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Universal Relation between the Variances of Distortions of Gravitational Waves owing to Gravitational Lensing

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Abstract

Gravitational waves from the distant sources are gravitationally lensed during their propagation through the intervening matter inhomogeneities before arriving at detectors. It has been proposed in the literature that the variance of the lensed waveform can be used to extract information of the matter power spectrum at very small scales and of low-mass dark halos. In this Letter, we show that the variance of the amplitude fluctuation and that of the phase fluctuation of the lensed waveform obey a simple relation irrespective of the shape of the matter power spectrum. We study conditions under which this relation can be violated and discuss some potential applications of the relation. This relation may be used to confirm the robustness of claimed observations of gravitational lensing of gravitational waves and the subsequent reconstruction of the matter power spectrum.

Unified Astronomy Thesaurus concepts: Gravitational lensing (670); Gravitational waves (678)

1. Introduction

Direct detections of gravitational waves (GWs) have opened the golden era of GW astronomy (Abbott et al. 2021). It is highly expected that many new discoveries by the GW experiments will excite us in the coming decades and progress our understanding about the universe (e.g., Bailes et al. 2021). One of such discoveries is the gravitational lensing of the GWs (Schneider et al. 1992). General relativity (GR) predicts that GWs propagating in the gravitational potential sourced by matter distribution are gravitationally lensed in a similar manner to light (Misner et al. 1973). Gravitational lensing typically magnifies gravitational-wave amplitude and may significantly contribute to the high mass tail of the mass distribution of the black hole binaries (Dai et al. 2017; Oguri 2018). One notable difference between the gravitational lensing of the GWs and that of light is that, due to the longwavelength nature of the GWs from astrophysical sources, wave effects such as diffraction become important in some cases for which wave optics must be used, while the geometrical optics is an excellent approximation for light (Ohanian 1974; Nakamura 1998; Nakamura & Deguchi 1999).¹

In the regime of wave optics, the amplification factor, which represents the amount of distortion of the wave by the gravitational lensing, depends on the wave frequency differently from what it does in the geometrical optics (Nakamura & Deguchi 1999). Because of this, the lensed waveform in the wave optics provides us with additional information about the lensing objects that the geometrical optics does not. For instance, even if the source position is outside the Einstein radius, for which there is only a single path in the geometrical optics, the mass of the lensing object may be extracted in the wave optics (Takahashi & Nakamura 2003). Another important feature of the wave effects is that the waves are insensitive to structures smaller than the Fresnel scale $r_{\rm F}$, which scales with the wave frequency f as $\propto 1/\sqrt{f}$ (Macquart 2004; Takaha-shi 2006). Thus, measurement of the amplification factor at different frequencies, which is possible for GWs whose frequency varies in time as is the case for the GWs emitted from the chirping binaries, allows us to probe the matter distribution at different Fresnel scales (Takahashi 2006; Oguri & Takahashi 2020; Gil Choi et al. 2021).

In Takahashi (2006), a novel idea was proposed that measurements of the variance of the amplitude and phase fluctuations of the amplification factor enable us to determine the matter power spectrum at the Fresnel scale. As an example, for the GWs in the decihertz range, which corresponds to the sensitivity range of the future space interferometers such as DECIGO (Seto et al. 2001), the Fresnel scale is about 1 pc (see Equation (12)). The matter inhomogeneities at this scale are supposed to be dominated by low-mass dark halos, and information of the matter power spectrum around that scale will provide us with the fundamental properties of dark matter and possibly new knowledge of the primordial power spectrum. In Oguri & Takahashi (2020), as an extension of the previous work (Takahashi 2006), detectability of the low-mass dark halos as well as the primordial black holes has been investigated, and it is concluded that the measurements of the gravitational lensing variance are a promising achievement in the future GW observations. Because of the fundamental importance of the variance of the amplification factor of the GWs, there is a good motivation to study its basic properties.

In this Letter, we point out that there is an intriguing consistency relation between the variance of the amplitude and the phase fluctuations of the amplification factor that has not been given in the literature. Remarkably, this relation holds true irrespective of the shape of the matter power spectrum. Thus, this relation may provide a consistency test to evaluate if the observational determination of the variance of the gravitational lensing has been done correctly and allow us to conduct the subsequent reconstruction of the matter power spectrum on a solid basis. We also study in which situations the consistency

¹ However, there is an exception. For the very compact lensing objects such as primordial black holes, the wave optics plays an important role even for light (Gould 1992; Sugiyama et al. 2020).

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THE ASTROPHYSICAL JOURNAL LETTERS, 918:L30 (5pp), 2021 September 10



Figure 1. Schematic configuration of the gravitational lensing of the GWs by the intervening matter inhomogeneities.

relation can be violated and discuss some potential applications of the consistency relation.

2. Weak Gravitational Lensing of GWs

2.1. Amplification Factor

In this subsection, we briefly overview the formulation of the gravitational lensing of GWs by the matter inhomogeneities. This overview is intended to make this Letter self-contained and hence the content is minimal. Those who want to know more about individual equations and statements are recommended to read Oguri & Takahashi (2020) and references therein.

Ignoring the tiny variation of the polarization of the GWs by the lensing objects, the amplitude ϕ of the lensed GWs at the detector's position is represented, in the frequency domain, by the product of the amplification factor *F* and the unlensed wave ϕ_0 as

$$\phi(f) = F(f)\phi_0(f). \tag{1}$$

Here f is the (comoving) frequency of the GWs. Throughout this Letter, we assume weak lensing for which the deviation of F from unity is given by the linear order in the gravitational potential Φ sourced by the matter inhomogeneities (Born approximation). In this regime, the amplification factor of the GWs emitted from the source at the comoving distance χ_s from the detector is given by (Takahashi et al. 2005)

$$F(f) - 1 = -4\pi f^2 \int_0^{\chi_s} d\chi \frac{\chi_s}{\chi(\chi_s - \chi)} \\ \times \int d^2 r \ \Phi(\chi, \mathbf{r}; t(\chi)) e^{2\pi i f \Delta t(\mathbf{r})}.$$
(2)

Here χ is the comoving distance from the detector, \mathbf{r} is the twodimensional vector perpendicular to the line of sight, $t(\chi)$ is the cosmic time when the wave is at χ , and Δt is the geometric time delay given by

$$\Delta t(\mathbf{r}) = \frac{\chi_s}{2\chi(\chi_s - \chi)} \mathbf{r}^2.$$
(3)

See Figure 1 as a schematic picture representing the configuration. For future convenience, we introduce the Fourier

transformation of Φ as

$$\Phi(\chi, \mathbf{r}; t) = \int \frac{dk_{\parallel}}{2\pi} \int \frac{d^2k_{\perp}}{(2\pi)^2} \,\tilde{\Phi}(k_{\parallel}, k_{\perp}; t) e^{ik_{\parallel}\chi + ik_{\perp}\cdot\mathbf{r}}.$$
 (4)

Substituting this expression into Equation (2) and performing integration over r, we obtain

$$F(f) - 1 = -4\pi i f \int_0^{\chi_s} d\chi \int \frac{dk_{\parallel}}{2\pi} \\ \times \int \frac{d^2 k_{\perp}}{(2\pi)^2} \tilde{\Phi}(k_{\parallel}, k_{\perp}; t(\chi)) e^{i k_{\parallel} \chi - i \frac{\chi(\chi_s - \chi)}{4\pi f \chi_s} k_{\perp}^2}.$$
(5)

...

The amplification factor is a complex number. In physical terms, the absolute value and the argument of complex of F give the magnification and the phase shift, respectively. In particular, in the high frequency limit (geometrical optics limit), the phase shift of F becomes proportional to the frequency with its coefficient being 2π times the Shapiro time delay Δt_g . We then define a quantity $\hat{\eta}$ by subtracting the Shapiro time delay from the phase shift, namely,

$$\hat{\eta}(f) \equiv F(f)e^{-2\pi i f \Delta t_g} - 1, \tag{6}$$

where the second term is added just to make $\hat{\eta}(f)$ vanish in the absence of the lensing potential. Observationally, it should be in principle possible to determine $\hat{\eta}$ for the lensed GWs if the waveform covers a frequency range including both the geometrical and wave optics regimes for which case we can set the phase shift to zero in the high-frequency side. Continuous measurements of GWs from the evolving binaries may be promising for this purpose.

We introduce K(f) and S(f) by²

$$\hat{\eta}(f) + 1 = (1 + K(f))e^{iS(f)}.$$
 (7)

In Takahashi (2006), K(f) was called amplitude fluctuation and S(f) was called phase fluctuation. In what follows, we adopt these terminology.

Since the matter inhomogeneities are randomly distributed, K(f) and S(f) also behave as stochastic variables for each GW event. Therefore, the statistical properties of these quantities are more useful to extract cosmological information than looking at the individual lensed events. The two-point correlation function of K(f) and S(f) is related to the power spectrum of Φ as³

$$\langle K^2(f) \rangle = (4\pi f)^2 \times \int_0^{\chi_s} d\chi \int \frac{d^2 k_\perp}{(2\pi)^2} P_{\Phi}(k_\perp; t(\chi)) \sin^2\left(\frac{r_{\rm F}^2 k_\perp^2}{2}\right), \tag{8}$$

² To the first order in the gravitational potential, Equation (7) leads to $K(f) = \text{Re}(\eta(f))$ and $S(f) = \text{Im}(\eta(f))$. The variances of K(f) and S(f) given by Equations (8) and (9) are derived under this first-order approximation. ³ In Takahashi (2006) and Oguri & Takahashi (2020), $\langle K^2(f) \rangle$ and $\langle S^2(f) \rangle$ are expressed in terms of the power spectrum of the matter density contrast δ . This can be done by going through the Poisson equation. In Equations (8) and (9), we instead use P_{Φ} since it is the metric perturbations that cause the gravitational lensing and Equations (8) and (9) manifest that they are free from the relation between the density contrast and the metric perturbations. Please refer to the discussions in Section 2.2 regarding this point.

THE ASTROPHYSICAL JOURNAL LETTERS, 918:L30 (5pp), 2021 September 10

$$\langle S^2(f) \rangle = (4\pi f)^2 \int_0^{\chi_s} d\chi \int \frac{d^2 k_\perp}{(2\pi)^2} P_{\Phi}(k_\perp; t(\chi))$$

$$\times \left[1 - \cos\left(\frac{r_{\rm F}^2 k_\perp^2}{2}\right) \right]^2.$$

$$(9)$$

Here P_{Φ} is the power spectrum of $\tilde{\Phi}$ defined by

$$\left| \Phi(\boldsymbol{k}, t) \Phi(\boldsymbol{k}', t) \right\rangle = (2\pi)^3 P_{\Phi}(\boldsymbol{k}, t) \delta(\boldsymbol{k} + \boldsymbol{k}'), \qquad (10)$$

and $r_{\rm F}$ is the Fresnel scale (Macquart 2004; Takahashi 2006)

$$r_{\rm F}^2 \equiv \frac{\chi(\chi_s - \chi)}{2\pi f \chi_s}.$$
 (11)

The Fresnel scale is determined by the ratio of the geometrical distance (3) to the GW wavelength. This means the Fresnel scale provides a rough indication of the length scale of the matter inhomogeneities below which GWs are not sensitive to. In mathematical language, this can be understood as the trigonometric functions appearing in the integration for the expressions of $\langle K^2(f) \rangle$ and $\langle S^2(f) \rangle$ and acting as erasing the contributions of the modes shorter than the Fresnel scale. Thus, frequency dependence of $\langle K^2(f) \rangle$ and $\langle S^2(f) \rangle$ contains information of the matter power spectrum at the Fresnel scale, and this fact enables us to probe the matter inhomogeneities on the Fresnel scale. For the cosmological GW sources, a typical value of the Fresnel scale is

$$r_{\rm F} \approx 130 \ {\rm pc} \left(\frac{f}{10^{-3} \,{\rm Hz}}\right)^{-1} \left(\frac{\chi(\chi_s - \chi)/\chi_s}{10 \,{\rm Gpc}}\right).$$
 (12)

Thus, both space- and ground-based GW detectors can probe the matter inhomogeneities on much smaller scales than on cosmological scales. In particular, it has been suggested that the matter perturbations around the parsec scales are dominated by dark low-mass halos. Measuring the amplitude and the phase fluctuations has the potential to probe how much the dark matter is in the form of the gravitationally bound objects and shed light on the nature of dark matter.

2.2. Universal Relation between the Amplitude and the Phase Fluctuations

In Section 2.1, we gave a brief overview of the amplitude and the phase fluctuations of the GWs lensed by the matter inhomogeneities. Now, we point out that there is a consistency relation between the variance of the amplitude fluctuation and that of the phase fluctuation as

$$\langle K^2(f) \rangle + \langle S^2(f) \rangle = \langle K^2(2f) \rangle.$$
 (13)

This relation can be straightforwardly derived from Equations (8) and (9). As far as we understand, this relation has not been given in the literature, and Equation (13) is one of the main result of this Letter. The purpose of this subsection is to discuss potential use and consequences of this relation.

Let us first clarify the fundamental assumptions we have made to arrive at the consistency relation to figure out the domain in which the relation holds true and how universal the relation is. The assumptions are as follows: (i) gravitational lensing is weak (i.e., Born approximation is valid), (ii) GR is valid, and (iii) statistical properties of the matter perturbations Inamori & Suyama

respect the homogeneity and isotropy. Let us consider these assumptions one by one.

First, as for the weak gravitational lensing, it has been demonstrated that the typical amplitude of K(f) and S(f) is $\mathcal{O}(10^{-2}) - \mathcal{O}(10^{-3})$ in a frequency range of our interests for the standard Λ CDM cosmology (Oguri & Takahashi 2020).⁴ These amplitudes are much smaller than unity, and corrections from higher orders in the gravitational potential are suppressed more than the leading-order one we have presented in Section 2.1. Therefore, it is reasonable to expect that most lensing events are in the weak gravitational lensing regime.

Second, as for point (ii), the formalism presented in the previous subsection has been developed within the framework of GR. However, the consistency relation can hold true even for some alternative theories of gravity. For instance, the simple class of the scalar-tensor theories only modifies the scalar part of the metric perturbations, and the propagation equation for the GWs is still the same as that in GR (De Felice & Tsujikawa 2012; Bellini & Sawicki 2014; Saltas et al. 2014). In such a case, the background metric on which the GWs propagate can be written as

$$ds^{2} = -(1+2\Phi)dt^{2} + a^{2}(t)(1-2\gamma\Phi)d\mathbf{x}^{2}, \qquad (14)$$

where γ represents deviation from GR ($\gamma = 1$ in GR if we ignore the anisotropic stress of the matter). In the scalar-tensor theories, γ is in general a function of the cosmic time and the length scales whose concrete form depends on the model under consideration (Amendola et al. 2008). Following the derivation of the amplification factor from the background metric given by Equation (14) in the case of GR ($\gamma = 1$) (Nakamura & Deguchi 1999) and applying it to the case with $\gamma \neq 1$, we find that the modification to the amplification factor is only to replace Φ with $\frac{1+\gamma}{2}\Phi$. Thus, both the amplitude fluctuation and the phase fluctuation are rescaled by the same factor, and the consistency relation is not modified. Generally, modifying the scalar sector changes a relation between the matter density contrast and the gravitational potential (i.e., Poisson equation; Amendola et al. 2008) and leads to the deviation of the growth rate of the matter inhomogeneities from GR, but change of the matter power spectrum due to the different growth rate does not affect at all the expressions of $\langle K^2(f) \rangle$ and $\langle S^2(f) \rangle$ since it is the gravitational potential that causes the gravitational lensing. In this sense, not all the elements in GR are equally crucial to derive the relation (13).

Third, the statistical homogeneity and isotropy, namely Equation (10), has been employed to arrive at Equations (8) and (9). Violation of this assumption will in general lead to the violation of the relation (13) but depending on how we abandon the assumption the consistency relation may still be satisfied. For instance, dropping the statistical isotropy only replaces $P_{\Phi}(k_{\perp}; t(\chi))$ appearing in the integrands of Equations (8) and (9) with the direction-dependent one $P_{\Phi}(k_{\perp}; t(\chi))$. We can easily verify that such a replacement does not change the consistency relation.

To summarize, there are a few assumptions imposed to derive the relation (13), but depending on how one violates those assumptions the consistency relation can be still satisfied under less restrictive conditions. It is also worth mentioning

 $[\]frac{1}{4}$ These typical magnitudes are values without accounting for selection of the lensing signal beyond a certain signal-to-noise ratio (S/N).



Figure 2. Differential merger rate detectable by DECIGO above the signal-tonoise ratio $\rho = 100, 500, 1000.$

that as it is clear from how the consistency relation has been derived, it holds irrespective of the shape of the matter power spectrum. Thus, the consistency relation does not rely on the physics that underlies the shape of the matter power spectrum such as the nature of dark matter and the early-universe models (inflation) characterizing the primordial power spectrum.

Having explained the underlying assumptions to obtain the consistency relation, let us next discuss its potential applications. One practical application would be to use the relation as an independent confirmation that the observational determination of the variances of K(f) and S(f) from the measurements of many lensing events has been done correctly. Observationally, measurements of the amplification factor is conducted by comparing the data with the unlensed template waveform. Thus, extracting the amplification factor correctly can be achieved successfully only when we have the correct theoretical modeling of the unlensed GW waveform that requires correct quantitative understanding of the GW sources. In other words, incorrect modeling of the sources generates spurious contributions to K(f) and S(f). Such undesired bias will lead to the violation of the consistency relation. Given that the expected amplitudes of K(f)and S(f)are $\mathcal{O}(10^{-3}) - \mathcal{O}(10^{-2})$ (Oguri & Takahashi 2020), we naively suppose that the understanding of the GW sources at the same level is requisite to avoid such bias. Conversely, confirmation that the observationally determined $\langle K^2(f) \rangle$ and $\langle S^2(f) \rangle$ satisfy the consistency relation provides us with a solid confidence that the information of the gravitational lensing has been obtained correctly, and we can safely use this observational result to determine/constrain the small-scale power spectrum of the matter inhomogeneities. In this sense, the consistency relation will be useful to establish the matter power spectrum by the measurements of the lensing of GWs.

A second potential application of the consistency relation is to infer $\langle K^2(f) \rangle$ at a higher frequency range that is out of the GW measurements. The relation tells us that knowledge of $\langle K^2 \rangle$ and $\langle S^2 \rangle$ at a frequency *f* enables us to infer $\langle K^2 \rangle$ at twice the frequency (2*f*). Thus, it is possible to observationally determine $\langle K^2 \rangle$ up to twice the maximum frequency that GW detectors can reach.

A third potential application is a test of GR. As we have already discussed, the consistency relation can be violated in alternative theories of gravity for which the propagation equation of the GWs deviates from that in GR. Therefore, observational verification of the relation provides a new test if the propagation of GWs obey what GR predicts. However, propagation of GWs has been already constrained to be very close to GR by the almost simultaneous detections of the GWs and gamma rays from the neutron-star mergers (Abbott et al. 2017a, 2017b), and it is not clear how significantly the relation can in principle be violated by changing the propagation properties of the GWs without conflicting the existing constraints. For the implications to the theories of modified gravity deduced from the recent GW observations, see, e.g., Creminelli & Vernizzi (2017) and Langlois et al. (2018). After all, it is possible that the new test is not as strong as the other ones. Even in that case, the consistency relation may be used as an independent confirmation of GR. Clarifying this issue quantitatively is beyond the scope of this Letter. To summarize this subsection, the universal relation between the amplitude and the phase fluctuations has some interesting applications to help our understanding of the universe.

Before closing this subsection, there is one comment that may be worth noting. The consistency relation is given in a simple form and fairly universal. Thus, we expect that there is a clear physical explanation behind it, although we were not able to find it.

2.3. On the Dependence of the Amplitude and the Phase Fluctuations on the Source Distance

Although not explicitly written (just for the sake of notational simplicity), both $\langle K^2(f) \rangle$ and $\langle S^2(f) \rangle$ are functions of not only f but also the (comoving) distance to the source χ_s . In other words, the ensemble average $\langle \cdots \rangle$ is performed for fixed f and χ_s . Observationally, the ensemble average is determined by measuring a sufficient number of the GW events. However, since each GW event has a different distance from us, strictly speaking, it is not possible to accumulate the GW events having exactly the same distance. Furthermore, there will be measurement errors of the distance for each GW event. As a compromise, we need to group the GW events having nearly the same source redshifts to compute the ensemble average for a particular source redshift z_s . This procedure may induce the error of the observational estimation of $\langle K^2(f, z_s) \rangle$ and $\langle S^2(f, z_s) \rangle$. (In this subsection, we explicitly show the dependence of $\langle K^2(f) \rangle$ and $\langle S^2(f) \rangle$ on χ_s in terms of the corresponding source redshift z_s .) According to the results presented in Takahashi (2006), $\langle K^2(f, z_s) \rangle$ and $\langle S^2(f, z_s) \rangle$ change by $\mathcal{O}(1)$ by changing z_s by $\mathcal{O}(1)$. Thus, crudely speaking, the relative change of $\langle K^2(f, z_s) \rangle$ and $\langle S^2(f, z_s) \rangle$ by shifting the source redshift by Δz_s is $\mathcal{O}(\Delta z_s)$. However, since the consistency relation holds true at any z_s , we may circumvent the above difficulty by considering the consistency relation integrated over some redshift range (e.g., $z_1 < z_s < z_2$) covering the scatter of the redshifts of the GW events. In other words, an estimator \mathcal{E} constructed out of data $\{K_i(f, z_{s,i}), S_i(f, z_{s,i})\}_{i=1}^N$ of N GW lensing events lying in the redshift range (z_1, z_2) by

$$\mathcal{E} \equiv \frac{1}{N} \sum_{i=1}^{N} X_i,$$

where $X_i = K_i^2(f, z_{s,i}) + S_i^2(f, z_{s,i}) - K_i^2(2f, z_{s,i}),$ (15)

yields, if the consistency relation holds, $\langle \mathcal{E} \rangle = 0$ and $\langle \mathcal{E}^2 \rangle \propto 1/N$. (Furthermore, for $N \gg 1$, \mathcal{E} obeys the Gaussian distribution thanks to the central limit theorem.) Therefore, if we alternatively consider the averaged consistency relation over some redshift range, the accuracy of its observational

confirmation is limited solely by N, i.e., the number of the GW lensing events.

In the above discussion, we have assumed that K and S for each GW lensing event are measured perfectly without errors. In reality, both K and S are determined with some errors caused by the instrumental noises, which is another crucial factor that hinders the observational verification of the consistency relation. Detectability of the amplitude and the phase fluctuations taking into account the noises is partially discussed in Oguri & Takahashi (2020), but more detailed investigations remain to be clarified. Since the main results in this work are to show the existence of the consistency relation and to propose its potential applications, we just give a crude estimation of Nfor DECIGO above a certain S/N ρ following the method presented in Ding et al. (2015) and Hou et al. (2021) before closing this subsection. Figure 2 shows the differential merger rate $\frac{dR}{dz}$ in the unit of yr⁻¹ for which $\int_0^z \frac{dR}{dz} dz$ gives the detectable number of the merger events of equal mass binary with a chirp mass $26.5 M_{\odot}$ within the redshift z above S/N $\rho = 100, 500, 1000.$ Given that the GW amplitude can be measured with the accuracy $\sim 1/\rho$, $\rho = 1000$ will be a representative value for determining $\langle K^2(f, z_s) \rangle$ and $\langle S^2(f, z_s) \rangle$, whose typical values are about 10^{-3} . As an example, from Figure 2, we expect that one year of operation of DECIGO will detect ~ 100 GW events having $\rho > 1000$ in the range $0.2 < z_s < 0.3$. (Since the relative error of the determination of the source distance is expected to be at most 0.01 (Camera & Nishizawa 2013), the error of z_s is not problematic if the redshift range is taken to be larger.) This suggests that with DECIGO we will be able to confirm the consistency relation or detect its violation at $\mathcal{O}(1)$ level in terms of the relative difference.

3. Conclusion

Detections of the gravitational lensing of GWs are promising in the near future. GWs from the cosmological distant sources are gravitationally lensed by traveling through the matter inhomogeneities before arriving at detectors. It has been proposed in the literature (Takahashi 2006) that the variance of the modulation of the GW waveform can be used to extract information of the matter power spectrum at very small scales and the low-mass dark halos. In this Letter, we have found that the variance of the amplitude fluctuation and that of the phase fluctuation of the amplification factor obey a consistency relation given by Equation (13). This relation is universal in the sense that it does not rely on the shape of the matter power spectrum. We then investigated how universal the relation is

and in which cases the relation can be violated. We also discussed some potential applications of the consistency relation that include the confirmation of the observational determination of the variances of the gravitational lensing. After the variances have been determined observationally over some frequency range, the consistency relation may be useful to confirm the robustness of their determinations and enables us to probe the matter spectrum at small scales with confidence by solving either Equation (8) or Equation (9).

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