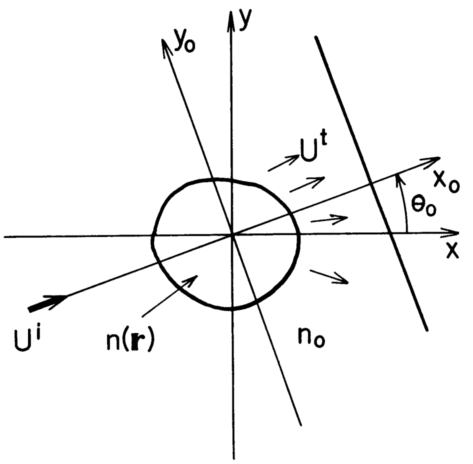

Fig. 3. Optical system and coordinates for the inversion using Rytov's approximation.

where

$$
\begin{aligned}
U &= (u-k_0)\cos\theta_0 - v\sin\theta_0,\\
V &= (u-k_0)\sin\theta_0 + v\cos\theta_0,\\
u^2+v^2 &= k_0^2.
\end{aligned} \tag{22}
$$

In these equations arguments of the Fourier transform $\hat{F}_1$ are in a rectangular coordinate (U, V). For a single incident wave, $\hat{F}_1$ can be calculated on a half circle expressed by eq. (22) and shown in Fig. 4. However, due to a large interval of practical sampling of ψ_1, its Fourier component can be calculated only in a region of small v. As a result, by varying θ_0, $\hat{F}_1$ is obtained in a region of rather small spatial frequency: *e.g.* a region enclosed by a broken circle in Fig. 4. Refractive index can be calculated by the Fourier inversion although the resolution is limited. The limited resolution does not harm its applicability if Rytov's approximation is valid also for large objects. If the sampling is made in an interval of the order of wavelength and also the back scattered wave is measured, accessible region becomes larger and resolution improves.

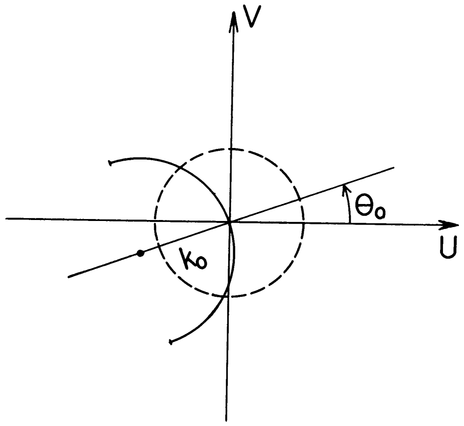

Fig. 4. Fourier plane and the region accessible by the inversion using Rytov's approximation.

3.3 *Relation to straight path approximation*

In the case when $k_0\to\infty$, as $v\ll k_0$, eqs. (21-b) and (22) are reduced to the equations

$$\hat{n}_1(U, V) = \frac{1}{2\pi}\left[\frac{1}{k_v i}\hat{\psi}_1(x_0, v, \theta_0)\right] \tag{23}$$

and

$$U = -v\sin\theta_0, \quad V = v\cos\theta_0, \tag{24}$$

where $n_1 = n - n_0$ and $\hat{n}_1$ is the Fourier transform of n_1. As the real part of the value in the brackets corresponds to optical path difference, this equation agrees with that of straight path approximation.[5] The imaginary part of eq. (23) has a similarity with the equation used in the three-dimensional reconstruction of absorption coefficient from radiographs and electron micrographs.[13,14]

Error Δx_0 in the determination of position of the measuring surface affects the value of $\hat{F}_1$ through the exponential factor in eq. (21-b). The condition that we can neglect the error is that

$$
\begin{aligned}
|(u-k_0)\Delta x_0| &= |U\cos\theta_0 + V\sin\theta_0|\\
&\quad \times |\Delta x_0| \ll 2\pi.
\end{aligned} \tag{25}
$$

In a qualitative expression, measuring surfaces should be located more accurately as we wish to calculate refractive index with higher resolution. This result is applicable also to ordinary "uni-directional interferometry," where refractive index distribution is calculated under the assumption of straight path of light and constant refractive index along the path.

§4. Conclusion

Two approximate inversion formulae in multi-directional interferometry were given for the two-dimensional case. Both are valid when variation of refractive index in an object is small. As for the size of the object, the formula using the Born approximation, eq. (10), seems more applicable to a small object. On the other hand, the formula using Rytov's approximation, eq. (21), is applicable to a large object when practical measurement is taken into consideration. In this sense the two procedures are complementary.

The shape of the measuring surface should conform to the respective situation in order to avoid computational difficulty: for the Born approximation and a small object the measuring surface is suggested to be a cylinder and for Rytov's approximation the measuring surfaces are planes which are parallel to the wavefronts of corresponding incident light.

Discussions on the relation of multi-directional interferometry to X-ray structure analysis clarified that they can be considered as different aspects of the same problem: an inverse problem for the wave equation having a variable coefficient.

Although the given inversion formulae are two-dimensional, the above conclusions will be applicable to three-dimensional case.

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