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Granular gravitational collapse and chute flow

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Abstract. – We argue that inelastic grains in a flow under gravitation tend to collapse into states in which the relative normal velocities of two neighboring grains is zero. If the time scale for this gravitational collapse is shorter than inverse strain rates in the flow, we propose that this collapse will lead to the formation of "granular eddies", large-scale condensed structures of particles moving coherently with one another. The scale of these eddies is determined by the gradient of the strain rate. Applying these concepts to chute flow of granular media (gravitationally driven flow down inclined planes), we predict the existence of a bulk flow region whose rheology is determined only by flow density. This theory yields the "Pouliquen flow rule", correlating different chute flows; it also accounts for the different flow regimes observed.

Introduction. – Flows of hard granular systems are ubiquitous in nature and technology, yet are still poorly understood [1]. Granular systems typically have a twofold separation of energy scales: the typical energy of a particle is determined by gravity or some other body force (in a few instances by initial conditions), and is much larger than the thermal scale $k_{\mathrm{B}}T$, yet much smaller than the scale required to appreciably deform the particle. Despite the smallness of $k_{\mathrm{B}}T$ on the scale of granular energies, many treatments use a pseudo-temperature connected to the random part of the kinetic energy of a particle. Such treatments often link granular phenomena to the kinetic theory of gases. The "granular gas" has an intrinsic rheology, and is driven by the external forcing.

One of the pioneering treatments of this rheology was by Bagnold, who discussed chute flows, the gravitationally driven flow of a granular material down an inclined surface [2]. It is simplest to consider a flow of constant, fixed depth H, with the average velocity of the particles parallel to the free surface. The particles are spheres of monodispersed mass M and radius R. We choose axes such that the direction of flow is $\hat{x}$, the direction perpendicular to the free surface of the flow is $\hat{z}$, and the direction parallel to vorticity is $\hat{y}$ (see fig. 1). We expect the momentum transfer communicated by collisions between particles at different depths to be of the order of $\Delta p = MR\partial_z v_x$. Furthermore, these collisions will occur at typical intervals of the order of $\Delta t = (\partial_z v_x)^{-1}$, from which we conclude that the typical collisional stress will be

$$\sigma_{xz} \sim \frac{1}{R^2}\frac{\Delta p}{\Delta t} = \frac{M}{R}(\partial_z v_x)^2, \tag{1}$$

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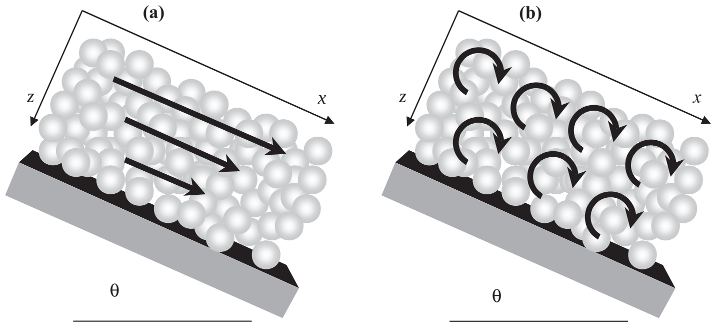

Fig. 1 – (a) Chute flow is flow down a surface inclined at an angle θ. The x-axis is chosen parallel to the flow, the z-axis perpendicular to the free surface. The y-axis is parallel to the vorticity of the flow, and is directed out of the page. (b) In the granular eddy picture, the motion of the particles is regarded as a superposition of the translation and rotation of granular eddies.

with a proportionality constant that depends on the local packing fraction of the spheres.

In a steady state flow down a surface inclined at an angle θ, the xz shear component of the stress tensor is determined by the gravitational acceleration g to be

$$\sigma_{xz} = \rho g z \sin\theta, \tag{2}$$

with ρ the (local) mass density, which we here assume to be independent of z (we will return to this point below). We are measuring the depth of the pile z from the free surface $z = 0$.

If $\sigma_{xz} \propto z \propto (\partial_z v_x)^2$, we obtain $\partial_z v_x = -A_{\mathrm{Bag}}\sqrt{z}$, defining the coefficient A_{Bag}, and if there is no slip at the base ($v_x = 0$ at $z = H$), then the depth-averaged velocity

$$u \equiv H^{-1}\int_0^H \mathrm{d}z\ v_x(z) = \frac{2}{5}A_{\mathrm{Bag}}H^{3/2}. \tag{3}$$

While there have been a variety of authoritative experimental studies of chute flow [3,4], as well as intricate theoretical discussions of the rheologies to be expected on general grounds [5], we have been particularly inspired by the recent work of Pouliquen [6]. Pouliquen studied the behavior of chute flows as a function of inclination angle θ and height of the flow H. He found that for small values of θ or height H, no flow took place. With the increase of either θ or H such that an angle of repose line $\theta_{\mathcal{R}}(H)$ was passed, a region of steady-state flow was entered. Finally, for values of θ above a maximum θ_{M}, the flows continuously accelerated, and no steady-state flow was observed.

The dominant observational fact about the steady-state flows is the "Pouliquen flow rule", which connects the average velocity u of a flow of height H with the height H_{stop} at which flow ceases for a chute of that inclination θ. (The angle of repose $\theta_{\mathcal{R}}(H)$ is the inverse of the function $H_{\mathrm{stop}}(\theta)$.) The Pouliquen flow rule gives a scaling form for u,

$$\frac{u}{\sqrt{gH}} = \beta\frac{H}{H_{\mathrm{stop}}}\,, \tag{4}$$

and accounts well for experimental data with $\beta = 0.136$.

The scaling $u \propto H^{3/2}$ in the Pouliquen flow rule is consistent with the Bagnold rheology (cf. eq. (3)]. But the Pouliquen flow rule also connects the coefficient $A_{\mathrm{Bag}}(\theta)$ with *the thickness of the pile at that inclination below which flow arrests.*

Note that we would have obtained dimensionally the Bagnold result for the rheology had we claimed that the stress should obey

$$\sigma_{xz} = \mu \partial_z v_x \sim \rho R^2 (\partial_z v_x)^2, \tag{5}$$

where we have made the substitution for the viscosity $\mu \sim \rho R^2 \partial_z v_x$ on grounds that a granular flow has no other obvious local length or time scales than R for the length scale and $(\partial_z v_x)^{-1}$ for the time scale. The Bagnold scaling result thus stems from the assumption that these are the only local scales. Note that the gravitational constant g does not figure directly in either of these scales. However, the Pouliquen flow rule implies that this rheology does depend both upon g and upon the thickness of the arresting pile H_{stop}, which is hardly local information. Thus the Pouliquen flow rule appears to be inconsistent with any assumption of a purely local rheology comparable to that of a granular gas [7]. This, and other considerations, have motivated some authors to build non-local models for the rheology [8].

The broad features of Pouliquen's conclusions have been confirmed by a series of numerical studies in which these authors have participated [9]. For relatively thin piles, the Bagnold rheology breaks down, but the thicker piles show a Bagnold rheology and obey the Pouliquen flow rule, albeit with a slightly larger value of β (the crossover is examined numerically in [10]). However, the assumption of the Bagnold or granular gas approach that the stress is mostly transmitted through collisions seems not to be true in these numerical studies; stress seems instead to be transmitted primarily through relatively long-lived contacts between particles. The density in the interior of the piles is independent of depth, consistent with the assertion made in eq. (2).

In this treatment, we eschew granular gas approaches, and we do not assume the existence of any rheology independent of the gravitational character of the flow. We show that gravitation combined with particle inelasticity and friction is able to dissipate a significant fraction of the system's kinetic energy over time scales short compared to the inverse strain rate. This suggests the existence of gravitationally collapsed clusters of particles ("granular eddies"), which move coherently and whose properties determine the flow rheology [11]. The rheology that follows from this picture agrees with the Pouliquen flow rule, and also gives a simple explanation for the different flow regimes observed by Pouliquen.

In this report, we first analyze the phenomenon of gravitational collapse for inelastic particles. We then introduce the granular eddy picture, and relate the eddy size to flow properties. We specialize to the case of chute flows, determining the rheology, and accounting for the principal features of the observed phenomenology of these flows.

Gravitational collapse. – Consider an inelastic ball with a coefficient of restitution of ϵ, bouncing on a rigid horizontal surface. It is elementary to show that if its normal velocity at first impact is v_0, then after a finite time τ_{b} it will come to rest, with

$$\tau_{\text{b}} = \frac{2v_0}{g} \frac{\epsilon}{1-\epsilon}\,. \tag{6}$$

(A similar result obtains if we take a more realistic ball with a Hertzian contact force and a visco-elastic dissipation; here we restrict ourselves to the simplest case.)

Now consider a particle in a granular flow. If it hits a cluster of neighboring particles with a relative normal velocity v_{n}, from dimensional grounds we expect that it will collapse onto this cluster in a time

$$\tau_{\text{gc}} = \frac{2v_{\text{n}}}{g} f(\epsilon), \tag{7}$$

where we have suppressed the complicated dependence of the dimensionless function $f(\epsilon)$ on dimensionless parameters such as the direction of impact, interparticle friction, cluster geometry and the way in which the cluster absorbs the shock of the impact. If τ_{gc} is short compared to the time scale for the particles in the cluster to rearrange themselves, we expect that the motion of the impacting particle will become strongly correlated with the particles in the cluster, *i.e.*, the particle will become a part of the cluster.

Thus we can envision large aggregations of particles coming into existence, each of whose motions with respect to its neighbors is at most of a rolling kind. We term these clusters "granular eddies" (see fig. 1), because we believe that they are analogous in their control of viscosity to eddies in turbulent systems.

Granular eddy size. – Consider an eddy of effective radius ℓ, whose center of mass is at a position z_0. The local average velocity can be expanded as

$$v_x(z) = v_x(z_0) + (z - z_0)\partial_z v_x(z_0) + \frac{1}{2}(z - z_0)^2 \partial_{zz} v_x(z_0) + \cdots . \tag{8}$$

While the first and second terms can be matched by an eddy whose center of mass moves at a velocity $v_x(z_0)$ and which rotates at an angular velocity $\omega = \partial_z v_x(z_0)$, the third term in this series is incompatible with the rigid-body rotation of an eddy. We write this incompatible velocity across the diameter of the eddy as

$$v_{\mathrm{ic}} = \ell^2 \partial_{zz} v_x . \tag{9}$$

A certain amount of this incompatible velocity will be absorbed into internal distortion of the eddy as it rotates, while the remainder will be reflected in deviations of the average horizontal velocity within the eddy from the average of the surrounding flow. We expect that the part of the incompatibility relieved by internal strains of the eddy will be relieved by strains on the scale ℓ of the eddy itself, thus the time scale corresponding to the variation of these "incompatibility strains" is

$$\tau_{\mathrm{ic}} \equiv \frac{\ell}{v_{\mathrm{ic}}} = (\ell \partial_{zz} v_x)^{-1} . \tag{10}$$

On this time scale the environment of particles at the boundary of an eddy will inevitably change as that eddy conforms with the surrounding flow.

At the boundary of an eddy, its particles will collide with those of neighboring eddies. The relative normal and tangential velocities of these collisions will be functions of the vertical offset of the eddies, as well as of their precise local geometries. However, the maximum scale of the collision relative velocity is $v_{\max} = \ell \partial_z v_x$. The characteristic gravitational collapse time associated with these most violent collisions is

$$\tau_{\mathrm{gc}} = \frac{2 v_{\max}}{g} f(\epsilon), \tag{11}$$

so that the criterion for the particles at the eddy surface to be able to gravitationally collapse before their environment is altered by incompatibility strains is

$$\frac{\tau_{\mathrm{gc}}}{\tau_{\mathrm{ic}}} = \left[\frac{2\ell \partial_z v_x}{g} f(\epsilon) \right] (\ell \partial_{zz} v_x) < \tilde{a}, \tag{12}$$

where $\tilde{a}$ is an unknown numerical constant of O(1). The maximum value of ℓ consistent with this relation is determined by

$$\ell_{\mathrm{e}}^2 \left[\partial_z (\partial_z v_x)^2 \right] = \tilde{a} g f(\epsilon). \tag{13}$$

We expect that this maximum value will set the scale of the eddies, since eddies smaller than this size will tend to grow as more and more particles collapse onto their surfaces, and eddies larger than this size will lose particles from their surfaces.

Phenomenology of chute flow. – Let us define an effective "viscosity length scale" ℓ_ν, using the Bagnold scaling relation given by eq. (5), but substituting this new length scale instead of R. This yields

$$\sigma_{xz} \equiv \rho \ell_\nu^2 (\partial_z v_x)^2. \tag{14}$$

We now make a different scaling assumption than that of Bagnold: The length scale appearing in Bagnold's argument should be set by the eddy size ℓ_e instead of the particle size R, *i.e.*, $\ell_\nu \approx \ell_\mathrm{e}$. In other words, we assume that this is the unique length scale that determines the bulk rheology of the granular dispersion. However, there is a subtle problem with making this relation an equality: while eq. (13) is expected to be valid for $\ell_\mathrm{e} \gg R$, it clearly breaks down for $\ell_\mathrm{e} < R$, since the smallest possible structure is of size R. In order to correct for this discrepancy between eq. (13) and the actual size of the structures that set the viscosity scale ℓ_ν, we introduce finite-size corrections into the relation between these two scales of the form $\ell_\nu - \ell_\mathrm{e} \sim R$, which in the opposite limit of ℓ_ν large implies the Laurent expansion

$$\ell_\nu^2 = \ell_\mathrm{e}^2 \left(1 + \tilde{b} \frac{R}{\ell_\mathrm{e}} + \cdots \right), \tag{15}$$

where the unknown numerical constant $\tilde{b}$ accounts for the leading-order finite-size corrections due to the existence of the particle size R. Then, for chute flow, ℓ_e and $\partial_z v_x$ are jointly determined by simultaneous solution of eqs. (2), (13), (14) and (15).

If the angle of inclination is too small, there is no solution, in particular, for

$$\theta < \theta_\mathrm{R} \equiv \sin^{-1}(\tilde{a} f(\epsilon)). \tag{16}$$

For $\theta > \theta_\mathrm{R}$, we have

$$\tilde{b} \frac{R}{\ell_\mathrm{e}} = \frac{\sin\theta}{\sin\theta_\mathrm{R}} - 1, \tag{17}$$

which fixes the eddy size ℓ_e. Finally, we find

$$\partial_z v_x = -A_\mathrm{Bag} \sqrt{z} \tag{18}$$

with

$$A_\mathrm{Bag} = \frac{\sqrt{g \sin\theta_\mathrm{R}}}{\ell_\mathrm{e}} = \frac{\sqrt{g \sin\theta_\mathrm{R}}}{\tilde{b} R} \left(\frac{\sin\theta}{\sin\theta_\mathrm{R}} - 1 \right). \tag{19}$$

To determine the flow density, let us suppose that the eddies themselves have a fixed density ρ_0, independent of their size. Then the medium as a whole can have a density that differs from this ("free volume") only due to a presumably lower density in the regions, of scale R, that separate different eddies from one another. Thus

$$\rho = \rho_0 \left(1 - \tilde{c} \frac{R}{\ell_\mathrm{e}} + \cdots \right) \tag{20}$$

or

$$\frac{\rho_0 - \rho}{\rho_0} = \frac{\tilde{c}}{\tilde{b}} \left(\frac{\sin\theta}{\sin\theta_\mathrm{R}} - 1 \right), \tag{21}$$

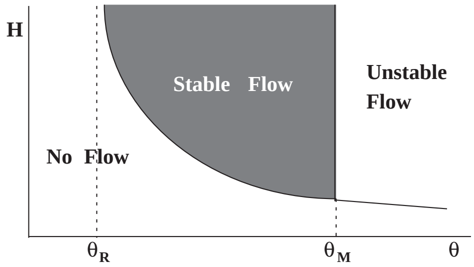

Fig. 2 – The granular eddy picture predicts that the angle-of-repose line separating the region of no flow from that of stable flow depends on H. The angle-of-repose line approaches a fixed angle θ_{R} as $H \to \infty$. The maximum angle of stable flow θ_{M} does not depend on H in this picture.

where $\tilde{c}$ is yet another unknown numerical constant. Note that for chute flow, the eddy size ℓ_{e} given by eq. (17) is independent of depth, consistent with the assumption of a depth-independent density ρ.

Of course, the eddy size ℓ_{e} is not entirely unrestricted. When the eddies get too large to be accommodated within the height H of the flow, *i.e.*, for $\ell_{\mathrm{e}} \sim H/2$, we expect flow arrest to occur since all free volume allowing flow has disappeared. This gives the thinnest flowing pile at a given angle θ in terms of a new constant $\tilde{d} \sim 1$,

$$H_{\mathrm{stop}}(\theta) = \tilde{d}\ell_{\mathrm{e}}(\theta) = \tilde{b}\tilde{d}R\left(\frac{\sin\theta}{\sin\theta_{\mathrm{R}}} - 1\right)^{-1}, \tag{22}$$

or, equivalently, the lowest possible angle of stable flow at a given pile height H,

$$\theta_{\mathcal{R}}(H) = \sin^{-1}\left([1 + (\tilde{b}\tilde{d}R/H)]\sin\theta_{\mathrm{R}}\right). \tag{23}$$

Note that we can now re-interpret θ_{R} as the limiting value of the angle of repose as the thickness of the flow $H \to \infty$, and θ_{R} can be used as an adjustable parameter of the theory in order to eliminate the unknown function $f(\epsilon)$ from the results. Note that since along the line $H_{\mathrm{stop}}(\theta)$ the eddy is the scale of the flow depth, we expect here a roughly linear velocity profile, corresponding to the rotation of eddies spanning the depth of the system.

On the other hand, gravitational collapse ceases to stabilize the flow when $\ell_{\mathrm{e}} \sim R$, corresponding to an upper limit of stability

$$\theta_{\mathrm{M}} = \sin^{-1}\left([1 + (\tilde{b}/\tilde{e})]\sin\theta_{\mathrm{R}}\right), \tag{24}$$

which is independent of the flow height H. Here $\tilde{e} \sim 1$ is a further unknown constant.

Since this is a scaling theory, it is not possible to make a quantitative comparison between these predictions and numerical results such as those of ref. [9]. However, for $\theta - \theta_{\mathrm{R}} \ll 1$, the dependences on tilt angle for a thick flow ($H \gg R$) such as $\partial_z v_x = -A_{\mathrm{Bag}}\sqrt{z}$ with $A_{\mathrm{Bag}} \propto (\theta - \theta_{\mathrm{R}})$, and $\rho_0 - \rho \propto (\theta - \theta_{\mathrm{R}})$ are borne out by these numerical results. Also, the Pouliquen flow rule eq. (4) is recovered with

$$\beta = \frac{2\tilde{d}}{5}\sqrt{\sin\theta_{\mathrm{R}}}\,. \tag{25}$$

Finally, the phase boundaries (see fig. 2) agree with both numerical and experimental results.

An interesting aspect of the result is that it is the finite-size correction in eq. (15) that determines the fundamental length scale ℓ_{e}. While this is perhaps surprising, we should remember that the angular range of steady-state flow is quite narrow; the more generic behaviors are the regions of no flow and that of accelerating flow. Without the finite-size correction we would not have been able to predict steady-state flow except precisely at $\theta = \theta_{\mathrm{R}}$; this correction thus broadens the possible angle of steady flow from this one angle to a narrow range.

In this letter we have addressed the form of the bulk rheology for chute flows; our conclusions regarding this rheology should hold in portions of the pile for which the computed eddy scale ℓ is less than the distance to the boundaries. Clearly, there will be both upper and lower boundary layers in which this is impossible. We have not addressed the structure of these boundary layers. Although chute flow does seem to have a bulk-rheology–dominated regime, this may not be the case with all flow geometries. In some flows the structure of the boundary layers may dominate in determining the characteristics of the flow.

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