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Financial markets as adaptive systems

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Abstract. – We show, by studying in detail the market prices of options on liquid markets, that the market has empirically corrected the simple, but inadequate Black-Scholes formula to account for two important statistical features of asset fluctuations: "fat tails" and correlations in the scale of fluctuations. These aspects, although not included in the pricing models, are very precisely reflected in the price fixed by the market as a whole. Financial markets thus behave as rather efficient adaptive systems.

Motivations. – Options markets offer an interesting example of the adaptation of a population (the traders) to a complex environment, through trial and errors and natural selection (inefficient traders disappear quickly). The problem is the following: an "option" is an insurance contract protecting its owner against the rise (or fall) of financial assets, such as stocks, currencies, etc. The problem of knowing the value of such contracts has become extremely acute ever since organized option markets opened twenty-five years ago, allowing one to buy or sell options much like stocks. Almost simultaneously, Black and Scholes (BS) proposed their famous option pricing theory, based on a simplified model for stock fluctuations, namely the (geometrical) continuous-time Brownian-motion model [1], [2]. The most important parameter of the model is the "volatility" σ, which is the standard deviation of the market price's fluctuations. The Black-Scholes model is known to be based on unrealistic assumptions but is nevertheless used as a benchmark by all market participants. Guided by the Black-Scholes theory, but constrained by the fact that "bad" prices lead to "arbitrage opportunities" (that is, an easy way to make money), the option market fixes prices which are close, but significantly and systematically different from the BS formula.

The aim of this paper is to exploit a recent reformulation of the Black-Scholes problem [3], [4] which allows one to incorporate non-Gaussian effects, and to compare directly option prices (as determined by the market) to their theoretical value. In order to compare theoretical prices to reality, we choose a very active market, such that

– reliable option prices are available (anomalies in the pricing process are rare);

– the non-Gaussian effects are small and can be treated perturbatively, using a cumulant expansion.

Surprisingly, our study clearly shows that, despite the lack of an appropriate model, traders have empirically adapted to incorporate some subtle information on the real statistics of price changes, that is

– the fact that the tails of the fluctuations are much "fatter" than for a Gaussian distribution;

– the fact that the volatility is itself time dependent, and reveals slowly decaying (power law) correlations.

Although this ability to price financial assets correctly is often assumed in the literature (the "efficient market" hypothesis), it is in general difficult to assess quantitatively, because the "true" value of a stock is difficult to determine (or might even be an empty concept). The case of option markets is interesting in that respect, because the "true" value of an option is, in principle, calculable.

Option theory: a brief summary. – Let us start by recalling that a "call" option is an "insurance contract" such that if the price $x(T)$ of a given asset at time T (the "maturity") exceeds a certain level x_s (the "strike" price), the owner of the option receives the difference $x(T)-x_\mathrm{s}$. Conversely, if $x(T) < x_\mathrm{s}$, the contract is lost. To make a long story short [1]-[4], if T is small enough (a few months) so that interest rate effects and average returns are negligible compared to fluctuations, the "fair" price $\mathcal{C}$ of the option today ($T=0$), knowing that the price of the asset now is x_0, is simply given by [5]

$$\mathcal{C}(x_0, x_\mathrm{s}, T) = \int_{x_\mathrm{s}}^{\infty} \mathrm{d}x' \; (x' - x_\mathrm{s}) P(x', T|x_0, 0)\,, \tag{1}$$

where $P(x', T|x_0, 0)$ is the conditional probability density that the stock price at time T will be equal to x', knowing that its present value is x_0. Equation (1) means that the option price is such that, on average, there is no winning party. Pricing correctly an option is thus tantamount to having a good model for the probability density $P(x', T|x_0, 0)$.

Price increments are uncorrelated, but not independent. – There is fairly strong evidence that beyond a time scale τ of the order of ten minutes, the fluctuations of prices in liquid markets are uncorrelated, but *not independent* variables [6]-[11]. In particular, it has been observed that although the signs of successive price movements seem to be independent, their magnitude —as represented by the absolute value or square of the price increments— is correlated in time [6], [10]: this is related to the so-called "volatility clustering" effect [12], [7]. In order to capture these features, one can represent the price $x(T)$ of the asset as

$$x(T) = x_0 + \sum_{k=0}^{\frac{T}{\tau}-1} \delta x_k\,, \tag{2}$$

where the price increments δx_k are obtained as the *product of two random variables*, *i.e.* $\delta x_k = \epsilon_k \gamma_k$. The $(\epsilon_k)_{k\geq 0}$ are independent, identically distributed random variables of mean zero and unit variance which describe the direction of the instantaneous "trend". The *scale* of the fluctuation is described by another random factor, which we denote as γ_k, and which we assume to be independent of the ϵ_k's. The sequence $(\gamma_k)_{k\geq 0}$ is considered to be a stationary random process but allowed to exhibit non-trivial correlations (see below), describing persistent "bursts" in the market activity. Under these hypotheses, the conditional distribution of δx_k, conditioned on γ_k, may be written in a scaling form, as

$$P(\delta x_k) \equiv \frac{1}{\gamma_k} P_0 \left(\frac{\delta x_k}{\gamma_k} \right) , \tag{3}$$

where P_0 is independent of k. Models with conditionally Gaussian increments —*i.e.* where P_0 is a Gaussian— have been extensively studied [12] both in discrete-time (ARCH models) and continuous-time (stochastic volatility models) settings. The present model is more general since we do not assume *a priori* that P_0 is Gaussian.

A simple case first. – Let us consider first the simple case where γ_k is independent of k, and equal to γ_0. Equation (2) then corresponds to the classical problem of a sum of independent, identically distributed variables. Although $P(\delta x)$ is strongly non-Gaussian (see, *e.g.*, [13]), it has a finite variance [9] and the central-limit theorem [14] tells us that for large $N = T/\tau$, $P(x', T|x_0, 0)$ will be close to a Gaussian. Using then eq. (1) essentially leads back to the BS formula [15]. For finite N, however, there are corrections to the Gaussian, and thus corrections to the BS price. More precisely, the difference between $P(x', T|x_0, 0)$ and the limiting Gaussian distribution $\mathcal{G}_{x_0,\sigma^2}$ may be calculated using a *cumulant* expansion [14]. The leading correction in the cumulant expansion turns out to be, for large N, proportional to the *kurtosis* κ, defined as $\kappa = \langle \delta x^4 \rangle / \langle \delta x^2 \rangle^2 - 3$ [14] —to a very good approximation, the distribution $P_0(\delta x)$ is symmetric [13], [4] for time scales less than a month; in particular the third cumulant, which measures the skewness of the distribution, is small. Note that the kurtosis κ vanishes if the increments δx are Gaussian random variables, and measures the "fatness" of the tails of the distribution as compared to a Gaussian.

Neglecting higher-order cumulants, the expansion takes the following form:

$$\int_{-\infty}^{x} \left\{ P(x', T|x_0, 0) - \mathcal{G}_{x_0,\sigma_T^2}(x') \right\} \mathrm{d}x' = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \left[\frac{\kappa_T}{24}(u^3 - 3u) + \ldots \right] , \tag{4}$$

where $u = (x - x_0)^2/\sigma_T^2$, σ_T^2 and κ_T being the variance and kurtosis corresponding to the scale T. $\mathcal{G}_{x_0,\sigma_T^2}$ is the Gaussian density centered at x_0 of variance σ_T^2.

Volatility "smile" and anomalous kurtosis. – It is then easy to show, using eq. (1), that the leading correction to the BS price can be reproduced by using the BS formula, but with a modified value for the volatility $\sigma = \sqrt{\langle \delta x^2 \rangle}$ (which traders call the "implied volatility" Σ), which depends both on the strike price x_s and on the maturity T through

$$\Sigma(x_\mathrm{s}, T) = \sigma \left[1 + \frac{\kappa_T}{24} \left(\frac{(x_\mathrm{s} - x_0)^2}{\sigma_T^2} - 1 \right) \right] . \tag{5}$$

The fact that implied volatility depends on the strike price x_s is known as the "smile effect", because the plot of Σ *vs.* x_s, for a given value of $T = N\tau$, has the shape of a smile.

That the volatility had to be smiled up was realized long ago by traders —this reflects the well-known fact that the elementary increments have fat-tailed distributions: large fluctuations occur much more often than for a Gaussian random walk.

The data reveals that the smile formula (5) indeed correctly reproduces the observed option prices on the "BUND" market provided the kurtosis κ_T in formula (5) becomes itself

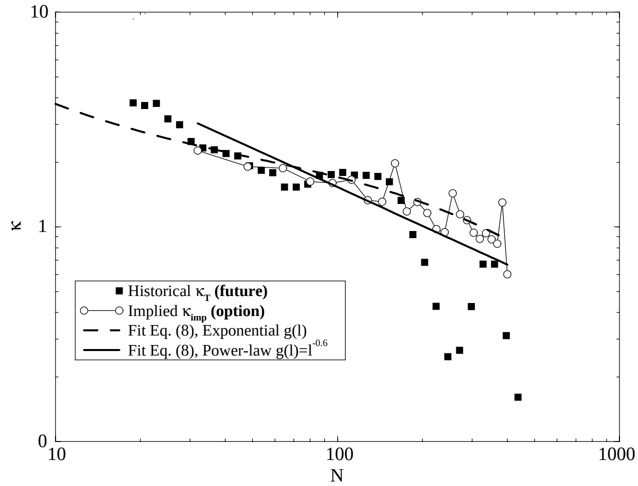

Fig. 1. – Plot (in log-log coordinates) of the average implied kurtosis κ_{imp} (determined by fitting the implied volatility for a fixed maturity by a parabola) and of the empirical kurtosis κ_T (determined directly from the historical movements of the BUND contract), as a function of the reduced time scale $N = T/\tau$, $\tau = 30$ minutes. All transactions of options on the BUND future from 1993 to 1995 were analyzed along with 5 minute tick data of the BUND future for the same period. We show for comparison a fit with formula (7), with $g(\ell) \simeq \ell^{-0.6}$, which leads to $\kappa_T \simeq T^{-0.6}$ (dark line). A fit with an exponentially decaying $g(\ell)$ is however also acceptable (dotted line).

T-dependent. The shape of the "implied" kurtosis $\kappa_{\text{imp}}(T)$ as a function of T is given in fig. 1; $\kappa_{\text{imp}}(T)$ is seen to decrease more slowly than κ_τ/T, as one should observe if the increments δx were independent and identically distributed (*i.e.* $\gamma_k \equiv \gamma_0$).

Let us then study directly the kurtosis of the distribution of the underlying stock, $P(x,T|x_0,0)$, as a function of $N \equiv T/\tau$. In fig. 1, we have also shown κ_T as a function of N. One can notice that not only κ_T does not decay as $1/T$, but actually κ_T matches quantitatively (at least for $N \leq 200$) the evolution of the implied kurtosis κ_{imp}! (Note that there is no adjustable overall factor.) In other words, the price over which traders agree capture rather precisely the anomalous evolution of κ_T. A similar agreement has been found on other liquid option markets, where bid-ask spreads are sufficiently small to ascertain that the quoted prices should indeed be set by a fair game condition. For "over the counter" options (*i.e.* options for which there is no organized market), this is likely not to be the case, since a rather high risk premium is generally included in the price.

Interpretation: volatility correlations. – As we shall show now, the non-trivial behaviour of κ_T is related to the fact that the *scale* of the fluctuations γ_k is itself a time-dependent random variable [12], [6], with rather long-range correlations.

We define the correlation function of the scale of fluctuations as

$$g(\ell) = \frac{\langle \delta x_{k+\ell}^2 \delta x_k^2 \rangle - \langle \delta x_k^2 \rangle^2}{\langle \delta x_k^4 \rangle - \langle \delta x_k^2 \rangle^2}\,, \tag{6}$$

$g(\ell)$ is normalized so that $g(0) = 1$. In this case, one can show that eq. (5) holds, with κ_T

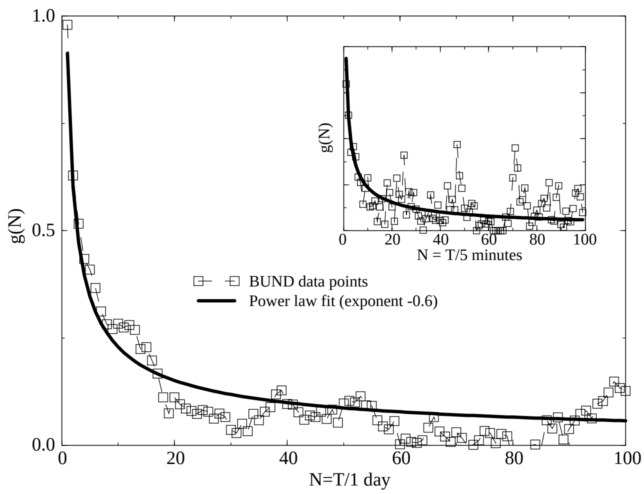

Fig. 2. – Plot of the daily volatility correlation function $g(N)$ for the BUND future market, from 1991 to 1995. A fit by $g(N) \simeq N^{-\lambda}$ with $\lambda = 0.6$ is shown for comparison. The same behaviour is found to persist for intra-day fluctuations (see Inset).

given by

$$\kappa_T = \frac{\tau}{T}\left[\kappa_\tau + 6(\kappa_\tau + 2)\sum_{\ell=1}^{N}\left(1 - \frac{\ell}{N}\right) g(\ell)\right] , \tag{7}$$

where κ_τ is the kurtosis of $\delta x = x(t+\tau) - x(t)$. We have computed from historical data on the BUND index the correlation function $g(\ell)$, which we show in fig. 2. Interestingly, $g(\ell)$ decreases rather slowly, as $\ell^{-\lambda}$, with $\lambda \simeq 0.6 \pm 0.1$, from minutes to several days (note, however, that the data is quite noisy). A similar decay of $g(\ell)$ was observed on other markets as well, with rather close values for λ, such as the S&P500 (for which $\lambda \simeq 0.37$) [10] and the DEM/$ market (for which $\lambda \simeq 0.57$). The fact that the correlation function decays slowly was also reported in [11]. This could be related to the presence of many relevant "periods" in financial markets (days, weeks, months, quarters, years).

Remarkably, eq. (7) with $g(\ell) \propto \ell^{-0.6}$ leads to $\kappa_T \propto T^{-0.6}$, in good agreement with both the direct determination of κ_T and the one deduced from the volatility smile, κ_{imp}. Note that the effect of a non-zero kurtosis on Black-Scholes prices was previously investigated in [16], [17]. However, the relation between κ_T and κ_{imp}, and their anomalous T-dependence, were not, to our knowledge, previously reported.

Conclusion. – In conclusion, we have shown, by studying in detail the market prices of options, that traders have evolved from the simple, but inadequate BS formula to an empirical know-how which encodes two important statistical features of asset fluctuations: "fat tails" (*i.e.* a rather large kurtosis) and the fact that the scale of fluctuations exhibits slowly decaying (power-law–like) correlations. These features, although not explicitly included in the theoretical pricing models used by traders, are nevertheless reflected rather precisely in the price fixed by the market as a whole. Option markets offer an interesting ground where "theoretical" and "experimental" prices can be systematically compared, and were found to agree rather well. This has enabled us to test quantitatively the idea that the trader population behaves as an efficient adaptive system.

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