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Random Walks on Cayley Trees: Temperature-Induced Transience-Recurrence Transition, Small Exponents and Logarithmic Relaxation.

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Abstract. – Random walks on tree structures are as useful tools in physics as they are interesting themselves. Here we show that for a certain class of models they can undergo a transition from being recurrent to being transient depending on the temperature. At the transition the relaxation is logarithmic. The significance of the pole in the relaxation exponent is also discussed.

Random walks on tree structures are useful modelling tools in physics, because they both possess a rich variety of behaviours, sometimes radically differing from those of Euclidean diffusion, and they are simple enough to be mathematically tractable.

Several solutions of related models have already appeared [1-9]. Since, however, the applicability of the models rests on their generality and robustness, we feel it is worthwhile reading the calculations in a new way which clarifies these issues. We also present some novel results, *i.e.*:

1) The walk can change from recurrent to transient as a function of the temperature for a certain class of models.

2) Close to the transition temperature the relaxation exponent is close to zero, and at the transition the relaxation is logarithmic rather than algebraic.

3) Finally, we discuss the significance of the pole [2, 3] in the relaxation exponent as a function of the temperature. We show that the pole cannot be observed in the frequency domain, which is appropriate for some applications involving generalized susceptibilities [10].

General theory. – We consider the tree displayed in fig. 1: a walker randomly moves on the nodes, jumping upwards from a level j node at rate k_j^{u} and down to any daughter site at

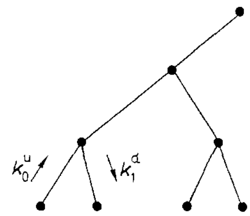

Fig. 1. – The hierarchical structure for the random walk.

rate k_j^d. The total rate down is thus zk_j^d, where z, the branching ratio, is assumed level independent for notational convenience. The rates k_j^d and k_j^u are arbitrary functions of j.

Let $\tilde{G}(f, s|i)$ be the Laplace transform of the probability of being at node f, with start point at i. As shown by Hoffmann, Grossmann and Wegner [2, 3], the basic relation

$$\tilde{G}(f, s \mid i) = \sum_{l=k_0}^{\infty} (Q^l(f, s \mid i) - Q^{l-1}(f, s \mid i)) z^{f-l} \tag{1}$$

holds true. In the above formula, the Q^l's are analogous quantities to $\tilde{G}$, except that they refer to the random walk on the levels obtained by projecting the original motion on the vertical axis. The arguments «f» and «i» of Q^l must thus be interpreted as the heights of the initial and final level. Finally, the superscript l indicates that we impose an absorbing boundary condition at level $l+1$, and k_0 is the level of the closest common ancestor of nodes f and i. The equation can be understood by observing that $Q^l(f, s|i) - Q^{l-1}(f, s|i)$ is the Laplace transform of the probability of being somewhere in the tree at level f, starting at a level-i node, and having reached but not exceeded level l. Since all the nodes with the same level index are equivalent, the probability of being at one particular level-f node is obtained by dividing $Q^l(f, s|i) - Q^{l-1}(f, s|i)$ by z^{l-f}, which is the number of level f descendant of a level-l node. In summary, each term in the sum of eq. (1) counts the contribution to G from the walks which reach but do not exceed level l.

Each Q^l is the solution of a random walk problem on a finite set of nodes, having a discrete set of eigenvalues, λ_i^l, $i = 0, 1, 2, \ldots, l$, in order of increasing magnitude. The small s/large time asymptotics is determined by the behaviour of the lowest eigenvalue λ_0^l (which appears as a pole in Q^l). In order to discuss its dependence on l, it is expedient to introduce a (free) energy function by the formula

$$F(j) - F(j+1) = T \ln\left(\frac{k_j^u}{zk_{j+1}^d}\right), \tag{2}$$

which is just detailed balance «the other way round». Given $F(0)$ and the rates, any $F(j)$ can be calculated by iteration, F acts as an «effective» potential well, in which the motion up and down the *levels* takes place.

In general the eigenvalues depend on the rates k_j^u and k_j^d, however, if we exclude cases where the time scale on energy level j is rescaled with a factor as in [2, 3], then we can derive the dependence of λ_0^l on l for any F by exact arguments, and we can also show that, if the energy F increases at least linearly with l, the spectrum remains discrete, *i.e.* $\lambda_0^l \to 0$ for $l \to \infty$ and $\lambda_i^l > \varepsilon > 0$, $\forall i > 0$. This will be shown in detail in a forthcoming paper. Here we assume the discreteness of the spectrum and find the form of $Q^l(f, s|i)$ for small s by a time-honoured qualitative argument, in essence due to Kramers [11], which avoids technicalities and appeals to the intuition. The result is

$$Q^l(f,\, s\,|\,i) = \frac{1}{s}\,\frac{\exp[-F(f)/T]/Z}{1+\exp[-F(l)/T]/s}\,, \tag{3}$$

from which $\lambda_0^l = \exp[-F(l)/T]$. The parameter Z is the partition function for the infinite system: $Z = \sum_{j=1}^{\infty} \exp[-F(j)/T]$. We assume that the series is convergent, which imposes obvious restrictions on F. The justification of eq. (3) can be phrased as follows: on time scales such that $1 \gg s \gg \exp[-F(l)/T]$, the system «does not notice» the boundary condition, and relaxes to the equilibrium distribution $\exp[-F(f)/T]/Z$. As a consequence of the flow of probability through the boundary, the total mass of the distribution decreases steadily, but the form (*i.e.* the spatial dependence) remains unchanged. Equation (3) is the one-pole approximation to Q^l which fit this picture. Of course, the Arrhenius factor $\exp[-F(l)/T]$ could be modified by some temperature-dependent prefactor, which, however, is not important at the present level of description.

Equation (1) can be reshuffled into

$$\tilde G(f,\, s\,|\,i) = (1-z^{-1}) \sum_{l=k_0}^{\infty} Q^l(f,\, s\,|\,i)\, z^{-l} - z^{-k_0} Q^{k_0-1}(f,\, s\,|\,i) =$$

$$= (1-z^{-1})\,\frac{\exp[-F(f)/T]}{Z} \sum_{l=k_0}^{\infty} \frac{z^{-l}}{s+\exp[-F(l)/T]}\,, \tag{4}$$

where we have neglected the term $z^{-k_0}Q^{k_0-1}$, which is unimportant for small s, and used eq. (3). It follows easily from eq. (4) that, if $\lim_{l\to\infty} F(l)/l < \infty$, then $\lim_{s\to 0} \tilde G(f,\, s\,|\,i) < \infty$, while if $\lim_{l\to\infty} F(l)/l = \infty$, then $\lim_{s\to 0} \tilde G(f,\, s\,|\,i) = \infty$. It is well known that the random walk is transient in the former case and recurrent in the latter [12]. The case in which F is asymptotically linear is at the borderline, and is discussed below in more detail.

Linear case. – We now think about the random motion as created by thermally activated hopping, and introduce a constant-energy difference Δ between nodes lying on contiguous levels. We then take the rates as

$$k_j^{\mathrm{u}} = k \exp[-\Delta/T]\,, \tag{5}$$

$$k_j^{\mathrm{d}} = 1\,, \tag{6}$$

where the parameter k is a positive number which describes the possibility of k parallel channels from a node to its parent $(k-1)$ of which can be transversed in one direction only. By eq. (2)

$$F(j) = (\Delta - T\ln(k/z))\,j\,. \tag{7}$$

The Boltzmann factor $\exp[-F(j)/T]$ is conveniently rewritten as A^j, where

$$A = \exp[-(\Delta/T + \ln(z/k))] < 1\,. \tag{8}$$

By eq. (3) we then get

$$Q^l(f,\, s\,|\,i) = \frac{(1-A)A^f}{s+A^l}\,. \tag{9}$$

In order to do the sum in eq. (4), we divide it into two parts:

$$\sum_{l=k_0}^{\infty} \frac{z^{-l}}{s+A^l} = \sum_{l=k_0}^{M-1} \frac{z^{-1}}{s+A^l} + \sum_{l=M}^{\infty} \frac{z^{-l}}{s+A^l}\,, \tag{9a}$$

where M is the least integer larger than $\ln s/\ln A$. This means that

$$M = \frac{\ln s}{\ln A} - \varepsilon(s)\,, \tag{10}$$

where $\varepsilon(s)$ is a periodic function of $\ln(s)$ which has unit amplitude [13]. For $l \leq M-1$ we have $s < A^l$, while for $l \geq M$, $s > A^l$. Therefore, $1/(s+A^l)$ can be expanded in powers of sA^{-l} and A^l/s in the two regions. Interchanging the order of summation in the resulting sum, one finally obtains

$$\sum_{l=k_0}^{\infty} Q^l(f,\, s\,|\,i)\, z^{-l} = (1-A)A^f \Bigg\{ s^{(\ln z/\ln A)-1} \sum_{l=0}^{\infty} (-1)^l \left[\frac{A^{-l\varepsilon} z^{\varepsilon}}{1-A^l/z} + (1-\delta_{l+1,n}) \frac{A^{(l+1)\varepsilon} z^{\varepsilon}}{(A^{l+1}z)^{-1}-1} \right] + \\ + \sum_{l=0}^{\infty} (-1)^l s^l \left[\left(\frac{\ln s}{\ln A} - \varepsilon(s) - k_0 \right) \delta_{l+1,n} - \frac{(A^{l+1}z)^{k_0}}{(A^{l+1}z)^{-1}-1} (1-\delta_{l+1,n}) \right] \Bigg\}, \tag{11}$$

where we have introduced a parameter

$$n = -\frac{\ln z}{\ln A} = \frac{T \ln z}{\Delta + T \ln(z/k)}, \tag{12}$$

which is, in general, a noninteger quantity. Accordingly, the Kroneker δ's which appear in eq. (11) are zero, except for special values of the temperature, at which logarithmic terms appear.

Let $T_1 = \Delta/\ln(k)$ and $T_2 = \Delta/\ln(k/z)$. Then $n(T_1) = 1$ and $n(T_2) = 2$. Furthermore, $T_1 < T_2$ when both quantities are positive, *i.e.* $k > z > 1$. The behaviour of G is qualitatively different in the regions $0 < T < T_1$, $T_1 < T < T_2$ and $T > T_2$, which we denote regions I, II and III, respectively.

We now disregard the s-dependence of $\varepsilon(s)$ in eq. (11), and denote the singular part of the propagator by $s^{-\beta}$. This defines the exponent

$$\beta(T) = \left(\frac{\ln z}{\ln A} + 1 \right) = \frac{\Delta - T \ln k}{\Delta + T \ln(z/k)}\,. \tag{13}$$

In region I, β is positive, $\tilde{G} \to \infty$ for $s \to 0$, and the walk is recurrent. In region II, β is negative, $\tilde{G} \to$ *constant* $< \infty$ for $s \to 0$ and the walk is transient. This applies also to region III. Furthermore, in this region the leading term of $\tilde{G}$ is linear in s, since $-\beta > 1$. Hence the exponent seen in the Fourier-Laplace domain is effectively one, and the pole of eq. (13) cannot be seen. In the time domain we have $G \sim t^{-(T \ln z)/(\Delta + T \ln(z/k))}$ (and exponentially decaying terms from the regular part of $\tilde{G}$). If the algebraic decay exponent is numerically large, the decay will be exponential over a long period, before the «true» asymptotic regime sets in. Hence, as the pole is approached from the left, the algebraic part of the decay becomes increasingly difficult to observe.

Finally, we look at the points $T = T_1$ and $T = T_2$. At $T = T_1$, $n = 1$, $\beta = 0$ and $\tilde{G} \sim \ln s/\ln A \to \infty$ for $s \to 0$. The walk is recurrent and the relaxation logarithmic. At $T = T_2$, $n = 2$, $\beta = -1$, the logarithmic term is $s \ln s$, which goes to zero for $s \to 0$. It is still larger than the linear term, and is therefore the leading term of $\tilde{G}$. The points at which

$n = 3, 4, \ldots$ etc., do not affect the leading term, which is $\sim s$. Figure 2 shows a comparison between the analytic results presented above and a numerical calculation for the exponent β. The deviations are due to the nature of the numerical method used.

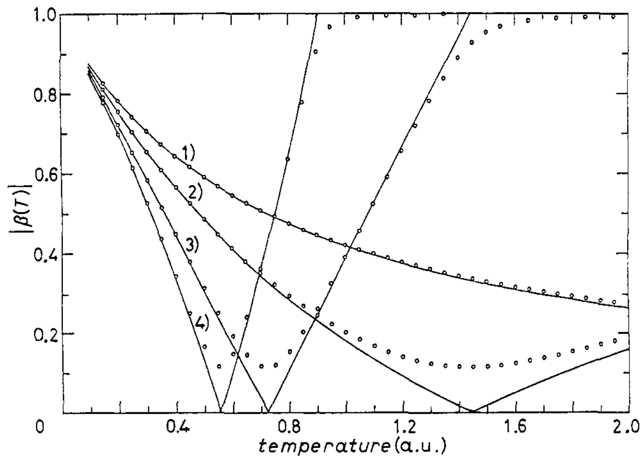

Fig. 2. – A comparison between a numerical calculation (circles) of the exponent β and the analytic results of the considered approximation (solid line). The analytic exponent is -1 for all $T > T_2$, for which it reaches this value. 1) $z = 4$, $k = 1$; 2) $z = 4$, $k = 2$; 3) $z = 4$, $k = 4$; 4) $z = 4$, $k = 6$.

Conclusion. – We have shown that the relaxation behaviour of a random walk on the tree is only determined by the least eigenvalue of a diffusion problem on the levels of the tree, with a potential energy $F(j)$. If $F(j)$ is asymptotically linear in j, there is the possibility of a transition from recurrent to transient behaviour when the temperature is varied.

While it has previously [14] been suggested that logarithmic relaxation and small exponents only would occur for $T \to 0$ or $z \to 1$ (the last limit means that the tree branches in an inhomogeneous fashion, and that most of the off-spring of a node only has a finite number of descendants), we show that they also occur at the transition and close to the transition, respectively, by a different mechanism, which involves a parameter k, entering the definition of the rates. The condition for having a transition is $k > 1$. When $k \neq 1$, some paths are effectively one-way roads, which is at variance with the principle of detailed balance. However, if the nodes of the tree represent complex states of some physical systems, with an «inner» dynamics, rather than simple pointlike entities, the condition $k \neq 1$ means that the «exits» of a node in the upwards direction (with respect to the tree structure) are more easily found than those in the downward direction. Thus both energy and entropy considerations determine the form of the rates out of a node.

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