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NONSINGULAR UNIVERSES IN GAUSS–BONNET GRAVITY'S RAINBOW

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ABSTRACT

In this paper, we study the rainbow deformation of Friedmann-Robertson-Walker (FRW) cosmology in both Einstein gravity and Gauss–Bonnet (GB) gravity. We demonstrate that the singularity in FRW cosmology can be removed because of the rainbow deformation of the FRW metric. We obtain the general constraints required for FRW cosmology to be free of singularities. We observe that the inclusion of GB gravity can significantly change the constraints required to obtain nonsingular universes. We use rainbow functions motivated by the hard spectra of gamma-ray bursts to deform FRW cosmology and explicitly demonstrate that such a deformation removes the singularity in FRW cosmology.

Key words: cosmological parameters – cosmology: theory – early universe

1. INTRODUCTION

Although it has not yet been possible to construct a quantum theory of gravity, there are many proposals for quantum gravity, and such proposals can have interesting physical consequences (Amelino-Camelia 2001, 2002; Mogueijo & Smolin 2003; Smolin 2006). In fact, many of these proposals have predicted similar physical consequences. One such nearly universal prediction of many of these different approaches to quantum gravity is the deformation of the standard relativistic dispersion relation (Kempf et al. 1995; Brau 1999). Such a deformation of the standard relativistic dispersion relation occurs in various different approaches to quantum gravity, such as spacetime discreteness ('t Hooft 1996), spacetime foam models (Amelino-Camelia et al. 1998), spontaneous symmetry breaking of Lorentz invariance in string field theory (Kostelecky & Samuel 1989), and spin-network in Loop quantum gravity (Cambini & Pullin 1999). The standard energy–momentum dispersion relation is deformed to a modified dispersion relation (MDR) near the Planck scale. It is possible to use an MDR to explain certain astronomical and cosmological observations, such as the threshold anomalies of ultra-high-energy cosmic rays and TeV photons (Amelino-Camelia et al. 1998; Colladay & Kostelecky 1998; Takeda et al. 1999; Finkbeiner et al. 2000; Myers & Pospelov 2003; Jacobson et al. 2004; Amelino-Camelia 2013).

The MDR is based on the existence of a maximum energy scale, and so it is possible to construct a theory with such an intrinsic maximum energy scale (Amelino-Camelia 2002; Mogueijo & Smolin 2003). This theory is called doubly special relativity. In this theory, the Planck energy ($E_{\rm P}$) and the velocity of light (c) are two universally invariant quantities. Just as it is not possible for a particle to attain a velocity greater than the velocity of light in special relativity, it is not possible for a particle to attain an energy larger than the Planck energy in doubly special relativity. In doubly special relativity, the Lorentz transformations are deformed to a set of nonlinear Lorentz transformations in momentum space. In fact, this deformation of the Lorentz transformations directly deforms the standard energy–momentum relation. It is possible to extend doubly special relativity to a curved spacetime and to obtain doubly general relativity (Mogueijo & Smolin 2004). In this theory, it is assumed that the geometry of spacetime depends on the energy of the test particle. So, we do not have a single metric describing the geometry of spacetime, but instead a one-parameter family of energy-dependent metrics. These metrics depend on the energy of the test particles. As we have a family of energy-dependent metrics in thisa theory, it is referred to as gravity's rainbow (Amelino-Camelia et al. 1998; Mogueijo & Smolin 2004).

Recently, gravity's rainbow has been used to study the high-energy behavior of various physical systems (Galan & Mena Marugan 2004; Aloisio et al. 2006; Hackett 2006; Ling et al. 2007; Garattini & Majumder 2014; Ali & Khalil 2015; Chang & Wang 2015; Santos et al. 2015). Rainbow deformation of various black hole solutions has been performed, and their properties have been studied (Galan & Mena Marugan 2006; Ali 2014; Gim & Kim 2015a, 2015b; Hendi et al. 2015b, 2016; Mu et al. 2015). Hydrostatic equilibrium for compact objects and the structure of neutron stars have also been investigated using gravity's rainbow (Hendi et al. 2015a; Garattini & Mandanici 2016). Furthermore, the effects of gravity's rainbow on wormholes have also been investigated (Garattini & Lobo 2015). Gravity's rainbow has also been used to analyze the effects of rainbow functions on gravitational force and the Starobinsky model of $f(R)$ gravity (Sefiedgar 2015; Chatrabhuti et al. 2016).

It should be noted that string theory can be regarded as a two-dimensional theory, and the target space metric can be regarded as a matrix of coupling constants for this two-dimensional theory. These coupling constants flow due to the renormalization group flow, and so the target space metric will depend on the scale at which spacetime is probed, but this scale would in turn depend on the energy of the test particle used to probe this spacetime. Thus, the target space metric in string theory would depend on the energy of the probe, and so gravity's rainbow is motivated by string theory (Mogueijo & Smolin 2004). It may be noted that the low-energy effective field theory approximation to heterotic string theory (Gross & Witten 1986; Metsaev & Tseytlin 1987a) produces Gauss–Bonnet (GB) gravity (Stelle 1978; Maluf 1987). It has been demonstrated that the low-energy expansion of string theory effective action contains the GB term and a scalar field

(Metsaev & Tseytlin 1987b). It is possible to neglect the effect of this scalar field as it can be regarded as a constant field. GB gravity contains curvature-squared terms and is free of ghosts. Furthermore, the corresponding field equations contain no more than second derivatives of the metric (Boulware & Deser 1985; Zumino 1986; Callan et al. 1989; Cai 2002). Black object solutions have also been studied in GB gravity (Myers & Simon 1988; Cho & Neupane 2002; Mignemi 2006; Chen et al. 2008; Bogdanos et al. 2009; Brihaye et al. 2010; Cai et al. 2010; Hendi & Eslam Panah 2010; Gaete & Hassaine 2013; Ayzenberg & Yunes 2014). As both GB gravity and gravity's rainbow can be motivated by string theory, there is a strong motivation to study the rainbow deformation of GB gravity. In fact, the thermodynamics of black holes has been studied using a combination of gravity's rainbow and GB gravity (Hendi & Faizal 2015). We also note that GB gravity has been used to analyze various cosmological models (Deruelle & Dolezel 2000; Kim & Myung 2004; Amendola et al. 2006; Elizalde et al. 2007; Leith & Neupane 2007; Brihaye & Radu 2008; Chingangbam et al. 2008; Bamba et al. 2014; Capozziello et al. 2014; Kanti et al. 2015). Friedmann-Robertson-Walker (FRW) cosmology in Einstein gravity's rainbow has also been analyzed (Ling 2007; Awad et al. 2013). It was observed that the universe is nonsingular in this model, however, it is important to analyze other cosmological models, so that we can know if this is a model-dependent effect or a general feature of gravity's rainbow. Furthermore, no work has been done on cosmological applications of GB gravity's rainbow, even though there are strong string theoretical motivations to perform such an analysis. Therefore, in this paper, we analyze a cosmological model using GB gravity's rainbow. It is observed that this model is also nonsingular, and so it seems that removal of the Big Bang singularity because of the rainbow deformation is a general feature of the rainbow deformation of any model of gravity.

2. FRW RAINBOW COSMOLOGY IN EINSTEIN GRAVITY

Here, we are going to modify the FRW universe in Einstein gravity's rainbow. We consider the Lagrangian of Einstein gravity with a matter field as

$$\mathcal{L}_{\mathrm{E}} = \mathcal{R} + \mathcal{L}_m, \tag{1}$$

where $\mathcal{R}$ is the Ricci scalar and $\mathcal{L}_m$ is the Lagrangian of matter. Variation of action (1) with respect to the metric tensor $g_{\mu\nu}$ leads to

$$G^{\mathrm{E}}_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

where $G^{\mathrm{E}}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ is the Einstein tensor. The energy–momentum tensor can be expressed as

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu), \tag{3}$$

where ρ and P are the energy density and the pressure of a perfect fluid, respectively. Here, u_μ is defined as

$$u_\mu = (f^{-1}(\varepsilon), 0, 0, 0, 0), \tag{4}$$

and is a unit vector,

$$g^{\mu\nu} u_\mu u_\nu = -1. \tag{5}$$

Now, since we want to analyze this model using gravity's rainbow, we will first review gravity's rainbow. Gravity's rainbow is based on the deformation of the standard energy–momentum dispersion relation,

$$E^2 f(\varepsilon)^2 - p^2 g_1(\varepsilon)^2 = m^2, \tag{6}$$

where $\varepsilon = E/E_p$ and the functions $f(\varepsilon)$ and $g_1(\varepsilon)$ are called rainbow functions, and m is the mass of the test particle. In the IR limit, we have $\lim_{\varepsilon\to 0} f(\varepsilon) = \lim_{\varepsilon\to 0} g_1(\varepsilon) = 1$, and so the standard energy–momentum dispersion relation is recovered in the IR limit of this theory. Thus, gravity's rainbow reduces to standard general relativity in the IR limit. As the Planck energy is the largest energy that a particle can attain, we can write

$$\varepsilon \leqslant 1. \tag{7}$$

The exact forms of the rainbow functions are constructed using various theoretical and observational motivations. In fact, the study of hard spectra from gamma-ray bursts has been used as motivation to construct the following rainbow functions (Amelino-Camelia et al. 1998):

$$f(\varepsilon) = \frac{e^\varepsilon - 1}{\varepsilon} \quad \text{and} \quad g(\varepsilon) = 1. \tag{8}$$

Now, after substituting this into Equation (6), we can find the corresponding MDR:

$$p^2 = E^2 \left(\frac{e^\varepsilon - 1}{\varepsilon}\right)^2. \tag{9}$$

In order to compare the results of Einstein gravity with GB theory, they should be eformulated with identical dimensions. Since the GB term does not contribute in four dimensions, we consider the following five-dimensional spacetime:

$$ds^2 = -\frac{dt^2}{f(\varepsilon)^2} + \frac{R(t)^2}{g(\varepsilon)} dx_i^2, \qquad i = 1, 2, 3, 4, \tag{10}$$

where $R(t)$ is scale factor, $g(\varepsilon) = g_1(\varepsilon)^2$, and we consider a flat universe with $k = 0$. Using the above metric, field Equation (2), and the energy–momentum tensor (3), the FRW equations in Einstein gravity's rainbow can be written as

$$\left(H - \frac{\dot g(\varepsilon)}{g(\varepsilon)}\right)^2 + \frac{\dot g(\varepsilon)}{g(\varepsilon)}\left(H - \frac{3\dot g(\varepsilon)}{4g(\varepsilon)}\right) = \frac{4\pi G\rho}{3f(\varepsilon)^2}, \tag{11}$$

$$\begin{aligned} g(\varepsilon)\Bigg[\dot H - \frac{\ddot g(\varepsilon)}{2g(\varepsilon)} - \frac{\dot g(\varepsilon)}{g(\varepsilon)}\left(H - \frac{\dot g(\varepsilon)}{g(\varepsilon)}\right) \\ + \left(H + \frac{\dot f(\varepsilon)}{2f(\varepsilon)}\right)\left(2H - \frac{\dot g(\varepsilon)}{g(\varepsilon)}\right)\Bigg] - 2\left(H - \frac{\dot g(\varepsilon)}{g(\varepsilon)}\right)^2 \\ + \frac{\dot g(\varepsilon)}{g(\varepsilon)}\left(\frac{3\dot g(\varepsilon)}{2g(\varepsilon)} - 2H\right) = -\frac{8\pi G(\rho + P)}{3f(\varepsilon)^2}, \end{aligned} \tag{12}$$

for which $H = \dot{R(t)}/R(t)$ is the Hubble parameter. It may be noted that we used the notations $\dot A = \frac{dA}{dt}$ and $\ddot A = \frac{d^2A}{dt^2}$. The conservation of energy–momentum tensor can be written as

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda + \Gamma^\mu_{\mu\lambda} T^\lambda_\nu = 0,$$

and this equation reduces to

$$\dot{\rho} + 2\left(2H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)(\rho + P) = 0. \tag{13}$$

We can consider a large range of ultra relativistic particles, which are in thermal equilibrium with an average energy $\epsilon \sim T$. The continuity equation leads to the first law of thermodynamics (as in standard cosmology),

$$d(\rho V) = -PdV, \tag{14}$$

where V is the volume and $V = [R(t)/g(\varepsilon)]^4$. Equation (14), along with the integrability condition $\frac{\partial^2 S}{\partial V \partial P} = \frac{\partial^2 S}{\partial P \partial V}$ (Kolb & Turner 1990), leads to constant entropy:

$$S = \frac{V(\rho + P)}{T} = \text{const}. \tag{15}$$

In this paper, we consider the following equation of state (EoS):

$$P = (\gamma - 1)\rho. \tag{16}$$

FRW spacetime is singular at $t = 0$ if this EoS is used in standard cosmology. In the above equation, γ is the EoS parameter. For this pressure, the average energy ϵ can be written as

$$\epsilon \sim T = c\gamma\rho V, \tag{17}$$

where T is the temperature and c is a constant (which is equal to $1/S$). Using the EoS (16) in Equation (13), we obtain the following equation:

$$\frac{d\rho}{d\ln[R(t)^2/g(\varepsilon)]} = -2\gamma\rho, \tag{18}$$

which can be solved to give a density of $\rho = [R(t)^2/g(\varepsilon)]^{-2\gamma}$. This leads to an average energy of

$$\epsilon = \frac{c\gamma}{R(t)^4}\rho^{\frac{\gamma-2}{\gamma}}. \tag{19}$$

3. MODIFIED FRW RAINBOW COSMOLOGY IN GB GRAVITY

Now, we analyze the FRW universe using gravity's rainbow with the GB term. We also study its effect on the early universe using a semi-classical approximation. The Lagrangian of Einstein-GB gravity can be written as

$$\mathcal{L}_{\text{tot}} = \mathcal{R} + \alpha\mathcal{L}_{\text{GB}} + \mathcal{L}_m, \tag{20}$$

where the parameter α in the second term of Equation (20) is the GB coefficient with dimension $(\text{length})^2$, and $\mathcal{L}_{\text{GB}}$ is the Lagrangian of GB gravity,

$$\mathcal{L}_{\text{GB}} = R_{\mu\nu\tau\sigma}R^{\mu\nu\tau\sigma} - 4R_{\mu\nu}R^{\mu\nu} + \mathcal{R}^2. \tag{21}$$

Variation of action (20) with respect to the metric $g_{\mu\nu}$ leads to

$$G^{\text{E}}_{\mu\nu} + \alpha G^{\text{GB}}_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{22}$$

where $G^{\text{GB}}_{\mu\nu} = 2(R_{\mu\tau\sigma\lambda}\ R_{\nu}^{\tau\sigma\lambda} - 2R_{\mu\tau\nu\sigma}\ R^{\tau\sigma} - 2R_{\mu\lambda}\ R_{\nu}^{\lambda} + \mathcal{R}\ R_{\mu\nu}) - \frac{1}{2}\mathcal{L}_{\text{GB}}g_{\mu\nu}$. Using metric (10), field Equation (22), and the energy–momentum tensor (3), we can write the FRW equations in gravity's rainbow with the GB term as

$$\left(H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)^2 + \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\left(H - \frac{3\dot{g}(\varepsilon)}{4g(\varepsilon)}\right) + \alpha\frac{f(\varepsilon)^2}{8}\left(2H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)^4 = \frac{4\pi G\rho}{3f(\varepsilon)^2}, \tag{23}$$

$$\begin{aligned} & g(\varepsilon)\left[\dot{H} - \frac{\ddot{g}(\varepsilon)}{2g(\varepsilon)} - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\left(H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)\right. \\ & \left. + \left(H + \frac{\dot{f}(\varepsilon)}{2f(\varepsilon)}\right)\left(2H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)\right] + \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\left(\frac{3\dot{g}(\varepsilon)}{2g(\varepsilon)} - 2H\right) \\ & - 2\left(H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)^2 + \frac{\alpha}{2}g(\varepsilon)f(\varepsilon)^2\left(2H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)^2 \\ & \times\left[2(\dot{H} + H^2) + \frac{\dot{g}(\varepsilon)^2}{2g(\varepsilon)^2} + \left(1 - \frac{1}{2g(\varepsilon)}\right)\right. \\ & \times\left(2H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)^2 - \frac{\ddot{g}(\varepsilon)}{g(\varepsilon)} - \left(2H - \frac{\dot{f}(\varepsilon)}{f(\varepsilon)}\right) \\ & \left. \times\left(2H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)\right] = -\frac{8\pi G(\rho + P)}{3f(\varepsilon)^2}. \end{aligned} \tag{24}$$

Here, for $\alpha = 0$, Equations (23) and (24) reduce to Equations (11) and (12), respectively. The conservation equation for GB gravity's rainbow can be written as

$$\dot{\rho} + 2\left(2H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)(\rho + P) = 0. \tag{25}$$

It may be noted that Equation (25) is the same as the conservation equation obtained in Einstein gravity's rainbow (Equation (13)). Now, using the same procedure with Equations (16) and (25), it can be demonstrated that the average energy has the same form for GB gravity (19),

$$\epsilon = \frac{c\gamma}{R(t)^4}\rho^{\frac{\gamma-2}{\gamma}}. \tag{26}$$

4. WHEN IS A NONSINGULAR RAINBOW UNIVERSE IN GB GRAVITY POSSIBLE?

Before analyzing how MDR (Equation (9)) leads to a nonsingular cosmology, it is useful to discuss the general conditions of the rainbow functions that lead to a nonsingular universe. Substituting Equation (16) and the modified Friedmann equation of gravity's rainbow (Equation (23)) into the conservation equation (Equation (25)), we obtain

$$\dot{\rho} = \pm 2\gamma\rho\left\{\frac{8}{\alpha f(\varepsilon)^2}\left[\frac{4\pi G\rho}{3f(\varepsilon)^2} - H\frac{\dot{g}(\varepsilon)}{g(\varepsilon)} - \left(H - \frac{\dot{g}(\varepsilon)}{g(\varepsilon)}\right)^2\right]\right\}^{\frac{1}{4}}. \tag{27}$$

A similar system has been studied in (Awad 2013), and this analysis was performed using the Hubble rate. However, it is also possible to study this model using density ρ instead of the Hubble rate H (Awad et al. 2013). Our analysis will be based on the approach used in Awad et al. (2013), and we will demonstrate that this cosmological model is free from finite-

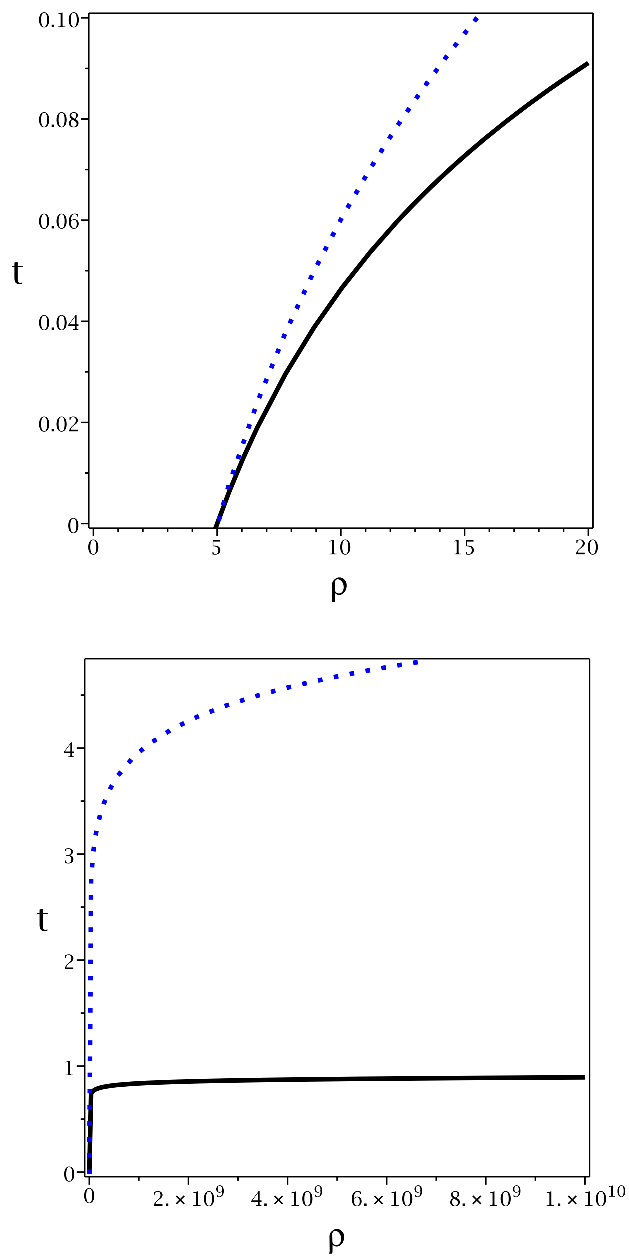

Figure 1. GB case: time vs. density, for $G = 2$, $\gamma = 4/3$, $\alpha = 0.1$, $\rho^* = 5$, $s = 1/5$ (continuous line), and $s = 1/3$ (dotted line): "up and down figures indicate various ρ ranges."

time singularities. This is because an upper bound for the density ρ is reached in an infinite time. Thus, there is a point at which the density diverges; however, that point exists at an infinite time.

To use this explanation, we need a differential equation for ρ with respect to time. If we write $f(\varepsilon)$ and $g(\varepsilon)$ as functions of ρ and R instead of ϵ according to Equation (26), then Equation (27) will be too complicated and we will not be able to solve it. So, to solve this problem, we choose the following form of the solution (separation of variables):

$$g(\varepsilon, t) = G(\varepsilon)R(t), \tag{28}$$

which has the following consequence:

$$\frac{\dot{g}(\varepsilon)}{g(\varepsilon)} = \frac{\dot{g}(\varepsilon, t)}{g(\varepsilon, t)} = \frac{G(\varepsilon)\dot{R(t)}}{G(\varepsilon)R(t)} = \frac{\dot{R(t)}}{R(t)} = H. \tag{29}$$

According to Equation (28), the first deformed FRW Equation (23) reduces to

$$H^2 + \frac{1}{2}f(\varepsilon)^2 H^4 \alpha = \frac{16\pi G\rho}{3f(\varepsilon)^2}. \tag{30}$$

In addition, considering Equation (28), one can show that Equations (25) and (26) become

$$\dot{\rho} + 2H(\rho + P) = 0, \tag{31}$$

$$\epsilon = \frac{c\gamma}{g(\varepsilon)^4}\rho^{\frac{\gamma-2}{\gamma}}. \tag{32}$$

Now, substituting Equations (16) and (30) into Equation (31), we obtain

$$\dot{\rho} = \pm 2\gamma\rho\left[\frac{1}{\alpha f(\rho)^2}\left(\pm\sqrt{1 + \frac{32}{3}\pi G\rho\alpha} - 1\right)\right]^{\frac{1}{2}}, \tag{33}$$

where we will choose the plus sign in parentheses to obtain a consistent equation in Einstein gravity, i.e., in the limit $\alpha \longrightarrow 0$. We expressed f as a function of ρ instead of ϵ according to Equation (32). Now, one can show that finite-time singularities (including Big Bang singularity) are absent if f grows asymptotically as $\rho^{1/4}$, or faster. For example, if $f \sim \rho^s$, where $s \geqslant 1/4$. In this case, one can calculate the time for reaching a potential singularity by integrating Equation (33) (starting from some initial finite density ρ^* to an infinite one). This integration leads to

$$t = \pm\frac{\sqrt{\alpha}}{2\gamma}\int_{\rho*}^{\infty}\rho^{s-1}\left(\sqrt{1 + \frac{32}{3}\pi G\rho\alpha} - 1\right)^{-\frac{1}{2}}d\rho. \tag{34}$$

After some calculations, we obtain

$$t = \pm\frac{\rho^s\{2(16s^2 - 32s + 15)\mathcal{H}_1 + \varrho\}}{\gamma(4s-1)(4s-3)(4s-5)(1+\mathcal{X})^s}\sqrt{\frac{\alpha\mathcal{X}}{2}}\,\Bigg|_{\rho^*}^{\infty}, \tag{35}$$

where

$$\varrho = -(4s-1)\mathcal{X}[(4s-5)\mathcal{H}_2 - (4s-3)\mathcal{X}\mathcal{H}_3],$$

and $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ are the following hypergeometric functions

$$\mathcal{H}_1 = {}_2F_1\left(\left[-s, \frac{1}{2} - 2s\right], \left[\frac{3}{2} - 2s\right], -\mathcal{X}\right),$$

$$\mathcal{H}_2 = {}_2F_1\left(\left[-s, \frac{3}{2} - 2s\right], \left[\frac{5}{2} - 2s\right], -\mathcal{X}\right),$$

$$\mathcal{H}_3 = {}_2F_1\left(\left[-s + 1, \frac{5}{2} - 2s\right], \left[\frac{7}{2} - 2s\right], -\mathcal{X}\right),$$

with

$$\mathcal{X} = \frac{6}{\sqrt{9 + 96\pi G\rho\alpha} - 3}.$$

Comparing various terms in Equation (35), one can show that time is infinite for $s \geqslant 1/4$. Thus, the time to reach the potential singularity is infinite, and so it is not a finite-time singularity, i.e., not physical. For more clarification, we consider the term with ρ as the dominant term in Equation (34), so that we have

$$\begin{aligned} t &= \pm\frac{\sqrt{\alpha}}{2\gamma}\int_{\rho^*}^{\infty} \rho^{s-1}\left(\frac{32}{3}\pi G\rho\alpha\right)^{-\frac{1}{4}} d\rho \\ &= \pm\frac{2\sqrt{3\alpha}}{\gamma(4s-1)}(96\pi G\alpha)^{-\frac{1}{4}}\rho^{s-\frac{1}{4}}\Bigg|_{\rho^*}^{\infty} = \infty, \quad s \geqslant \frac{1}{4} \end{aligned} \tag{36}$$

and here obtain the same result for s. We conclude that the rainbow function $f(\varepsilon)$ plays an important role in possible resolution of the Big Bang singularity, but it has to grow asymptotically as $\rho^{1/4}$, or faster.

Now, we can discuss Equation (36) and plot the $t-\rho$ diagram for $s < 1/4$ and $s > 1/4$ in Figure 1. Considering this figure, one can find an initial finite density at $t = 0$ (present time), as expected. In addition, for $s < 1/4$, we obtain a finite value for time (to backward) when the density of the universe goes to infinity (Big Bang singularity). However, in the case of $s > 1/4$, there is no finite (backward) time to obtain infinite density, and therefore there is no Big Bang singularity at any finite time in the past.

It is not necessary to perform similar analysis to investigate the rainbow deformation of FRW cosmology in Einstein gravity. To do so, it is sufficient to expand the function in Equation (34) for $\alpha \to 0$. So, using rainbow deformation of the Einstein theory, we obtain

$$\begin{aligned} t &= \pm\frac{1}{8\gamma}\sqrt{\frac{3}{\pi G}}\int_{\rho^*}^{\infty} \rho^{s-\frac{3}{2}} d\rho \\ &= \pm\frac{1}{4\gamma(2s-1)}\sqrt{\frac{3}{\pi G}}\rho^{s-\frac{1}{2}}\Bigg|_{\rho^*}^{\infty} = \infty, \quad s \geqslant \frac{1}{2}, \end{aligned} \tag{37}$$

where it shows that the value of f for having a nonsingular universe in Einstein gravity has to grow asymptotically as $\rho^{1/2}$, or faster.

Here, we plot the $t-\rho$ diagram in the Einstein case (Equation (37)) for both $s < 1/2$ and $s > 1/2$ in Figure 2. Similar to the GB case, Figure 2 shows that there is an initial finite density at $t = 0$ (present time). There is no infinite density (Big Bang singularity) in a finite (backward) time for $s > 1/2$. However, for $s < 1/2$, there is a finite (backward) time at which an infinite density (Big Bang singularity) exists.

Here, we have investigated the possibility of obtaining a nonsingular rainbow universe in the Einstein and GB gravities. In the coming section, we use MDR (Equation (9)) to analyze such a nonsingular FRW-like cosmology.

5. NONSINGULAR RAINBOW UNIVERSES

Using Equations (8), (16), and (25), one can show that the average energy (26) can be expressed as

$$\epsilon = c\gamma\rho^{\frac{\gamma-1}{\gamma}}. \tag{38}$$

Now, using Equations (8) and (38), the function $f(\varepsilon)$ will be

$$f(\varepsilon) = \frac{\exp(\gamma\mathcal{G}^{\frac{\gamma-1}{\gamma}}) - 1}{\gamma\mathcal{G}^{\frac{\gamma-1}{\gamma}}}, \tag{39}$$

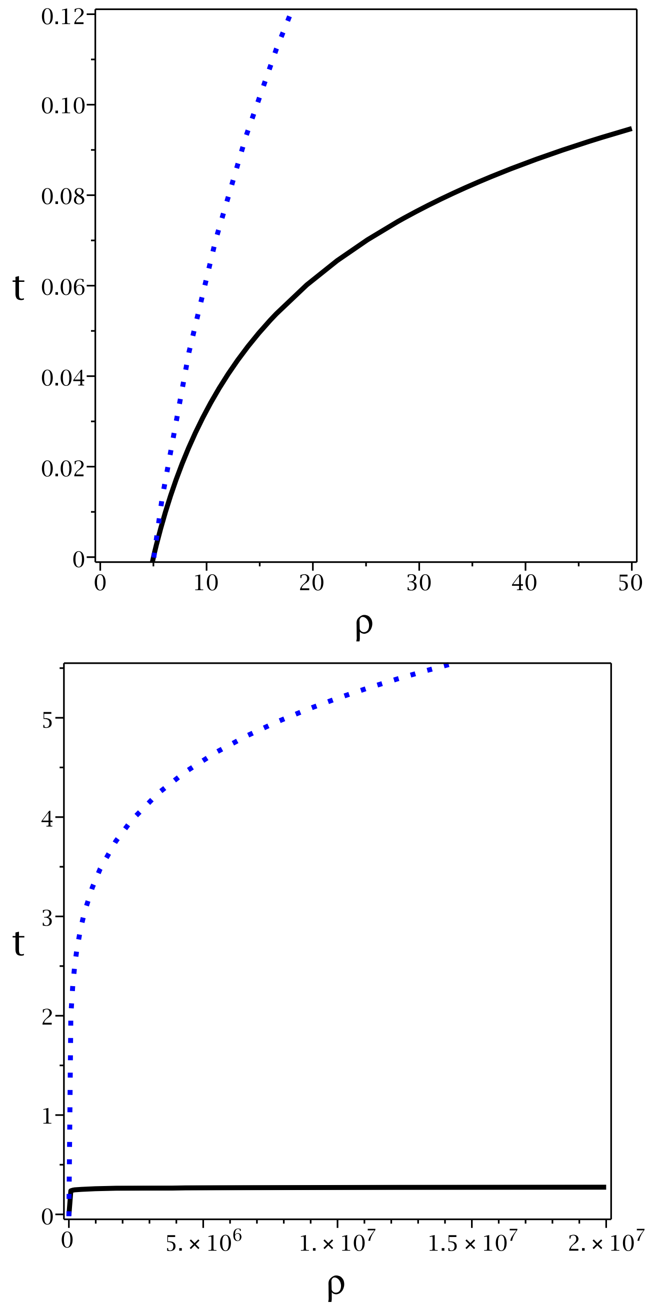

Figure 2. Einstein case: time vs. density, for $G = 2$, $\gamma = 4/3$, $\rho^* = 5$, $s = 1/3$ (continuous line), and $s = 2/3$ (dotted line): "up and down figures indicate various ρ ranges."

where $\mathcal{G} = \rho/\rho_{\rm P}$ and $E_{\rm P} = c\rho_{\rm P}^{\frac{\gamma-1}{\gamma}}$ is the Planck energy versus density $\rho_{\rm P}$. Using the above equation and the MDR relation, one can show that the modified Friedmann Equation (23) will be given by

$$H = \pm\frac{\gamma\mathcal{G}^{\frac{\gamma-1}{\gamma}}\left[-9 \pm 3\sqrt{9 + 96\pi\alpha\mathcal{G}\rho_{\rm P}^{\frac{2-\gamma}{\gamma}}}\right]^{\frac{1}{2}}}{6\sqrt{\alpha}\,[\exp(\gamma\mathcal{G}^{\frac{\gamma-1}{\gamma}}) - 1]}. \tag{40}$$

We choose the plus sign in the parentheses to obtain a consistent equation in Einstein gravity, i.e., in the limit $\alpha \longrightarrow 0$. We can investigate a possible singular solution of the Big Bang singularity using the discussion of Section 4. Substituting Equation (16) and the modified Friedmann Equation (40) into (25) and using Equation (8), we can obtain the following equation:

$$\dot{\mathcal{G}} = \pm \frac{2\gamma^2 \mathcal{G}^{\frac{2\gamma-1}{\gamma}}\left[-9+3\sqrt{9+96\pi\alpha\mathcal{G}\rho_{\mathrm{P}}^{\frac{2-\gamma}{\gamma}}}\right]^{\frac{1}{2}}}{3\sqrt{\alpha}\,[\exp(\gamma\mathcal{G}^{\frac{\gamma-1}{\gamma}})-1]}, \qquad (41)$$

where $\dot{\mathcal{G}} = \dot{\rho}/\rho_{\mathrm{P}}$.

Now, we want to show that the time is infinite when we go from an initial finite density $\mathcal{G}^*$ to an infinite one in the special case $\gamma = 4/3$, (i.e., radiation). This can be done by integrating Equation (41),

$$t = \pm\frac{27\sqrt{\alpha}}{32}\int_{\mathcal{G}^*}^{\infty} \frac{\exp\left(\frac{4}{3}\mathcal{G}^{\frac{1}{4}}\right)-1}{\mathcal{G}^{\frac{5}{4}}\left[-9+3\sqrt{9+96\pi\alpha\mathcal{G}\rho_{\mathrm{P}}^{\frac{1}{2}}}\right]^{\frac{1}{2}}}d\mathcal{G}, \qquad (42)$$

in which it is too hard to compute this integration analytically; however, one can use numerical calculations to show that it does not converge on $[\mathcal{G}^*, \infty)$, and so the time to reach infinite density is infinite. For more clarification, we consider the term with $\mathcal{G}$ to be the dominant term in the denominator, and so we obtain

$$\begin{aligned} t &= \pm\frac{27}{32}\left[\frac{3}{\alpha}\sqrt{96\pi\alpha\rho_{\mathrm{P}}^{\frac{1}{2}}}\right]^{-\frac{1}{2}}\int_{\mathcal{G}^*}^{\infty}\frac{\exp\left(\frac{4}{3}\mathcal{G}^{\frac{1}{4}}\right)-1}{\mathcal{G}^{\frac{3}{2}}}d\mathcal{G} \\ &= \pm\frac{27}{32}\left[\frac{3}{\alpha}\sqrt{96\pi\alpha\rho_{\mathrm{P}}^{\frac{1}{2}}}\right]^{-\frac{1}{2}} \\ &\quad\times\left(\frac{2}{\mathcal{G}^{\frac{1}{2}}}-\frac{32\mathcal{E}\left(1,-\frac{4}{3}\mathcal{G}^{\frac{1}{4}}\right)}{9}-\frac{2\mathcal{G}^{\frac{-1}{2}}(3+4\mathcal{G}^{\frac{1}{4}})}{3\exp\left(\frac{-4\mathcal{G}^{\frac{1}{4}}}{3}\right)}\right)\Bigg|_{\mathcal{G}^*}^{\infty} \\ &= \infty, \end{aligned} \qquad (43)$$

where $\mathcal{E}$ is the exponential integration. This shows that the time to reach this infinite density is infinite. Thus, there are no finite-time singularities and this result confirms the consequence of Equation (42).

In order to investigate the rainbow deformation of Einstein gravity, one can follow the same procedure using Equations (8), (11), (13), and (38), or just expand the function in integration (42) for $\alpha \to 0$. Here, we use the second method and obtain

$$\begin{aligned} t &= \pm\frac{9}{128}\sqrt{\frac{3}{\pi\rho_{\mathrm{P}}^{\frac{1}{2}}}}\int_{\mathcal{G}^*}^{\infty}\frac{\exp\left(\frac{4}{3}\mathcal{G}^{\frac{1}{4}}\right)-1}{\mathcal{G}^{\frac{7}{4}}}d\mathcal{G} \\ &= \pm\frac{9}{128}\sqrt{\frac{3}{\pi\rho_{\mathrm{P}}^{\frac{1}{2}}}}\left(\frac{4}{3\mathcal{G}^{\frac{3}{4}}}+\zeta\right)\Bigg|_{\mathcal{G}^*}^{\infty} = \infty, \end{aligned} \qquad (44)$$

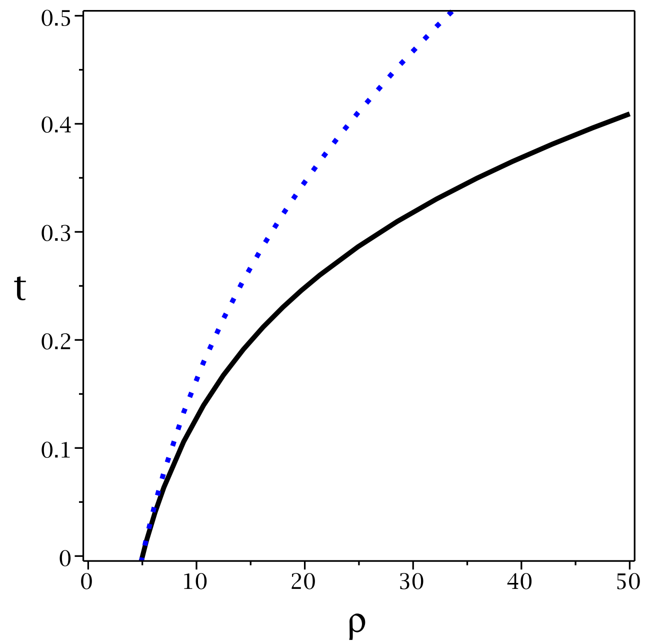

Figure 3. Time vs. density for $\mathcal{G}^* = 5$ and $\rho_{\mathrm{P}} = 0.2$. Einstein gravity's rainbow ($\alpha = 0$: continuous line) and Gauss-Bonnet gravity's rainbow ($\alpha = 0.1$: dotted line).

where

$$\zeta = -\frac{128\mathcal{E}\left(1,-\frac{4}{3}\mathcal{G}^{\frac{1}{4}}\right)}{81}-\frac{4(9+8\mathcal{G}^{\frac{1}{2}}+6\mathcal{G}^{\frac{1}{4}})}{27\mathcal{G}^{\frac{3}{4}}\exp\left(\frac{-4}{3}\mathcal{G}^{\frac{1}{4}}\right)}.$$

This result shows that there are no finite-time singularities.

We also plot the $t - \rho$ diagram for both Einstein and GB gravities (Equations (43) and (44)) in Figure 3. This figure shows that for both Einstein and GB gravities, there is an initial finite density at $t = 0$, however (backward) time goes to infinity as density goes to infinity, and therefore there is no Big Bang singularity.

Finally, it is interesting to investigate the behavior of the density of states at the Planck scale to analyze divergences (Ling 2007; Ling & Wu 2010). Using the MDR, the density of states can be written as

$$a(E)dE \simeq p^3dp \simeq f(\varepsilon)^4\left(1+E\frac{f(\varepsilon)'}{f(\varepsilon)}\right)E^3dE. \qquad (45)$$

Here, by substituting the MDR and using the fact that energy cannot be larger than the Planck energy, the density of states has a finite value $e(e-1)^3$ with regular behavior without any divergences.

6. CONCLUSIONS

In this work, we investigated the effect of gravity's rainbow in Einstein and GB gravities for the early universe. We analyzed the rainbow deformation of the five-dimensional FRW solution in both Einstein and GB gravity. We observed that although the GB term contributes to the field equations, it does not change the conservation equation and average energy. We also demonstrated that the rainbow functions modify both the conservation equation and average energy. In addition, we discussed the general conditions for nonsingular FRW cosmologies using the rainbow deformation of both Einstein and GB gravities.

We used the rainbow functions defined by Amelino-Camelia et. al. (1998; Amelino-Camelia 2004) to investigate the effect of the rainbow deformation of FRW-like cosmology. We have demonstrated that it is possible to obtain nonsingular cosmological solutions by using a rainbow deformation of Einstein and GB gravities. The Friedmann equations were modified using gravity's rainbow by suitable rainbow functions. We also identified the rainbow functions with the MDR introduced by Amelino-Camelia et al. (1998; Amelino-Camelia 2004) and studied the rainbow-modified Friedmann equations of a perfect fluid. We found nonsingular solutions for a wide range of values for the EoS parameter $\gamma > 4/3$ in both Einstein and GB gravities. We also found that GB gravity has a considerable effect on the constraint for having nonsingular universes. Using the analysis in Awad et al. (2013), we found that the universe takes infinite time to reach $\rho \to \infty$ from a finite value of ρ. We have also found that the density of states do not diverge at the Planck scale. So, for both cases, we found a possible resolution of the Big Bang singularity. Hence, it seems that the removal of singularities by rainbow deformation is not a model-dependent effect. It would be interesting to perform this analysis in other models of Lovelock gravity.

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