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Rankine–Hugoniot Shock Conditions for Space and Astrophysical Plasmas Described by Kappa Distributions

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Abstract

This paper provides the set of Rankine–Hugoniot (R–H) jump conditions for shocks in space and astrophysical plasmas described by kappa, distributions. The characteristic result is the development of a new R–H condition that transforms the values of kappa upstream and downstream the shock. The kappa index parameterizes and labels kappa distributions, and it is necessary for characterizing the thermodynamics of space plasmas. This first approach is restricted to non-magnetized plasmas, and the whole achievement is derived by following first principles of statistical mechanics and thermodynamics. The results show that, depending on the shock strength, the kappa indices across the shock may decrease or increase, indicating cases of shock acceleration or deceleration, respectively.

Unified Astronomy Thesaurus concepts: Plasma astrophysics (1261); Space plasmas (1544); Shocks (2086); Heliosphere (711)

1. Introduction

The Rankine–Hugoniot (R–H) jump conditions transform the thermodynamic properties of space and astrophysical plasmas during their passage through shock discontinuities (Rankine 1870; Hugoniot 1887, 1889). These conditions describe the relationships between thermodynamic variables of the plasma flow, such as, density, velocity, temperature, and magnetic field, on both sides of the shock. These relationships enable us to link the thermodynamic properties upstream and downstream of shocks and derive their characteristics, such as the shock speed and normal angles (e.g., see: Colburn & Sonett 1966; Zhuang & Russell 1981; Winterhalter et al. 1984; Szabo 1994; Petrinec & Russel 1997; Owen 2004; Zank et al. 2010; Janvier et al. 2014).

In particular, the R–H conditions connect the upstream and downstream values of (i) the density n, (ii) the bulk velocity of the flow V, (iii) the angle of the velocity flow to the shock normal a, and (iv) the temperature T, based on the following four conservations:

1. Mass conservation:

$$n_1 V_1 \cos a_1 = n_2 V_2 \cos a_2, \tag{1a}$$

2. Momentum conservation (parallel to the shock normal):

$$n_1(mV_1^2 \cos^2 a_1 + k_B T_1) = n_2(mV_2^2 \cos^2 a_2 + k_B T_2), \tag{1b}$$

3. Momentum conservation (perpendicular to the shock normal):

$$V_1 \sin a_1 = V_2 \sin a_2, \tag{1c}$$

4. Energy conservation:

$$mV_1^2 + (2+d)k_B T_1 = mV_2^2 + (2+d)k_B T_2, \tag{1d}$$

where the subscript notation indicates "1" for upstream and "2" for downstream; m is the particle mass (that is, the proton mass in this paper); the polytropic index γ is the exponent associated with a polytropic relationship between pressure—density $P \propto n^{\gamma}$ or density—temperature $n \propto T^{1/(\gamma-1)}$ (e.g., Parker 1963; Chandrasekhar 1967; Totten et al. 1995; Newbury et al. 1997; Kartalev et al. 2006; Nicolaou et al. 2014, 2015); the polytropic index γ can be written in terms of an effective dimensionality, d, that is, $\gamma = 1 + 2/d$ or $d \equiv 2/(\gamma - 1)$ (e.g., Sanderson & Uhrig 1978); thus, $P \propto n^{1+2/d}$ or $n \propto T^{d/2}$. Note: for the adiabatic case (only), the effective d coincides with the actual dimensionality of particles.

The solutions of the above relationships are the four explicit expressions of the downstream variables, expressed in terms of the upstream variables (e.g., Liepmann & Roshko 1957; Livadiotis 2015a):

$$n_2 = n_1 \cdot R, \tag{2a}$$

$$V_2 = V_1 \cdot \sqrt{1 - \cos^2 a_1 (1 - R^{-2})}, \tag{2b}$$

$$T_2 = T_1 \cdot \frac{d + 1 - R^{-1}}{d + 1 - R}, \tag{2c}$$

$$\tan a_2 = R \cdot \tan a_1. \tag{2d}$$

The above Equations 1(a)–(d) provide also the solution for the compression ratio (or shock strength), R, that is,

$$R = \frac{1 + d}{1 + (1 + \frac{1}{2}d)\beta_{TV1}}, \quad \text{or } R = 1, \tag{3}$$

where the thermal ratio β_{TV1} is defined by:

$$\beta_{TV} \equiv \frac{2k_B T}{mV^2 \cos^2 a}. \tag{4}$$

To date, the R–H conditions have been exclusively developed for plasmas described by the classical Maxwell–Boltzmann distributions (e.g., Cairns & Grabbe 1994; Zank et al. 1994, 2010; Livadiotis 2015a). However, space plasmas are characterized by a non-Maxwellian behavior, typically manifested by kappa distributions. Indeed, kappa distributions have become increasingly widespread across the physics of space and astrophysical plasmas, describing particles in the heliosphere, from the solar wind and planetary magnetospheres to the distant heliospheric boundaries and beyond, to the interstellar and intergalactic plasmas (Livadiotis 2017; Ch.1).

The origin of kappa distributions in space plasmas comes from its solid connection with statistical mechanics and thermodynamics. It has been shown that the most generalized form of particle distributions that can be assigned with a temperature, and thus, be consistent with thermodynamics, is that of kappa distributions (or combinations thereof) (Livadiotis 2018a, 2018b). The concept of "thermalization" is the characterization of a particle system residing in any stationary state assigned by a temperature; in these states, the particle velocities or energies are stabilized into kappa distributions.

Kappa distributions are consistent with thermodynamics, but this fact alone cannot justify the generation and existence of these distributions. Once a kappa distribution of velocities or energies is generated by a certain mechanism, the preservation—or not—of this distribution is a matter of thermodynamics alone. Nevertheless, there are a number of mechanisms generating kappa distributions in space plasmas; among others, some important examples are the following: macroscopic extensivity of entropy (Livadiotis 2018c), superstatistics (Beck & Cohen 2003; Schwadron et al. 2010; Hanel et al. 2011; Livadiotis et al. 2016); local correlations among the particles induced by long-range interactions (Livadiotis & McComas 2011a); effect of shock waves (Zank et al. 2006); weak turbulence (Yoon et al. 2006; Yoon 2014, 2019); turbulence with a diffusion coefficient inversely proportional to velocity (Bian et al. 2014); effect of pickup ions (Livadiotis & McComas 2011b); pump acceleration mechanism (Fisk & Gloeckler 2014); polytropic behavior (Livadiotis 2019a, 2019b, 2019c, 2018d, 2016; Nicolaou & Livadiotis 2019); and effects of Debye shielding and magnetic coupling (Livadiotis et al. 2018); for all the above, see Livadiotis (2017, Chapters 5, 6, 8, 10, 15, 16).

The kappa index is the characteristic parameter that governs the kappa distributions, and it is necessary for characterizing the thermodynamics of space plasmas. The physical meaning of the kappa index is interwoven with the particle correlations; the stronger the correlations, the smaller the kappa. The largest value of kappa, i.e., $\kappa \to \infty$, corresponds to the system residing at the classical thermal equilibrium, described by a Maxwell–Boltzmann distribution, and characterized by the absence of any correlations; on the other hand, the smallest possible kappa value, $\kappa \to \frac{3}{2}$, corresponds to the furthest state from classical thermal equilibrium, a state called anti-equilibrium (Livadiotis & McComas 2013), and characterized by the highest correlations. In fact, a simple relationship exists between the correlation coefficient ρ and the kappa index κ, that is, $\rho = \frac{3}{2}/\kappa$ (Livadiotis & McComas 2011a; Livadiotis 2015b, 2017, Chapter 5; for particles with $d = 3$ degrees of freedom).

As it was shown (Livadiotis 2018a), the kappa index is a thermodynamic quantity independent of temperature, where both the temperature and kappa are primarily defined by the zeroth law of thermodynamics and the concept of thermal equilibrium. Thereafter, the kappa index, together with density, temperature, and pressure, constitute the set of basic thermal observables for characterizing the thermodynamics of plasmas described by kappa distributions, such as space and astrophysical plasmas. It is, then, obvious that the set of R–H conditions, which is based on classical statistical mechanics and thermodynamics, needs to be modified to incorporate the concept of kappa distributions.

The set of R–H conditions incorporating kappa distributions must keep the mass, momentum, and energy conservations, i.e., Equations 1(a)–(d). As it is argued in Livadiotis (2015a), the statistical averages of mass, momentum, and energy, are independent of the kappa index, and thus, all kappa indices (infinite or finite) are physically equivalent for describing the thermodynamic nature of the plasma. The respective R–H conditions, corresponding to the conservation of mass, momentum, and energy, can be equivalently used in particle systems described either by Maxwell–Boltzmann or kappa distributions.

Therefore, for space plasmas described by kappa distributions, (i) the density n, (ii) the bulk velocity of the flow V, (iii) its angle to the shock normal a, and (iv) the temperature T, are still given by the standard R–H conditions, given by Equations 2(a)–(d); even so, the shock strength R is still given by Equation (3).

What is actually missing is an additional equation that will connect the kappa value upstream and downstream the shock, $\kappa_2 = f(\kappa_1; R)$. As mentioned by Livadiotis (2015a), "the values of the kappa index upstream and downstream the shock must be connected through a new R–H condition, yet to be discovered."

There were several attempts to resolve the problem of incorporating kappa distributions in the R–H conditions. Unfortunately, these were based on temperature misinterpretations. In the earlier years of the theory of kappa distributions, there was the impression that the temperature was dependent on the kappa index, i.e., $T = [\kappa/(\kappa - \frac{3}{2})] \cdot t$, where t was interpreted as a temperature-related quantity that remains invariant for all kappas, frequently called the Maxwellian temperature. Given such a relationship between temperature and kappa index, $\kappa(T) = \frac{3}{2}T/(T - t)$, and having already known the R–H condition for temperature, i.e., Equation 2(c), then it could be trivially used to develop the R–H condition for kappa index, i.e., $\kappa_2/\kappa_1 = T_2(T_1 - t)/[T_1(T_2 - t)]$ (e.g., Vogl et al. 2003). However, the development of the theory the previous two decades showed that the temperature and kappa index are both independent thermodynamic parameters (and that t is depended on both κ and T). (For further details, see: Livadiotis 2014, 2018a, 2018b, 2018e; see also the book of kappa distributions, Livadiotis 2017, Chapter 1.)

The paper attempts to develop a new set of R–H conditions incorporating kappa distributions, which is based on physical principles and consistent with thermodynamics. Specifically, the paper constructs the relationship between the upstream and downstream values of kappa indices, $\kappa_2 = f(\kappa_1; R)$, based on the following physical frameworks: (i) nonextensive statistical mechanics; (ii) connection of kappa distributions with statistical mechanics and thermodynamics; and (iii) mechanism of "superstatistics." The presented first approach is restricted to non-magnetized plasmas. In Section 2 we presented the method followed, which was based on the above physical concepts, in order to find the R–H condition for the kappa index. The results of our theoretical analysis are shown in Section 3, while they are discussed in detailed in Section 4. Finally, Section 5 summarizes the findings of this paper.

2. Method

The presented theoretical analysis is based on the following theoretical frameworks, (i) nonextensive statistical mechanics (e.g., Tsallis 1988, 2009; Tsallis et al. 1998); (ii) connection of kappa distributions with statistical mechanics and thermodynamics (e.g., Livadiotis & McComas 2009; Livadiotis 2017, Ch.1; 2018a); and (iii) mechanism of "superstatistics," that is, the analysis of a kappa distribution to superposition of Maxwell–Boltzmann distributions of variant temperatures (e.g., Beck & Cohen 2003; Schwadron et al. 2010; Hanel et al. 2011; Livadiotis et al. 2016), in order to develop the evolution of kappa distributions across a shock, thus deriving the relationship between the upstream and downstream values of the kappa index, $\kappa_2 = f(\kappa_1; R)$.

Nonextensive statistical mechanics is a generalization of the classical statistical physical theory of Boltzmann and Gibbs; among others, it describes particle systems with local correlations such as space and astrophysical plasmas. The involved entropy is a mono-parametrical function of the probability distribution. This entropic parameter q can attain any value, while $q \to 1$ recovers the Boltzmannian entropy and the Boltzmann–Gibbs statistical mechanics. The nonextensive statistical mechanics is the background of kappa distributions. The theory of kappa distributions shows that the kappa and the entropic q indices are connected through $\kappa = 1/(q-1)$.

The kappa index depends on the particle dimensionality or degrees of freedom d, as $\kappa = \kappa(d) = \text{const.} + \frac{1}{2}d$, so that the difference $\kappa(d) - \frac{1}{2}d$ remains invariant under variations of d. The invariant kappa index κ_0 is defined by $\kappa_0 \equiv \kappa(d) - \frac{1}{2}d$; hence, we obtain $\kappa(d) = \kappa_0 + \frac{1}{2}d$. The physical meaning of the thermodynamic parameter kappa is better carried by its invariant value κ_0, as this is independent of the degrees of freedom (Livadiotis & McComas 2011a, 2013; Livadiotis 2015b, 2015c, 2017). Therefore, throughout this paper, we use the notion of the invariant kappa index κ_0, but the typical three-dimensional index can be easily retrieved,

$$\kappa = \kappa_0 + \frac{3}{2}. \tag{5}$$

Specifically, the mechanism of "superstatistics" (e.g., Beck & Cohen 2003; Schwadron et al. 2010; Hanel et al. 2011; Livadiotis et al. 2016) deals with the analysis of the canonical distribution (derived within nonextensive statistical mechanics) as a superposition of Maxwell–Boltzmann distributions. It shows that the kappa index can be expressed in terms of thermal fluctuations, i.e., $\kappa_0 = F(\delta T/T)$. The infogram in Figure 1 shows the path of the analytical derivations.

3. Results

3.1. Derivation of the Kappa Index Expression in Terms of the Temperature Standard Deviation

First, we derive the relationship $\kappa_0 = F(\delta T/T)$. This derivation is based on the theory of superstatistics, one of the possible mechanisms that can generate kappa distributions in space plasmas. Below, we expose in more detail the theory of superstatistics and how it is involved in the derivation of $\kappa_0 = F(\delta T/T)$.

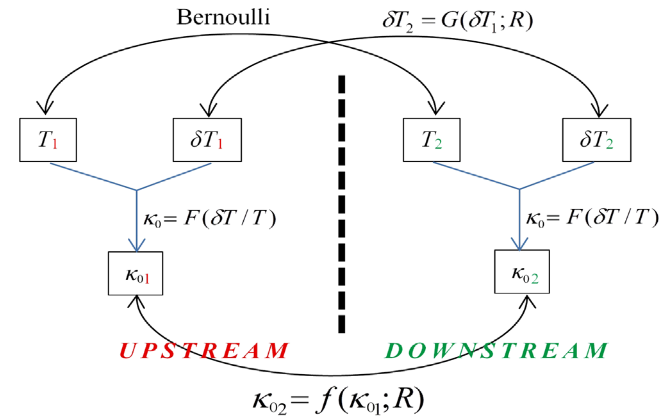

Figure 1. Infogram showing the method for deriving the R–H condition for the kappa index, $\kappa_{02} = f(\kappa_{01}; R)$.

Let the d-dimensional Boltzmannian distribution of kinetic energy ε, for temperature denoted by its inverse $\beta \equiv k_B T^{-1}$, i.e.,

$$P_M(\varepsilon;\, \beta) = \frac{\beta^{\frac{1}{2}d}}{\Gamma\left(\frac{1}{2}d\right)} \cdot \exp(-\beta\, \varepsilon) \cdot \varepsilon^{\frac{1}{2}d-1}. \tag{6}$$

If the temperature is not fixed and characterized by fluctuations, then the system does not reside in the classical thermal equilibrium described by Maxwell distributions. Instead, the superposition of Maxwell distributions with different temperatures can lead to a stationary state that is described by a kappa distribution,

$$P_K(\varepsilon;\, T,\, \kappa_0) = \frac{(\kappa_0 k_B T)^{-\frac{1}{2}d}}{B\left(\frac{1}{2}d,\ 1+\kappa_0\right)} \times \left(1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon}{k_B T}\right)^{-\kappa_0 - 1 - \frac{1}{2}d} \cdot \varepsilon^{\frac{1}{2}d-1}, \tag{7}$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ denotes the Beta function. As we will see, the definition of the global temperature can be related with the inverse of the average value of β, i.e.,

$$k_B T \equiv q_0 \langle\beta\rangle^{-1}, \ \text{ with } q_0 \equiv 1 + 1/\kappa_0, \tag{8}$$

so that

$$P_K(\varepsilon;\, \langle\beta\rangle,\, \kappa_0) = \frac{[(\kappa_0+1)/\langle\beta\rangle]^{-\frac{1}{2}d}}{B\left(\frac{1}{2}d,\ 1+\kappa_0\right)} \times \left(1 + \frac{1}{\kappa_0+1} \cdot \langle\beta\rangle\varepsilon\right)^{-\kappa_0 - 1 - \frac{1}{2}d} \cdot \varepsilon^{\frac{1}{2}d-1}. \tag{9}$$

This is the generalization of the Maxwellian distribution in Equation (6) for finite values of the kappa index κ_0, while Equation (6) recovers for $\kappa_0 \to \infty$.

It has been shown that particle systems with variable temperature are consistent with a particular distribution function of temperatures, that is, the gamma distribution (a generalization of the chi-square distribution; see: Abramowitz & Stegun 1972), with average $\langle\beta\rangle$ and degrees of freedom

$1 + \frac{1}{2}d_0$,

$$D(\beta) = \frac{\left(\dfrac{1 + \frac{1}{2}d_0}{\langle\beta\rangle}\right)^{1+\frac{1}{2}d_0}}{\Gamma(1 + \frac{1}{2}d_0)} \cdot \beta^{\frac{1}{2}d_0} \cdot \exp\left(-\frac{1 + \frac{1}{2}d_0}{\langle\beta\rangle} \cdot \beta\right). \tag{10}$$

According to superstatistics, the kappa distribution can be generated by the following integral transformation

$$P_K(\varepsilon) = \int_0^\infty P_M(\varepsilon, \beta)D(\beta)d\beta, \tag{11}$$

which expresses the transition from the equilibrium state, described by the Maxwell–Boltzmann distribution, to the non-equilibrium state, described by the kappa distribution (of kappa index κ_0). Then, the integral in Equation (11) is rewritten as:

$$\begin{aligned} P_K(\varepsilon) =& \frac{\left(\dfrac{1 + \frac{1}{2}d_0}{\langle\beta\rangle}\right)^{1+\frac{1}{2}d_0} \varepsilon^{\frac{1}{2}d-1}}{\Gamma\left(\frac{1}{2}d\right)\Gamma\left(1 + \frac{1}{2}d_0\right)} \\ &\times \int_0^\infty \beta^{\frac{1}{2}d+\frac{1}{2}d_0} \exp\left[-\left(\frac{1 + \frac{1}{2}d_0}{\langle\beta\rangle} + \varepsilon\right)\beta\right]d\beta \\ =& \left(\frac{1 + \frac{1}{2}d_0}{\langle\beta\rangle}\right)^{1+\frac{1}{2}d_0} \cdot \frac{\Gamma\left(\frac{1}{2}d_0 + 1 + \frac{1}{2}d\right)}{\Gamma\left(\frac{1}{2}d\right)\Gamma\left(1 + \frac{1}{2}d_0\right)} \\ &\times \left(\frac{1 + \frac{1}{2}d_0}{\langle\beta\rangle} + \varepsilon\right)^{-\frac{1}{2}d_0-1-\frac{1}{2}d} \cdot \varepsilon^{\frac{1}{2}d-1}, \end{aligned} \tag{12}$$

hence,

$$\begin{aligned} P_K(\varepsilon) =& \frac{\left(\dfrac{1 + \frac{1}{2}d_0}{\langle\beta\rangle}\right)^{-\frac{1}{2}d}}{B\left(\frac{1}{2}d, 1 + \frac{1}{2}d_0\right)} \\ &\times \left(1 + \frac{1}{1 + \frac{1}{2}d_0}\langle\beta\rangle\varepsilon\right)^{-\frac{1}{2}d_0-1-\frac{1}{2}d} \cdot \varepsilon^{\frac{1}{2}d-1}. \end{aligned} \tag{13}$$

Then, we include the notion of mean energy:

$$\begin{aligned} \langle\varepsilon\rangle =& \frac{\int_0^\infty \varepsilon\, P_K(\varepsilon)d\varepsilon}{\int_0^\infty P_K(\varepsilon)d\varepsilon} \\ =& \frac{\int_0^\infty \varepsilon^{\frac{1}{2}d} \cdot \left(1 + \frac{1}{1+\frac{1}{2}d_0}\langle\beta\rangle\varepsilon\right)^{-\frac{1}{2}d_0-1-\frac{1}{2}d} d\varepsilon}{\int_0^\infty \varepsilon^{\frac{1}{2}d-1} \cdot \left(1 + \frac{1}{1+\frac{1}{2}d_0}\langle\beta\rangle\varepsilon\right)^{-\frac{1}{2}d_0-1-\frac{1}{2}d} d\varepsilon}, \end{aligned} \tag{14}$$

which leads to the kinetic definition of global temperature, mentioned in Equation (8):

$$\langle\varepsilon\rangle = \frac{\frac{1}{2}d \cdot (1 + \frac{1}{2}d_0)}{\frac{1}{2}d_0\langle\beta\rangle} \equiv \frac{1}{2}d\ k_{\mathrm{B}}T \text{ or } \langle\varepsilon\rangle/\tfrac{1}{2}d = \frac{1 + \frac{1}{2}d_0}{\frac{1}{2}d_0\langle\beta\rangle} \equiv k_{\mathrm{B}}T, \tag{15}$$

thus, the energy distribution is written as

$$\begin{aligned} P_K(\varepsilon) =& \frac{\left(\frac{1}{2}d_0\dfrac{\langle\varepsilon\rangle}{\frac{1}{2}d}\right)^{-\frac{1}{2}d}}{B(\frac{1}{2}d, 1 + \frac{1}{2}d_0)} \\ &\times \left(1 + \frac{1}{\frac{1}{2}d_0} \cdot \frac{\varepsilon}{\langle\varepsilon\rangle/\frac{1}{2}d}\right)^{-\frac{1}{2}d_0-1-\frac{1}{2}d} \cdot \varepsilon^{\frac{1}{2}d-1}, \end{aligned} \tag{16}$$

or substituting the temperature $k_{\mathrm{B}}T \equiv \langle\varepsilon\rangle/\frac{1}{2}d$, we end up with the typical d-dimensional kappa distribution

$$\begin{aligned} P_K(\varepsilon) =& \frac{\left(\frac{1}{2}d_0 k_{\mathrm{B}}T\right)^{-\frac{1}{2}d}}{B(\frac{1}{2}d, 1 + \frac{1}{2}d_0)} \\ &\times \left(1 + \frac{1}{\frac{1}{2}d_0} \cdot \frac{\varepsilon}{k_{\mathrm{B}}T}\right)^{-\frac{1}{2}d_0-1-\frac{1}{2}d} \cdot \varepsilon^{\frac{1}{2}d-1}, \end{aligned} \tag{17}$$

that is, the kappa distribution, as shown in Equation (7), with kappa index $\kappa_0 \equiv \frac{1}{2}d_0$.

The moments of the gamma distribution, Equation (10), are given by:

$$\begin{aligned} \langle\beta^a\rangle = \int_0^\infty \beta^a D(\beta)d\beta =& \frac{\left(\dfrac{1 + \frac{1}{2}d_0}{\langle\beta\rangle}\right)^{1+\frac{1}{2}d_0}}{\Gamma(1 + \frac{1}{2}d_0)} \\ &\times\int_0^\infty \beta^{\frac{1}{2}d_0+a} \cdot \exp\left(-\frac{1 + \frac{1}{2}d_0}{\langle\beta\rangle} \cdot \beta\right)d\beta, \end{aligned} \tag{18}$$

thus,

$$\langle\beta^a\rangle = \frac{\Gamma(1 + \frac{1}{2}d_0 + a)}{\Gamma(1 + \frac{1}{2}d_0)} \cdot \left(\frac{\langle\beta\rangle}{1 + \frac{1}{2}d_0}\right)^a. \tag{19a}$$

Hence,

$$\langle\beta^1\rangle = \frac{\Gamma(1 + \frac{1}{2}d_0 + 1)}{\Gamma(1 + \frac{1}{2}d_0)} \cdot \left(\frac{\langle\beta\rangle}{1 + \frac{1}{2}d_0}\right) = \langle\beta\rangle, \tag{19b}$$

$$\langle\beta^2\rangle = \frac{\Gamma(1 + \frac{1}{2}d_0 + 2)}{\Gamma(1 + \frac{1}{2}d_0)} \cdot \left(\frac{\langle\beta\rangle}{1 + \frac{1}{2}d_0}\right)^2 = \frac{2 + \frac{1}{2}d_0}{1 + \frac{1}{2}d_0}\langle\beta\rangle^2. \tag{19c}$$

Then, the variance is given by:

$$(\delta\beta)^2 \equiv \langle\beta^2\rangle - \langle\beta\rangle^2 = \frac{1}{1+\frac{1}{2}d_0}\langle\beta\rangle^2, \tag{20a}$$

$$\frac{\delta\beta}{\langle\beta\rangle} = \frac{1}{\sqrt{1+\frac{1}{2}d_0}}, \tag{20b}$$

thus, given $\kappa_0 \equiv \frac{1}{2}d_0$, we obtain from Equations (20b) and (15), respectively:

$$\frac{\delta\beta}{\langle\beta\rangle} = \frac{1}{\sqrt{\kappa_0+1}}, \tag{21}$$

$$k_B T = \frac{1+\kappa_0}{\kappa_0}\langle\beta\rangle^{-1} = q_0\langle\beta\rangle^{-1}. \tag{22}$$

Next, we express the normalized standard deviation of temperature, $\delta T/T$, as a function of the normalized standard deviation of the inverse temperature, $\delta\beta/\langle\beta\rangle$. We consider the deviation of the inverse temperature $\beta \to \beta + \Delta\beta$, which corresponds to a propagated deviation to temperature, $T \to T + \Delta T$, hence

$$\frac{\Delta T}{T} = \frac{\Delta\langle\beta\rangle^{-1}}{\langle\beta\rangle^{-1}} = \frac{(\langle\beta\rangle+\Delta\beta)^{-1} - \langle\beta\rangle^{-1}}{\langle\beta\rangle^{-1}}$$
$$= \left(1+\frac{\Delta\beta}{\langle\beta\rangle}\right)^{-1} - 1 = -\frac{\frac{\Delta\beta}{\langle\beta\rangle}}{1+\frac{\Delta\beta}{\langle\beta\rangle}}. \tag{23}$$

Then, using the standard deviations, δT and $\delta\beta$, which are in an absolute value in contrast to the deviations, ΔT and $\Delta\beta$, we find the relationship $\kappa = F(\delta T/T)$, i.e.,

$$\frac{\delta T}{T} = \frac{\frac{\delta\beta}{\langle\beta\rangle}}{1 \pm \frac{\delta\beta}{\langle\beta\rangle}} = \frac{1}{\frac{\langle\beta\rangle}{\delta\beta} \pm 1} \text{ or } \frac{\delta T}{T} = \frac{1}{\sqrt{\kappa_0+1} \pm 1}. \tag{24}$$

The Maxwell–Boltzmann distribution is characterized by zero temperature deviation, while the latter must be larger as the kappa index decreases. It is therefore required to have zero deviation for $\kappa_0 \to \infty$ and infinite deviation when $\kappa_0 \to 0$; hence, we keep "−1" in (24):

$$\frac{\delta T}{T} = \frac{1}{\sqrt{\kappa_0+1} - 1}. \tag{25}$$

3.2. Propagation of the Temperature Standard Deviation

Next, we derive the error propagation relationship $\delta T_2 = G(\delta T_1; R)$. We derive the relationship that connects the thermal fluctuations upstream and downstream of the shock. This is achieved by perturbing the energy conservation condition (that is, the Bernoulli integral), in Equation 2(c), ending up with $\delta T_2 = G(\delta T_1; R)$.

The energy is conserved upstream and downstream of the shock, $\langle E\rangle_1 = \langle E\rangle_2$, where E is the particle kinetic energy in the shock reference frame. Indeed, we have

$$E = \tfrac{1}{2}m\boldsymbol{u}^2 = \tfrac{1}{2}m[(\boldsymbol{u}-\boldsymbol{V})+\boldsymbol{V}]^2$$
$$= \tfrac{1}{2}m[(\boldsymbol{u}-\boldsymbol{V})^2 + \boldsymbol{V}^2 + 2\boldsymbol{V}\cdot(\boldsymbol{u}-\boldsymbol{V})], \tag{26}$$

so that its average becomes

$$\langle E\rangle = \tfrac{1}{2}m\boldsymbol{V}^2 + \tfrac{1}{2}m\langle(\boldsymbol{u}-\boldsymbol{V})^2\rangle = \tfrac{1}{2}m\boldsymbol{V}^2 + (1+\tfrac{1}{2}d)k_B T. \tag{27}$$

Hence, Equation 2(c) is written as $\langle E\rangle_1 = \langle E\rangle_2$. However, for the standard deviation we have also the equality $\delta\langle E\rangle_1 = \delta\langle E\rangle_2$. The summation of standard deviations follows the L^2 norm (sum of squares), so that

$$(\delta\langle E\rangle)^2 = \left[\delta(\tfrac{1}{2}m\boldsymbol{V}^2)\right]^2 + \{\delta[(1+\tfrac{1}{2}d)k_B T]\}^2, \tag{28}$$

that is,

$$(\delta\langle E\rangle)^2 = (mV\delta V)^2 + \left[(1+\tfrac{1}{2}d)k_B\delta T\right]^2. \tag{29}$$

Then, the first term to the right becomes

$$(mV\delta V)^2 = 4(\tfrac{1}{2}mV^2)[\tfrac{1}{2}m(\delta V)^2] = 4\frac{\frac{1}{2}mV^2}{k_B T}\cdot\frac{\frac{1}{2}m(\delta V)^2}{k_B T}$$
$$\times (k_B T)^2 = 4r\beta_{TV}^{-1}\cdot(k_B T)^2, \tag{30}$$

where $\frac{1}{2}mV^2/(k_B T) = \beta_{TV}^{-1}$, as shown in Equation (4), while with r we denote the ratio of the fluctuation energy per thermal energy:

$$r \equiv \frac{\frac{1}{2}m(\delta V)^2}{k_B T}. \tag{31}$$

The limited values of r can be calculated as follows: (i) when the momentum fluctuation $m\delta V$ is mostly transferred to the more mobile electrons (e.g., low ion versus high electron temperature), then we have $m\delta V = m_e u_{\mathrm{th}_e} = \sqrt{2m_e k_B T}$, with $u_{\mathrm{th}_e} = \sqrt{2k_B T/m_e}$, where m_e is the electron mass; in this case, we have $r = m_e/m$. (ii) When the momentum fluctuation $m\delta V$ is mostly transferred among ions as thermal energy, then it is proportional to $\delta V \propto \theta = \sqrt{2k_B T/m}$ or $\frac{1}{2}m\,(\delta V)^2 = \frac{1}{2}d\ k_B T$; in this case, we have $r = \frac{1}{2}d$. Various physical processes and phenomena may contribute to the momentum fluctuation $m\delta V$ and the ratio r, e.g., turbulence and magnetic field, where the values of r lie between $m_e/m \leqslant r \leqslant \frac{1}{2}d$. However, we understand that $r = \frac{1}{2}d$ is the most profound case, thus, we adopt this value for the examples illustrated by this paper.

Therefore, the energy variance is

$$(\delta\langle E\rangle)^2 = 4r\beta_{TV}^{-1}(k_B T)^2 + \left(1+\tfrac{1}{2}d\right)^2(k_B T)^2 s^2$$
$$= [4r\beta_{TV}^{-1} + \left(1+\tfrac{1}{2}d\right)^2 s^2]\cdot(k_B T)^2, \tag{32}$$

where we included the factor $s \equiv \delta T/T$.

Then, from the conservation of energy deviation, $\delta\langle E\rangle_1 = \delta\langle E\rangle_2$, we derive

$$\left[4r\beta_{TV1}^{-1} + \left(1+\tfrac{1}{2}d\right)^2 s_1^2\right]\cdot T_1^2$$
$$= \left[4r\beta_{TV2}^{-1} + \left(1+\tfrac{1}{2}d\right)^2 s_2^2\right]\cdot T_2^2. \tag{33}$$

The thermal parameter β_{TV1} is determined by:

$$\beta_{TV1} \equiv \frac{2k_B T_1}{mV_1^2} \text{ with } R = \frac{1+d}{1+(1+\frac{1}{2}d)\beta_{TV1}} \text{ and}$$
$$\beta_{TV1}^{-1} = \frac{1+\frac{1}{2}d}{(1+d)R^{-1}-1}. \quad (34)$$

We then derive the same parameter downstream of the shock:

$$\beta_{TV2} = \frac{2k_B T_2}{mV_2^2} = \beta_{TV1} \cdot \left(\frac{V_1}{V_2}\right)^2 \cdot \frac{T_2}{T_1} = \beta_{TV1} \cdot R^2 \times \frac{d+1-R^{-1}}{d+1-R} = \beta_{TV1} \cdot \frac{(d+1)R-1}{(d+1)R^{-1}-1}, \quad (35)$$

$$\frac{1}{\sqrt{\kappa_{02}+1}-1} = \sqrt{\left(\frac{1}{\sqrt{\kappa_{01}+1}-1}\right)^2 \cdot \left(\frac{d+1-R}{d+1-R^{-1}}\right)^2 + \frac{4r}{1+\frac{1}{2}d} \cdot \frac{(R-R^{-1})\cdot[d+1-(R+R^{-1})]}{(d+1-R^{-1})^2}}, \quad (41a)$$

from Equations 2(b), (c). Its inverse is given by

$$\beta_{TV2}^{-1} = \beta_{TV1}^{-1} \cdot \frac{(d+1)R^{-1}-1}{(d+1)R-1}. \quad (36)$$

Returning to Equation (33) and substituting β_{TV2}^{-1} from Equation (36) and T_2/T_1 from Equation 2(c), we obtain

$$4r\beta_{TV1}^{-1} + \left(1+\tfrac{1}{2}d\right)^2 s_1^2 = \left[4r\beta_{TV1}^{-1} \cdot R^{-2}\frac{d+1-R}{d+1-R^{-1}} + \left(1+\tfrac{1}{2}d\right)^2 s_2^2\right] \cdot \left(\frac{d+1-R^{-1}}{d+1-R}\right)^2, \quad (37)$$

and solving in terms of s_2:

$$s_2 = \frac{d+1-R}{d+1-R^{-1}} \times \sqrt{s_1^2 + \frac{4r\beta_{TV1}^{-1}}{\left(1+\frac{1}{2}d\right)^2} \cdot \left(1 - R^{-2} \cdot \frac{d+1-R^{-1}}{d+1-R}\right)}. \quad (38)$$

After some calculus using Equation (3), we derive:

$$s_2 = \sqrt{s_1^2 \cdot \left(\frac{d+1-R}{d+1-R^{-1}}\right)^2 + \frac{4r}{1+\frac{1}{2}d} \cdot \frac{(R-R^{-1})\cdot[d+1-(R+R^{-1})]}{(d+1-R^{-1})^2}}, \quad (39a)$$

while in the case of $r \ll 1$, the above is approximated to

$$s_2 \cong \left(\frac{d+1-R}{d+1-R^{-1}}\right) \cdot s_1. \quad (39b)$$

Equation 39(a), or 39(b), constitutes the relationship $s_2 = G(s_1; R)$, as set in the infogram of Figure 1. Also, we may readily derive the condition for the temperature deviation, that is,

$$\delta T_2 = \sqrt{\delta T_1^2 + \frac{4r}{1+\frac{1}{2}d} \cdot \frac{(R-R^{-1})\cdot[d+1-(R+R^{-1})]}{(d+1-R)^2} T_1^2}. \quad (40)$$

3.3. Connection between Upstream and Downstream Kappa Indices

At this last step, using the relationships $\kappa_0 = F(\delta T/T)$ (25) and $\delta T_2 = G(\delta T_1; R)$ (40), we end up with the final relationship that connects the upstream and downstream kappa indices, $\kappa_{02} = f(\kappa_{01}; R)$.

Using Equations (25), (40) we obtain

or

$$\kappa_{02} = \left\{\left[\left(\frac{1}{\sqrt{\kappa_{01}+1}-1}\right)^2 \cdot \left(\frac{d+1-R}{d+1-R^{-1}}\right)^2 + \frac{4r}{1+\frac{1}{2}d} \times \frac{(R-R^{-1})\cdot[d+1-(R+R^{-1})]}{(d+1-R^{-1})^2}\right]^{-\frac{1}{2}} + 1\right\}^2 - 1. \quad (41b)$$

For small values of $r \ll 1$, Equations 41(a), (b) are approximated to

$$\frac{1}{\sqrt{\kappa_{02}+1}-1} \cong \frac{d+1-R}{d+1-R^{-1}} \cdot \frac{1}{\sqrt{\kappa_{01}+1}-1}, \quad (42a)$$

or

$$\kappa_{02} = f(\kappa_{01}; R) \cong \left[1 + \frac{d+1-R^{-1}}{d+1-R} \cdot \left(\sqrt{\kappa_{01}+1}-1\right)\right]^2 - 1. \quad (42b)$$

Figure 2 plots the downstream kappa index, κ_{02}, as a function of the upstream kappa index, κ_{01}, and for various values of the shock strength $1 \leqslant R < d+1$.

4. Discussion

4.1. Shock Acceleration or Deceleration

The family of curves $\kappa_{02} = f(\kappa_{01}; R)$ can be separated to those with $\kappa_{02} < \kappa_{01}$, and those with $\kappa_{02} > \kappa_{01}$, where the separatrix $\kappa_{02} = \kappa_{01}$ stands for the specific case where the downstream and upstream kappa indices are equal. This is derived by setting $\kappa_{02} = \kappa_{01} \equiv \kappa_{0\mathrm{iso}}$ in Equations 41(a) or 41(b), leading to $\kappa_{0\mathrm{iso}} = f(\kappa_{0\mathrm{iso}}; R)$:

$$\frac{1}{\sqrt{\kappa_{0\mathrm{iso}}(R)+1}-1} = \frac{4r}{1+\frac{1}{2}d} \cdot \frac{d+1-(R+R^{-1})}{2(d+1)-(R+R^{-1})}, \text{ or} \quad (43a)$$

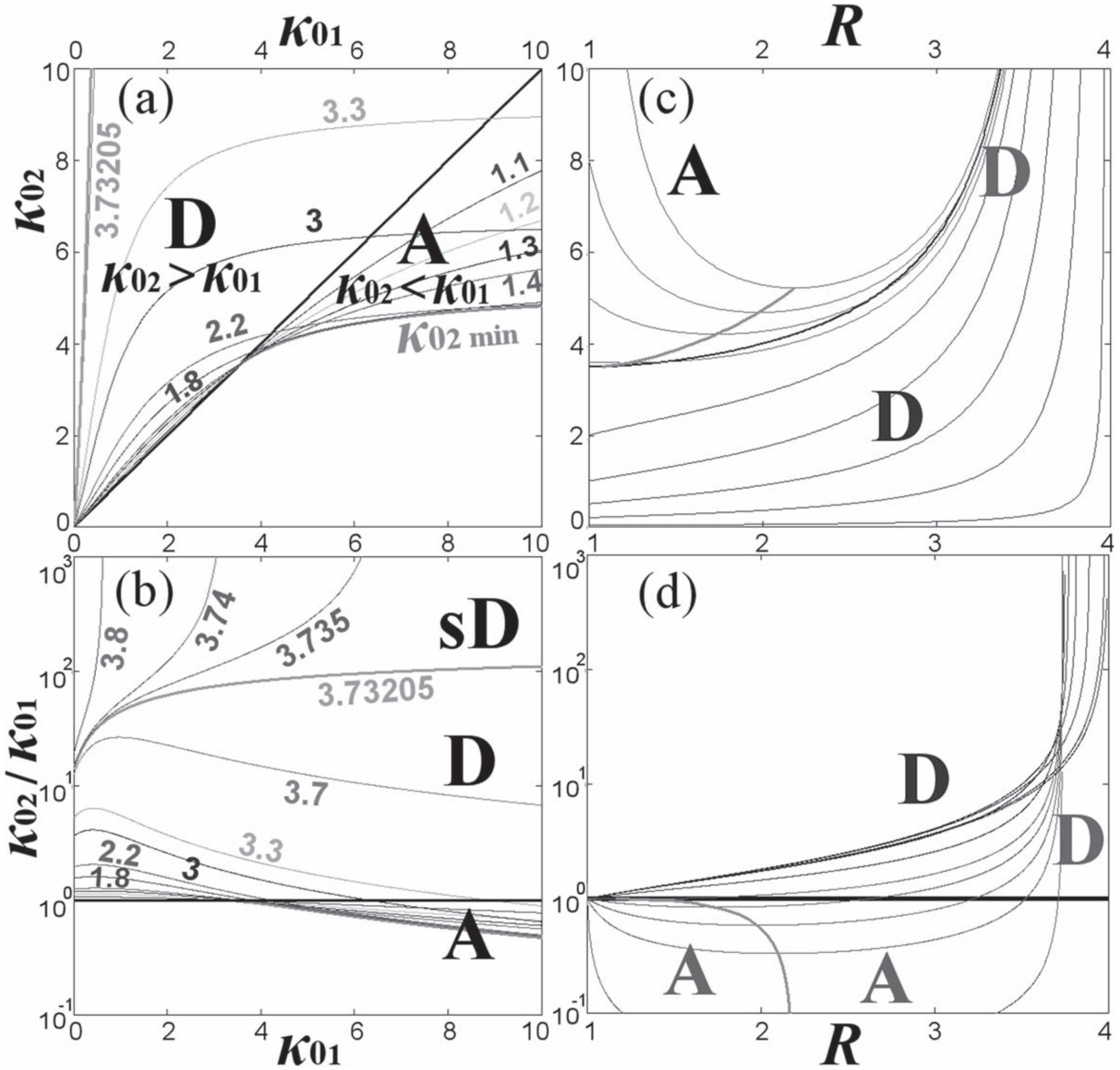

Figure 2. Relationship between the downstream and upstream kappa indices for various shock strength values R. The downstream kappa index κ_{02} (a), and its ratio with the upstream kappa index κ_{02}/κ_{01} on a semi-log scale (b), are plotted against κ_{01}, for various shock strengths R. In (c) and (d), the same quantities are plotted against R, for various upstream kappa indices κ_{01}. (Note: the upstream kappa κ_{01} in (c) can be deduced from the vertical axis for $R = 1$.) Explicitly shown are the regions of acceleration (A) $\kappa_{02} < \kappa_{01}$, deceleration (D) $\kappa_{02} > \kappa_{01}$, and the separatrix $\kappa_{02} = \kappa_{01} \equiv \kappa_{0\mathrm{iso}}$, for shock strength R smaller than the critical $R_c \approx 3.73205$; for $R \geqslant R_c$, the super-deceleration (sD) region appears.

$$\kappa_{0\mathrm{iso}}(R) = \left\{\left[\frac{4r}{1+\frac{1}{2}d} \cdot \frac{d+1-(R+R^{-1})}{2(d+1)-(R+R^{-1})}\right]^{-\frac{1}{2}} + 1\right\}^2 - 1. \tag{43b}$$

The critical curve (plotted with thick black line in Figure 2(a)) separates shocks that accelerates the plasma flow (i.e., $\kappa_{02} < \kappa_{01}$) from those shocks that decelerate the plasma flow (i.e., $\kappa_{02} > \kappa_{01}$). Indeed, when the kappa index decreases ($\kappa_{01} \to \kappa_{02} < \kappa_{01}$), then the suprathermal tail at the high energy part of the distribution becomes harder (more flattened), thus, a larger number of particles correspond to higher energies and a smaller number to moderate energies (from which particles are accelerated). When the kappa index increases ($\kappa_{01} \to \kappa_{02} > \kappa_{01}$), then the suprathermal tail becomes softer (more sharpened), thus a larger number of particles correspond to moderate energies and a smaller number to higher energies (from which particles are decelerated). (For more details, see, Figures 6 and 7 in Livadiotis & McComas 2011a; Figure 1 in Livadiotis 2014; Figure 1 in Livadiotis 2019b.)

Setting $R = 1$ in Equation 43(b), we obtain the minimum possible upstream kappa index, $\kappa_{01\,\min}$, for which the downstream kappa index is smaller than the upstream index, $\kappa_{02} < \kappa_{01}$; hence, the acceleration is possible only for $\kappa_{01} > \kappa_{01\min}$, where:

$$\kappa_{01\min}(d, r) = \left\{\left[4r \cdot \frac{d-1}{d(d+2)}\right]^{-\frac{1}{2}} + 1\right\}^2 - 1. \tag{44}$$

In the case of $r = d/2$ and $d = 3$, we find $\kappa_{01\min} = 3.486\,\ldots$.

The acceleration (A) and deceleration (D) regions are shown in Figures 2(a), (b). In panel (a) the downstream kappa index κ_{02} is plotted as a function of the upstream kappa index κ_{01}, for various shock strength values, R. Region D {A} is above {below} the diagonal $\kappa_{02} = \kappa_{01}$. We observe that for small upstream kappa indices, e.g., $\kappa_{01} < 1$, the corresponding downstream indices for the same shock strength are sharply larger, $\kappa_{02} \gg \kappa_{01}$, with $d\kappa_{02}/d\kappa_{01} \gg 1$; however, for quite larger values of κ_{01}, the downstream indices increase less, with $d\kappa_{02}/d\kappa_{01} \ll 1$ (function $\kappa_{02} = f(\kappa_{01}; R)$ is concave). In panel (b) the ratio of the two indices, κ_{02}/κ_{01}, is plotted against κ_{01}, on a semi-log scale (larger scales are better illustrated by this panel). The ratio of the two indices increases for small upstream kappa indices κ_{01}, while for larger indices it reaches a maximum, and then declines and intercepts the horizontal line $\kappa_{02}/\kappa_{01} = 1$ at a certain index $\kappa_{01\text{iso}}(R)$, as shown by Equation 43(b); the values of κ_{02}/κ_{01} above {below} the horizontal line correspond to deceleration {acceleration}. The smallest intercept of κ_{02}/κ_{01} with the horizontal line at 1, is given by $\kappa_{01\min} = 3.486\ldots$. The described behavior occurs for shock strengths smaller than a critical value given by $R_c \approx 3.73205$; exactly for this shock strength, we have $d\kappa_{02}/d\kappa_{01} \to 0$ as $\kappa_{01} \to \infty$, while for $R > R_c$, the downstream kappa index is dramatically increasing, even for very small upstream kappa indices, a behavior we call super-deceleration (sD).

The behavior of the downstream kappa index, κ_{02}, as a function of the shock strength, R, is illustrated in Figures 2(c), (d). Panel (c) plots the downstream index κ_{02} against R, for various upstream kappa indices κ_{01}, while panel (d) plots the corresponding ratio, κ_{02}/κ_{01}, against R on a semi-log scale. The strongest acceleration is achieved at the local minimum of $\kappa_{02} = f(R)$, as shown in Equation 41(b); the shock strength corresponding to this minimum is given by the quartic solution of:

$$R_{\min}:\ \left(s_1 - \frac{4r}{1+\frac{1}{2}d}\right)R^4 + \frac{1}{d+1}\left[-[(d+1)^2+2]s_1 + \frac{2r}{1+\frac{1}{2}d}[(d+1)^2+4]\right]R^3 - 3\left(s_1 - \frac{2r}{1+\frac{1}{2}d}\right)R^2 - (d+1)\left(s_1 - \frac{2r}{1+\frac{1}{2}d}\right)R - \frac{2r}{1+\frac{1}{2}d} = 0. \quad (45)$$

The minimum shock strength, $R_{\min}$, corresponds to the smallest possible downstream kappa index, κ_{02}, thus, to the strongest acceleration for a given upstream kappa index, κ_{01}; note: s_1 is expressed in terms of κ_{01}, as shown in Equation (25).

4.2. Symmetry

Strictly mathematically, the two sides of the shock must be equivalent (due to mirror symmetry). The shock compresses the plasma with strength R, while on the other side rarefies the plasma with strength R^{-1}. There is no mathematical preference for any side of the shock, upstream or downstream; then, any R–H shock condition should obey to this symmetry: $1 \leftrightarrow 2$ and $R \leftrightarrow R^{-1}$. Indeed, Equation 39(a) is characterized by this symmetry

$$s_2^2 = s_1^2 \cdot \left(\frac{d+1-R}{d+1-R^{-1}}\right)^2 + \frac{4r}{1+\frac{1}{2}d} \cdot \frac{(R-R^{-1})\cdot[d+1-(R+R^{-1})]}{(d+1-R^{-1})^2}$$
$$\Leftrightarrow$$
$$s_1^2 = s_2^2 \cdot \left(\frac{d+1-R^{-1}}{d+1-R}\right)^2 + \frac{4r}{1+\frac{1}{2}d} \cdot \frac{(R^{-1}-R)\cdot[d+1-(R^{-1}+R)]}{(d+1-R)^2}. \quad (46a)$$

For the approximation shown in Equation 39(b), and applying the same symmetry $1 \leftrightarrow 2$ and $R \leftrightarrow R^{-1}$, we obtain

$$s_2 \cong s_1 \cdot \frac{d+1-R}{d+1-R^{-1}} \Leftrightarrow s_1 \cong s_2 \cdot \frac{d+1-R^{-1}}{d+1-R}. \quad (46b)$$

4.3. Connection with the Polytropic Index

In the case where the plasma particles are subject to a potential energy, the polytropic behavior has been shown to have a one-to-one relationship with kappa distributions. Namely, not just the old-known result that the kappa distributions can lead to the polytropic relationship, $P \propto n^{\gamma}$ or $n \propto T^{1/(\gamma-1)}$ (Meyer-Vernet et al. 1995; Moncuquet et al. 2002; Livadiotis 2017, Ch. 5; 2018d; Nicolaou & Livadiotis 2019), but the reverse derivation has also been shown, that is, the polytropic behavior requires the particle velocities to be described by the kappa distributions. This means that the polytropic behavior has the role of a mechanism generating kappa distributions (Livadiotis 2019a, 2016). Then, the developed relationship between the kappa indices upstream and downstream of the shock leads to the corresponding relationship between the polytropic indices.

4.4. What's Next: Incorporation of the Magnetic Field and Turbulence

It is our next goal to work on the generalization of the R–H conditions for particle velocities described by kappa distributions and oblique shocks in magnetized space plasmas. In this case, the temperature fluctuation δT will have to be connected with both the momentum fluctuation $m\delta\boldsymbol{V}$ and the magnetic fluctuation $\delta\boldsymbol{B}$ (Zhuang & Russell 1981; Cairns & Grabbe 1994). Finally, further investigation is necessary for understanding the role and contribution of turbulence in the fluctuations of $m\delta\boldsymbol{V}$ and $\delta\boldsymbol{B}$, as well as of the turbulent energy (e.g., Bavvasano et al. 2000; Adhikari et al. 2015), derived from the variance of the Elsässer variables (Tu & Marsch 1995).

5. Summary and Conclusions

This paper developed the R–H jump conditions for shocks in space and astrophysical non-magnetized plasmas described by kappa distributions. The set of R–H conditions incorporating kappa distributions must keep the known mass, momentum, and energy conservations. Therefore, for space plasmas described by kappa distributions, (i) the density n, (ii) the bulk velocity of the flow V, (iii) its angle to the shock normal a, and (iv) the temperature T, are still given by the standard R–H conditions. The challenge was to derive the R–H jump condition for the new thermodynamic parameter, the kappa index.

This was achieved using the following theoretical frameworks: (i) nonextensive statistical mechanics; (ii) connection of kappa distributions with statistical mechanics and thermodynamics; and (iii) mechanism of "superstatistics" (that is, the analysis of a kappa distribution to superposition of Maxwell–Boltzmann distributions of variant temperatures). The result was to develop the evolution of kappa distributions across a shock, that is, to derive the relationship between the upstream and downstream values of the kappa index, $\kappa_{02} = f(\kappa_{01}; R)$. The condition $\kappa_{02} = f(\kappa_{01}; R)$ was derived following three steps: (i) used of the theory of superstatistics, in order to derive the kappa index expression in terms of the temperature standard deviation, $\kappa_0 = F(\delta T/T)$; (ii) apply the propagation theory of the temperature standard deviation, in order to derive the expression between the downstream and upstream kappa indices, $\delta T_2 = G(\delta T_1; R)$; and (iii) combine the derived relationships to connect the upstream and downstream kappa indices, $\kappa_{02} = f(\kappa_{01}; R)$ (e.g., see Figure 1).

Therefore, the four R–H jump conditions incorporating the kappa distributions are:

1. Mass conservation:

$$n_1 V_1 = n_2 V_2, \tag{47a}$$

2. Momentum conservation:

$$n_1(mV_1^2 + k_B T_1) = n_2(mV_2^2 + k_B T_2), \tag{47b}$$

3. Energy conservation, that is,
   3.1. Average Energy, $\langle E\rangle_1 = \langle E\rangle_2$:

$$\tfrac{1}{2}mV_1^2 + (1 + \tfrac{1}{2}d)k_B T_1 = \tfrac{1}{2}mV_2^2 + (1 + \tfrac{1}{2}d)k_B T_2, \tag{47c}$$

   3.2. Energy deviation, $\delta\langle E\rangle_1 = \delta\langle E\rangle_2$:

$$\left[4r\beta_{TV1}^{-1} + \left(1 + \tfrac{1}{2}d\right)^2 s_1^2\right]\cdot(k_B T_1)^2 = \left[4r\beta_{TV2}^{-1} + \left(1 + \tfrac{1}{2}d\right)^2 s_2^2\right]\cdot(k_B T_2)^2, \tag{47d}$$

where $\beta_{TV}^{-1} = mV^2/(2k_B T)$ and $s = \delta T/T$, leading to the desired relationship, $\kappa_2 = f(\kappa_1; R)$:

$$\left(\frac{1}{\sqrt{\kappa_{01}+1}-1}\right)^2\cdot(d+1-R)^2 + \frac{4r}{1+\frac{1}{2}d} \times R\,[d+1-(R+R^{-1})] = \left(\frac{1}{\sqrt{\kappa_{02}+1}-1}\right)^2 \times(d+1-R^{-1})^2 + \frac{4r}{1+\frac{1}{2}d}\cdot R^{-1}[d+1-(R^{-1}+R)]. \tag{48}$$

The presented theoretical analysis was restricted under the following simplifications: non-magnetized plasma; plasma flow parallel to shock normal; isotropic dimensionality and polytropic indices equal for both sides of the shock; and absence of potential energy. It is therefore straightforward to follow the method presented here in extended future theoretical analyses, in order to include oblique shocks in magnetized and anisotropic space and astrophysical plasmas.

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