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Fate of superconductivity in disordered Dirac and semi-Dirac semimetals

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E-mail: gqliu@ustc.edu.cn and zhangcj@hmf.ac.cn**Keywords:** semimetal, superconductivity, disorder, quantum phase transition

Abstract

The influence of weak disorder on the superconductivity in ordinary metals can be formally described by the Abrikosov-Gorkov diagrammatic approach. The vertex correction is ignored in this approach because an inequality $k_F l \gg 1$, where k_F is the Fermi momentum and l mean free path, is satisfied in ordinary metals with a large Fermi surface. In a Dirac semimetal that has discrete Fermi points, this inequality may break down even for arbitrarily weak disorder since $k_F \rightarrow 0$, and thus the vertex correction could be important. We incorporate the vertex correction into the self-consistent equations of the superconducting gap and the disorder scattering rate, and then apply the generalized approach to study how s -wave superconductivity is affected by random chemical potential in two- and three-dimensional Dirac semimetals, as well as two-dimensional semi-Dirac semimetal. In the clean limit, superconductivity is formed only when the pairing interaction strength is greater than some critical value in these materials. Adding random chemical potential to the system promotes superconductivity by generating a finite fermionic density of states at the Fermi level. In three-dimensional Dirac semimetal, the critical attraction strength is reduced by weak disorder, but remains finite. In the other two cases, superconductivity is induced by arbitrarily weak attraction. Including the vertex correction does not change these qualitative results, and actually could further promote superconductivity in the weak-attraction regime. Bilayer graphene is quite special in that its zero-energy density of states is nonzero despite the existence of Fermi points. Due to this peculiar property, superconductivity is always slightly suppressed by random chemical potential, and the impact of vertex correction is nearly negligible.

1. Introduction

The advance made in the fabrication of various types of semimetals (SMs) has stimulated a huge number of research works in the past decade [1–16]. Graphene, made of one layer of carbon atoms on a honeycomb lattice, is a typical two-dimensional (2D) Dirac SM (DSM) and has been extensively studied [1–4]. Moreover, the gapless surface electronic state of three-dimensional (3D) topological insulators can also be regarded as 2D DSM [5–7]. Recently, a stable 3D DSM state that is protected by crystal symmetry was found in Na_3Bi and Cd_3As_2 [8–10]. In addition, Weyl SM (WSM), in which the low-energy fermionic excitations display linear dispersion around pairs of nodes with opposite chirality, was observed in TaAs, NbAs, TaP, and NbP by the angle-resolved photoemission spectroscopy (ARPES) [10–14]. There also exist other types of SMs, such as 3D nodal line SM (NLSM) [15–18], 2D semi-DSM [19–46], 3D double-WSM [47–59], 3D triple-WSM [49, 53, 56–62], 3D anisotropic-WSM [49, 63–65], and 3D Luttinger SM [66–70].

Most of these SMs share a common feature: the Fermi surface is composed of a number of zero-dimensional points. This implies that the fermion density of states (DOS) vanishes at the Fermi level, which is different from

conventional metals whose DOS takes a finite value at the Fermi level. This difference is responsible for many intriguing properties of SM materials that cannot occur in ordinary metals.

Cooper pairing instability in 2D DSM has attracted considerable interest in recent years [71–92]. It has been proposed that phonon or plasmon may mediate an effective attraction between Dirac fermions [4, 72, 75]. For an undoped 2D DSM, superconductivity can be realized only when the net attraction is sufficiently strong [71–92]. This is quite different from the conventional metal superconductors in which even an arbitrarily weak attraction suffices to trigger Cooper pairing [93–95]. The difference is owing to the fact that the fermion DOS vanishes at zero energy, namely $\rho(0) = 0$. Cooper pairing instability may be achieved in other SMs, including 2D semi-DSM [43, 44], 3D DSM [96], 3D WSM [97, 98], 3D NLSM [17, 18], and 3D Luttinger SM [67, 69, 70]. In these materials, there is also a threshold value for the strength of attraction.

In terms of industrial and commercial applications, the existence of a threshold for net attraction appears to be a negative result because it makes it difficult to realize intrinsic superconductivity in undoped DSMs. However, from the perspective of theoretical study, it provides us with a unique opportunity to investigate various novel properties that do not exist in ordinary metal superconductors. The critical value of attraction defines a genuine quantum critical point (QCP) between the SM and superconducting (SC) phases at zero temperature, and the system exhibits a wealth of attractive quantum critical behaviors around such QCP [69, 99–104]. For instance, an effective space-time supersymmetry was recently argued to emerge at this QCP at low energies [99–104].

An interesting problem is how superconductivity formed in various DSMs is affected by weak disorder. Among the possible disorders, random chemical potential is most frequently encountered in realistic materials [3, 105, 106]. In the case of conventional metal superconductors, Anderson theorem states that weak random chemical potential does not alter the SC gap Δ and the critical temperature T_c if the pairing is *s*-wave [107–111]. This result can be obtained by the diagrammatic approach developed by Abrikosov and Gorkov (AG) [111, 112]. For such result to be valid, an important precondition is that the low-energy fermion DOS is not sensitive to weak disorder [110, 111]. While this condition is satisfied in ordinary metals with a large Fermi surface, it breaks down in DSMs that have only isolated Fermi points. Renormalization group (RG) analysis showed that random chemical potential is a relevant perturbation to 2D DSM [87, 92, 105, 113–115], thus arbitrarily weak disorder eventually flows to the strong coupling regime and then drives the system to enter into a compressible diffusive metal (CDM) state, in which disorder generates a finite scattering rate Γ . In addition, the fermions acquire a finite zero-energy DOS $\rho(0)$ that is a function of Γ [87, 92, 105, 113–118]. It is obvious that the zero-energy DOS of the 2D DSM is very sensitive to weak disorder, in stark contrast to the case of ordinary metals. As a result, weak random chemical potential might have significant influence on Δ and T_c . The question is whether superconductivity is enhanced or suppressed.

In a recent work [92], the influence of random gauge potential and random mass on superconductivity in 2D DSM was studied by using the perturbative RG method. It was found there that the critical attraction for Cooper pairing is increased by random gauge potential or random mass [92]. The strength parameter of random gauge potential can be fixed at a small value, whereas the strength parameter of random mass flows to zero logarithmically as the zero-energy limit is reached [92, 105, 113–115]. Therefore, the perturbative RG analysis is reliable in these two cases.

If random chemical potential and pairing interaction are considered simultaneously, RG analysis revealed [92] that both of the two strength parameters tend to diverge at some finite energy scale. This indicates that the system becomes unstable and will be turned into a new phase. In such a strong coupling regime, the RG method cannot be used to determine whether the system enters into a CDM phase or a SC phase. To address this issue, we need to analyze the possible generation of disorder scattering rate Γ and the possible generation of SC gap Δ . These two processes strongly affect each other. Therefore, the quantities Γ and Δ should be calculated in a self-consistent and unbiased way.

Dyson-Schwinger equations (DSEs) provide a suitable framework to treat disorder scattering and Cooper pairing on an equal footing. The essence of this approach is to solve the self-consistently coupled DSEs for Γ and Δ . In the most generic form, DSEs are complete and contain all the information about disorder scattering and Cooper pairing. In practice, however, it is always necessary to properly truncate the complete set of DSEs. The original AG method and its generalization to be presented below in this work are actually two different truncations of the complete set of DSEs.

Nandkishore *et al* [87] investigated the effect of random chemical potential on *s*-wave superconductivity by taking the surface state of 3D topological insulator as an example. After solving the mean-field gap equation, which is a simplified version of AG method, in combination with a RG analysis [87], they found that superconductivity can be induced by arbitrarily weak attraction. Subsequently, Potirniche *et al* [88] analyzed the interplay of superconductivity and random chemical potential by considering a Hubbard model defined on honeycomb lattices with an on-site attractive coupling U by means of self-consistent Bogoliubov-de Gennes equation method [88]. In the clean limit, they argued that the system remains a SM if U is smaller than certain

critical value U_c , but becomes SC when U exceeds U_c . Adding disorder to the clean system results in a complicated behavior of superconductivity. In the strong coupling regime $U > U_c$, disorder can suppress superconductivity. In the weak coupling regime $U < U_c$, weak disorder enhances superconductivity, but strong disorder eventually destroys superconductivity.

In [92], the diagrammatic AG method was applied to study the impact of random chemical potential on superconductivity in the context of a 2D DSM, yielding results that are qualitatively consistent with those of Nandkishore *et al* [87] and Potirniche *et al* [88]. However, here we emphasize that the applicability of the original AG method to 2D DSM is actually questionable [87, 92], because it entirely neglects the vertex correction to the fermion-disorder coupling. Such correction would be small if the inequality $k_F l \gg 1$, where k_F is the Fermi momentum and l the mean free path, is satisfied. Since the disorder scattering rate Γ is inversely proportional to l , the above inequality can be re-expressed as $k_F \gg \Gamma$. In ordinary metals with a finite Fermi surface, the Fermi momentum is usually large and the scattering rate caused by weak disorder is small, thus this inequality is generically satisfied. However, 2D DSM contains only discrete Fermi points, which means that $k_F \rightarrow 0$. Additionally, random chemical potential is a relevant perturbation in 2D DSM and always induces a finite disorder scattering rate Γ . In this case the inequality $k_F \gg \Gamma$ breaks down, so the vertex correction may be very important. It is interesting to verify whether random chemical potential still promotes superconductivity after including the vertex correction. To obtain a reliable relation between superconductivity and random chemical potential, one should go beyond the original AG approximation and study the role of vertex correction. This is the first motivation of this work.

We notice an important principle that the impact of random chemical potential in various SMs is closely related to the dimensionality and the specific fermion dispersion. For example, weak random chemical potential is irrelevant in a 3D DSM/WSM, but becomes relevant if its strength is sufficiently large. Accordingly, for a 3D DSM, there is a QCP between the SM and CDM phases with varying disorder strength [106, 119–124]. This behavior is apparently different from the case of 2D DSM, where an arbitrarily weak random chemical potential leads to the CDM transition. For 2D semi-DSM [45, 46], in which the fermion dispersion is linear along one direction and quadratic along the other direction, the CDM state is achieved by an arbitrarily weak disorder, analogous to 2D DSM. The second motivation of this work is to examine whether the conclusion obtained previously in the case of 2D DSM is applicable to other SMs.

We will show that arbitrarily weak attraction suffices to induce an *s*-wave superconductivity in 2D DSM when random chemical potential is present. If the attraction is weak, the magnitude of SC gap increases with growing disorder strength, but begins to decrease when the disorder strength becomes large enough. For relatively strong attraction, the SC gap is always suppressed by the increasing disorder strength. We include the vertex correction into the self-consistent equations for the SC gap and the disorder scattering rate, and show that the above behavior is not altered qualitatively by the vertex correction. In the case of weak attraction and weak disorder, the magnitude of SC gap is further amplified once the vertex correction is incorporated.

The generalized AG method can be also applied to other SM systems to examine whether or not the vertex correction plays a vital role. To make our analysis more comprehensive, and also to examine how superconductivity relies on the dimensionality and the energy dispersion of the fermion excitations, we then study the fate of superconductivity in disordered 3D DSM and 2D semi-DSM, which both have a point-touching band structure.

In the case of slightly disordered 3D DSM, the critical pairing interaction strength g_c is smaller than that obtained in the clean limit, but remains finite. Thus, there is still a SC QCP, at which interesting quantum critical phenomena might occur. When the 3D DSM contains strong random chemical potential, the critical value g_c vanishes, and superconductivity is formed by arbitrarily weak pairing interaction. An apparent conclusion is that superconductivity is promoted by random chemical potential in 3D DSM. In a 2D semi-DSM, the disorder effect on superconductivity is very similar to 2D DSM: superconductivity occurs only when the pairing interaction strength exceeds a threshold g_c in the clean limit, but is triggered by arbitrarily weak attraction when random chemical potential is introduced. In both of these two systems, including vertex correction results in more significant enhancement of SC gap in the weak-attraction regime.

Comparing to the above three SMs, the bilayer graphene is special in that its Fermi surface is composed of discrete points but the DOS is finite at the Fermi level. We find that *s*-wave superconductivity in bilayer graphene is always slightly suppressed by random chemical potential, and that the vertex correction plays a minor role. This result is qualitatively different from the other three kinds of SM. The difference stems from the fact that the zero-energy DOS is nonzero only in bilayer graphene but vanishes otherwise.

The rest of paper is organized as follows. In section 2, we present the model Hamiltonian for 2D DSM, analyze the influence of disorder on superconductivity by means of the AG method, and examine the role played by the vertex correction. The disorder effect on superconductivity in 3D DSM and 2D semi-DSM is studied in section 3 and section 4, respectively. In section 5, we apply the same approach to the case of bilayer graphene. In

section 6, we give a remark on several related questions, including truncation of DSEs, the effect of rare region, and Anderson localization. We summarize our main results in section 7.

2. Superconductivity in 2D Dirac semimetal

In this section, we consider the case of 2D DSM and study the impact of random chemical potential on superconductivity by means of the AG approach and its proper generalization. The framework employed in this section is quite general and will be utilized in the following several sections to study the fate of superconductivity in 3D DSM, 2D semi-DSM, and bilayer graphene.

2.1. Model Hamiltonian

We consider a single species of massless Dirac fermions that emerge, for instance, on the surface of a 3D topological insulator. Following the notations adopted in [87], we write the Hamiltonian in the form

$$H = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} (-\mu\sigma_0 + v k_x \sigma_1 + v k_y \sigma_2) \psi_{\mathbf{k}} - g \sum_{\mathbf{k}, \mathbf{q}} \psi_{\mathbf{k}}^{\dagger} (-i\sigma_2) \psi_{-\mathbf{k}} \psi_{\mathbf{q}} i\sigma_2 \psi_{-\mathbf{q}}, \quad (1)$$

where μ is the chemical potential and g is the coupling constant for the BCS-type attractive interaction. At a finite μ , the DSM has a finite Fermi surface, and many of its low-energy properties are very similar to ordinary metals. In the following, we only consider the most interesting case of zero chemical potential, $\mu = 0$, corresponding to undoped SM. The spinor field is defined as $\psi_{\mathbf{k}} = (c_{\uparrow\mathbf{k}}, c_{\downarrow\mathbf{k}})^T$, whose conjugate is $\psi_{\mathbf{k}}^{\dagger} = (c_{\uparrow\mathbf{k}}^{\dagger}, c_{\downarrow\mathbf{k}}^{\dagger})$. Moreover, $\sigma_{1,2}$ are standard Pauli matrices, and σ_0 is identity matrix. The SC order parameter is defined as

$$\Delta = g \sum_{\mathbf{k}} \langle \psi_{\mathbf{k}} (i\sigma_2) \psi_{-\mathbf{k}} \rangle. \quad (2)$$

The system enters into a SC phase when Δ acquires a nonzero value.

Our analysis starts from the partition function

$$Z = \int D[\psi^{\dagger}, \psi] \exp \left(- \int_0^{\beta} d\tau \int d^2\mathbf{x} L[\psi^{\dagger}, \psi] \right), \quad (3)$$

where $\beta = 1/T$ and the Lagrangian is related to the Hamiltonian H via the Legendre transformation

$$L = \int \frac{d^2\mathbf{k}}{(2\pi)^2} \psi_{\mathbf{k}}^{\dagger} \partial_{\tau} \psi_{\mathbf{k}} + H. \quad (4)$$

To express the action in a more compact form, it is convenient to introduce a four-component Nambu spinor:

$$\Psi = (\psi_{\mathbf{k}}, \psi_{-\mathbf{k}}^{\dagger})^T. \quad (5)$$

After decoupling the quartic attractive interaction by means of Hubbard-Stratonovich transformation, we can re-write the partition function as

$$Z = \int D[\Psi^{\dagger}, \Psi, \Delta^*, \Delta] \exp \left(- \int_0^{\beta} d\tau \int d^2\mathbf{x} L[\Psi^{\dagger}, \Psi, \Delta^*, \Delta] \right), \quad (6)$$

where L takes the form

$$L = T \sum_{\omega_n} \sum_{\mathbf{k}} \Psi_{\omega_n \mathbf{k}} G_{\omega_n \mathbf{k}}^{-1} \Psi_{\omega_n \mathbf{k}} - \frac{|\Delta|^2}{g}. \quad (7)$$

In the above expression, we have defined a fermion propagator $G_{\omega_n \mathbf{k}} \equiv G(\omega_n, \mathbf{k})$ that is given by

$$G_{\omega_n \mathbf{k}}^{-1} = \begin{pmatrix} i\omega_n & vk_+ & 0 & \Delta \\ vk_- & i\omega_n & -\Delta & 0 \\ 0 & -\Delta^* & i\omega_n & vk_- \\ \Delta^* & 0 & vk_+ & i\omega_n \end{pmatrix}, \quad (8)$$

where $k_{\pm} = k_x \pm ik_y$. Within the Matsubara formalism, the fermion frequency is $\omega_n = (2n + 1)\pi T$ with n being integers. The above expression of partition function is consistent with [87].

2.2. Clean case

The SC gap equation can be derived by integrating over fermion fields Ψ and Ψ^{\dagger} . For this purpose, the partition functions is further written as

$$Z = \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi e^S = \prod_{\omega_n} \prod_{\mathbf{k}} \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi e^{\Psi_{\omega_n, \mathbf{k}}^\dagger G_{\omega_n, \mathbf{k}}^{-1} \Psi_{\omega_n, \mathbf{k}}} e^{-\int_{\tau, \mathbf{x}} \frac{|\Delta|^2}{g}}, \quad (9)$$

where $\int_{\tau, \mathbf{x}} \equiv \int_0^\beta d\tau \int d^2\mathbf{x}$. Performing the functional integration over Ψ and Ψ^\dagger yields

$$Z = \prod_{\omega_n} \prod_{\mathbf{k}} \beta^4 \det(G_{\omega_n, \mathbf{k}}^{-1}) e^{-\int_{\tau, \mathbf{x}} \frac{|\Delta|^2}{g}}. \quad (10)$$

It is then easy to get

$$\ln Z = \sum_{\omega_n} \sum_{\mathbf{k}} \ln [\beta^4 \det(G_{\omega_n, \mathbf{k}}^{-1})] - \int_{\tau, \mathbf{x}} \frac{|\Delta|^2}{g}. \quad (11)$$

Through direct calculation, we obtain

$$\det G_{\omega_n, \mathbf{k}}^{-1} = (\omega_n^2 + v^2 k^2 + |\Delta|^2)^2. \quad (12)$$

For a sample of volume V , the free energy density is

$$f = \frac{F}{V} = -\frac{1}{\beta V} \ln Z = -\frac{1}{\beta V} \sum_{\omega_n} \sum_{\mathbf{k}} \ln [\beta^4 (\omega_n^2 + v^2 k^2 + |\Delta|^2)^2] + \frac{|\Delta|^2}{g}. \quad (13)$$

Making variation with respect to infinitesimal change of Δ , $\frac{\delta f}{\delta \Delta} = 0$, we finally obtain the gap equation:

$$2T \sum_{\omega_n} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\omega_n^2 + v^2 k^2 + \Delta^2} = \frac{1}{g}. \quad (14)$$

We have already fixed the phase factor of the gap function Δ , and in the following will take Δ as a real variable. At zero temperature $T = 0$, the gap equation becomes

$$2 \int \frac{d\omega}{2\pi} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\omega^2 + v^2 k^2 + \Delta^2} = \frac{1}{g}. \quad (15)$$

Performing the integration of ω and k , we obtain

$$\frac{g}{2\pi v^2} (\sqrt{v^2 \Lambda^2 + \Delta^2} - \Delta^2) = 1, \quad (16)$$

where Λ is the cutoff of the momentum. Setting $\Delta = 0$ yields the critical coupling $g_{c0} = 2\pi v/\Lambda$. The gap Δ acquires a nonzero value only when $g > g_{c0}$, and there is a SM-SC QCP at $g = g_{c0}$ [72, 74, 77, 87, 92].

2.3. Analysis by AG method without vertex correction

In the presence of random chemical potential, the dynamics of Dirac fermions is modified due to the disorder scattering. Consequently, the fermion propagator can be expressed by the matrix

$$(G_{\omega_n, \mathbf{k}}^F)^{-1} = \begin{pmatrix} A_1 i\omega_n & A_2 vk_+ & 0 & A_3 \Delta \\ A_2 vk_- & A_1 i\omega_n & -A_3 \Delta & 0 \\ 0 & -A_3 \Delta^* & A_1 i\omega_n & A_2 vk_- \\ A_3 \Delta^* & 0 & A_2 vk_+ & A_1 i\omega_n \end{pmatrix}, \quad (17)$$

where $A_i \equiv A_i(\omega_n)$ with $i = 1, 2, 3$ are the renormalization functions induced by the disorder scattering. The propagators $G_{\omega_n, \mathbf{k}}^F$ and $G_{\omega_n, \mathbf{k}}$ are connected with each other through the DS

$$(G_{\omega_n, \mathbf{k}}^F)^{-1} = G_{\omega_n, \mathbf{k}}^{-1} - \Sigma_{\text{dis}}(\omega_n), \quad (18)$$

where

$$\Sigma_{\text{dis}}(\omega_n) = n_{\text{imp}} u^2 \int \frac{d^2\mathbf{k}}{(2\pi)^2} G_{\omega_n, \mathbf{k}}^F, \quad (19)$$

where n_{imp} is the impurity concentration and u is the strength of one single impurity. The Feynman diagram used in the AG approach is shown in figure 1. In the case of random chemical potential, one can verify that $A_2 = 1$. If long-range correlated disorder [3] is considered, A_2 would receive a non-trivial correction. Making use of the fact $A_2 = 1$, the self-consistent equations become

$$A_1 = 1 + n_{\text{imp}} u^2 A_1 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{A_1^2 \omega_n^2 + v^2 k^2 + A_3^2 \Delta^2}, \quad (20)$$

$$A_3 = 1 + n_{\text{imp}} u^2 A_3 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{A_1^2 \omega_n^2 + v^2 k^2 + A_3^2 \Delta^2}. \quad (21)$$



Figure 1. Feynman diagram for the fermion self-energy. Thin line represents the free fermion propagator, and thick line the full fermion propagator. Dotted line represents disorder scattering. The vertex correction is neglected in the original AG method, which is valid if $k_F l \gg 1$.

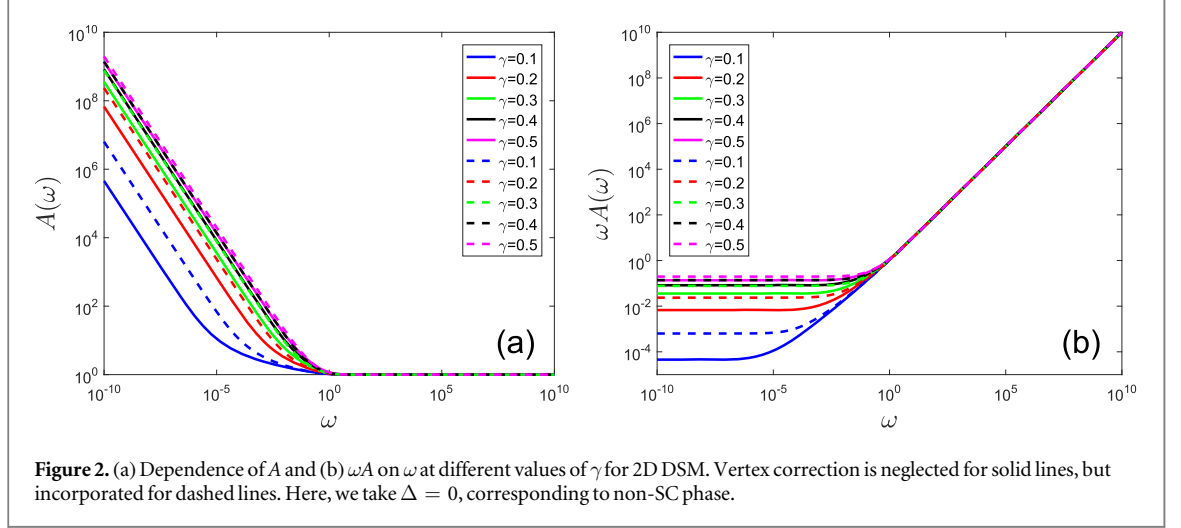


Figure 2. (a) Dependence of A and (b) ωA on ω at different values of γ for 2D DSM. Vertex correction is neglected for solid lines, but incorporated for dashed lines. Here, we take $\Delta = 0$, corresponding to non-SC phase.

It is clear that $A_1 = A_3 = A$. The gap equation becomes

$$\Delta = 2gT \sum_{\omega_n} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{A\Delta}{A^2 \omega_n^2 + v^2 k^2 + A^2 \Delta^2}. \quad (22)$$

Carrying out the integration over momenta, we get two coupled equations

$$A = 1 + \frac{n_{\text{imp}} u^2}{4\pi v^2} A \ln \left(1 + \frac{v^2 \Lambda^2}{A^2 \omega_n^2 + A^2 \Delta^2} \right), \quad (23)$$

$$\Delta = \frac{gT}{2\pi v^2} \sum_{\omega_n} A \Delta \ln \left(1 + \frac{v^2 \Lambda^2}{A^2 \omega_n^2 + A^2 \Delta^2} \right). \quad (24)$$

We now take the zero temperature limit, namely $T = 0$, and then make the re-scaling transformations $\frac{\omega}{v\Lambda} \rightarrow \omega$ and $\frac{\Delta}{v\Lambda} \rightarrow \Delta$, which leads us to

$$A = 1 + \frac{\gamma}{2} A \ln \left(1 + \frac{1}{A^2 \omega^2 + A^2 \Delta^2} \right), \quad (25)$$

$$1 = \frac{g}{g_{c0}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega A \ln \left(1 + \frac{1}{A^2 \omega^2 + A^2 \Delta^2} \right), \quad (26)$$

where

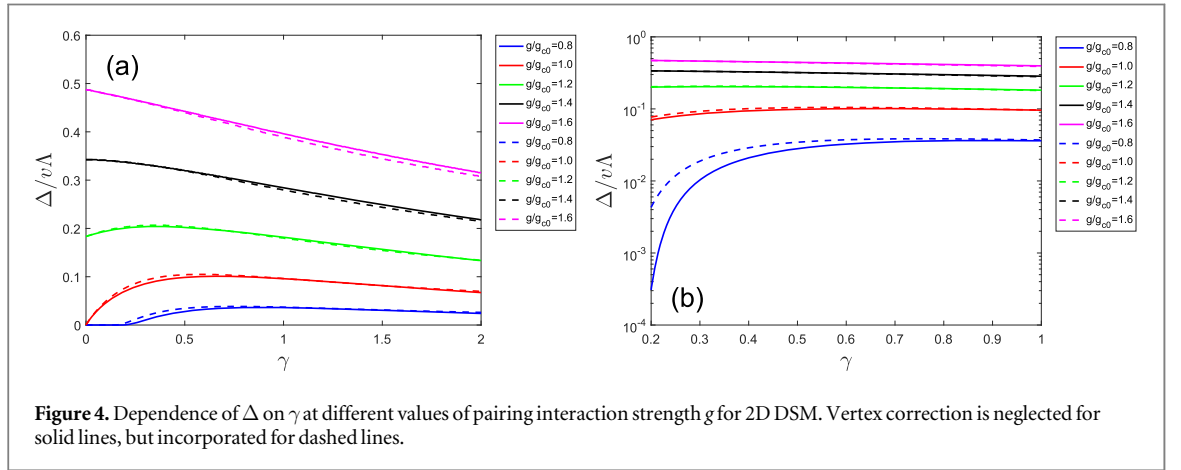
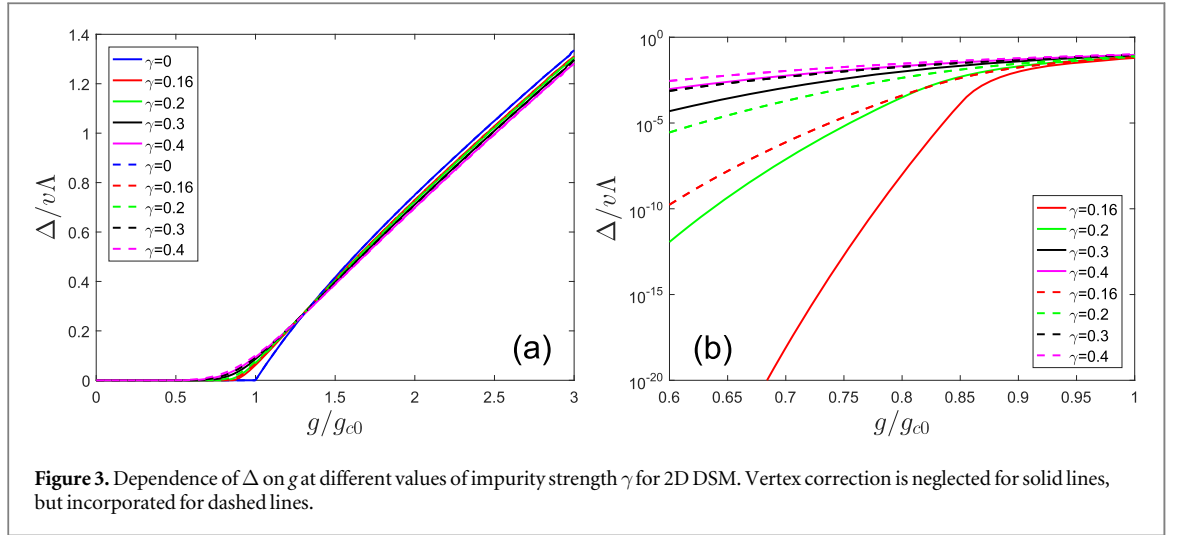
$$\gamma = \frac{n_{\text{imp}} u^2}{2\pi v^2}. \quad (27)$$

Upon approaching the QCP, $g \rightarrow g_c$, so the above equations are simplified to

$$A = 1 + \frac{\gamma}{2} A \ln \left(1 + \frac{1}{A^2 \omega^2} \right), \quad (28)$$

$$1 = \frac{g_c}{g_{c0}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega A \ln \left(1 + \frac{1}{A^2 \omega^2} \right). \quad (29)$$

Numerical results of equation (28) are shown in figures 2(a) and (b) by the solid lines. In the zero-energy limit, ωA approaches to a constant Γ , which is the disorder scattering rate. If ω decreases from the energy scale of Γ , ωA approaches to a constant. Above the scale of Γ , $A \rightarrow 1$. The asymptotic behavior of A is approximately described by



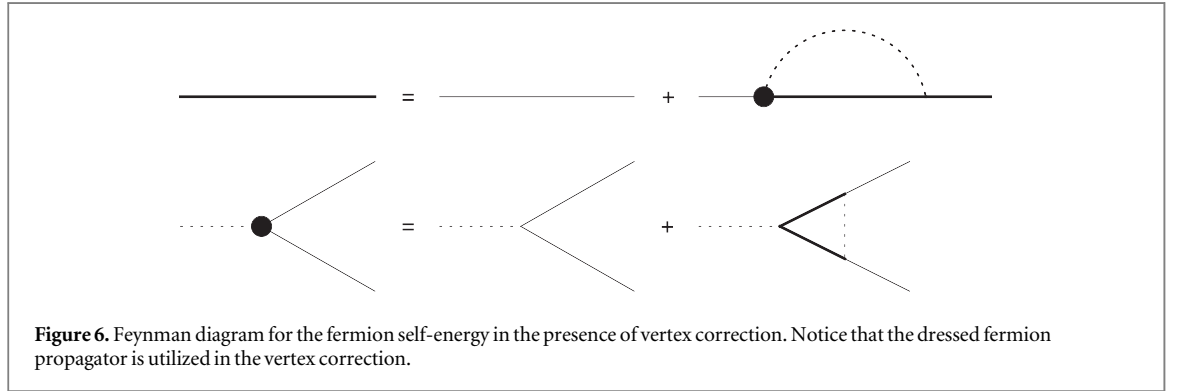
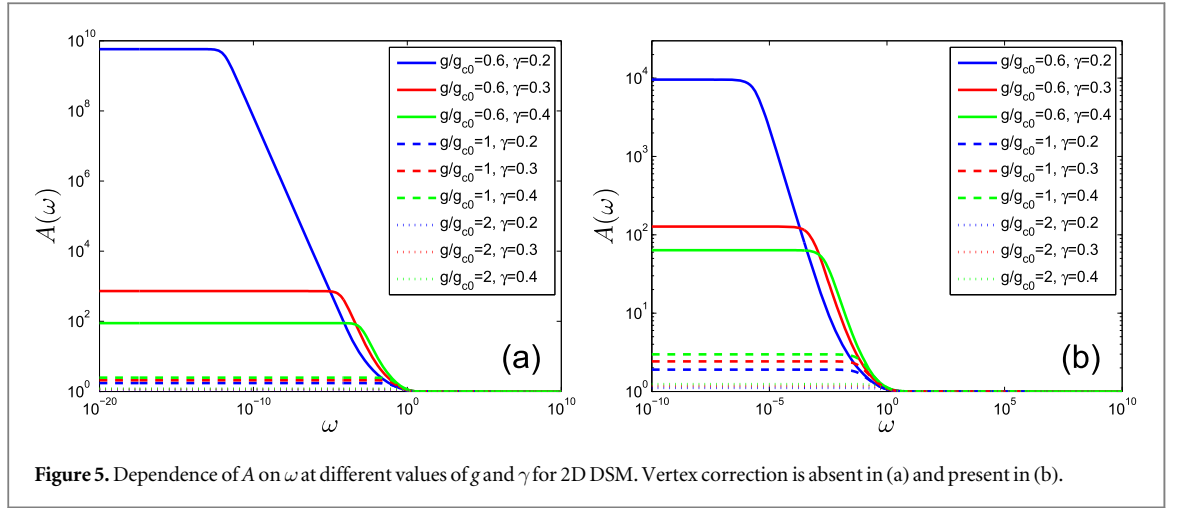
$$A \sim \begin{cases} \frac{\Gamma}{|\omega|} & \text{if } |\omega| \ll \Gamma, \\ 1 & \text{if } |\omega| \gg \Gamma. \end{cases} \quad (30)$$

The integral appearing in equation (29) is divergent, which implies that $g_c \rightarrow 0$. Therefore, even an arbitrarily weak attraction suffices to trigger Cooper pairing instability. In this regard, random chemical potential tends to promote the formation of Cooper pairing, in agreement with the result of Nandkishore *et al* [87]. The dependence of the gap Δ on g at different values of γ is displayed in figures 3(a) and (b) by the solid lines. We present the relation between Δ and γ with different values of g in figures 4(a) and (b) by the solid lines. The figures exhibit that Δ is enhanced by weak disorder and then suppressed gradually by strong disorder if the attraction is weak. However, for sufficiently large attraction, the gap Δ is suppressed by disorder monotonously. These results are qualitatively the same as that obtained by Potirniche *et al* through using self-consistent BdG equations [88].

The ω -dependence of $A(\omega)$, obtained from the solutions of equations (25) and (26), are shown in figure 5(a). The function $A(\omega)$ always approaches a finite value in the low energy regime, which is expected since the gap Δ provides a cutoff. According to figure 5(a), if the pairing interaction is relatively weak, $A(\omega)$ goes to a larger constant value at low energies for smaller γ . If the pairing interaction is relatively strong, however, $A(\omega)$ takes a larger constant value at low energies for larger γ .

2.4. Beyond AG approximation

We emphasize that the gap equation analysis of the disorder effects based on the original AG formalism is actually problematic. The validity of AG method relies on a crucial assumption that $k_F l \gg 1$, which is equivalent to $k_F \gg \Gamma$ since $\Gamma \sim 1/l$. This inequality is satisfied in an ordinary metal in which the Fermi momentum is large and the scattering rate Γ can be made sufficiently small if random chemical potential is supposed to be weak enough. For a 2D DSM, however, we know that $k_F = 0$. Additionally, even an arbitrarily weak random chemical



potential is able to drive 2D DSM to become a CDM [105, 113–118], which in turn generates a finite Γ . Thus, the above inequality certainly breaks down, and the vertex correction can no longer be regarded as unimportant.

To examine whether the interesting conclusion reached by employing the original AG approximation is robust, we need to incorporate the vertex correction explicitly in the self-consistent gap equations. The Feynman diagram for fermion self-energy that contains the vertex correction is shown in figure 6.

The correction to the fermion-disorder coupling vertex is given by

$$\Xi(\omega, \mathbf{p}, \mathbf{q}) = 1 + n_{\text{imp}} u^2 \int \frac{d^2 \mathbf{k}}{(2\pi)^2} G^F(\omega, \mathbf{k} + \mathbf{p}) G^F(\omega, \mathbf{k} + \mathbf{q}), \quad (31)$$

which can be further written as

$$\begin{aligned} \Xi(\omega, \mathbf{p}, \mathbf{q}) = & 1 + n_{\text{imp}} u^2 \int \frac{d^2 \mathbf{k}}{(2\pi)^2} [-A_1^2 \omega^2 + v^2(\mathbf{k} + \mathbf{p}) \cdot (\mathbf{k} + \mathbf{q}) + A_3^2 \Delta^2] \\ & \times [A_1^2 \omega^2 + v^2(\mathbf{k} + \mathbf{p})^2 + A_3^2 \Delta^2]^{-1} [A_1^2 \omega^2 + v^2(\mathbf{k} + \mathbf{q})^2 + A_3^2 \Delta^2]^{-1}. \end{aligned} \quad (32)$$

Using the transformation $\mathbf{k} + \mathbf{q} \rightarrow \mathbf{k}$, and employing the Feynman parameterization

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}, \quad (33)$$

we convert the vertex correction into the form

$$\begin{aligned} \Xi(\omega, \mathbf{p}, \mathbf{q}) = & 1 + n_{\text{imp}} u^2 \int_0^1 dx \int \frac{d^2 \mathbf{k}}{(2\pi)^2} [-A_1^2 \omega^2 + v^2(\mathbf{k} + \mathbf{p} - \mathbf{q}) \cdot \mathbf{k} + A_3^2 \Delta^2] \\ & \times [A_1^2 \omega^2 + v^2(\mathbf{k} + x(\mathbf{p} - \mathbf{q}))^2 + A_3^2 \Delta^2 + x(1-x)v^2(\mathbf{p} - \mathbf{q})^2]^{-2}. \end{aligned} \quad (34)$$

We further make the replacement $\mathbf{k} + x(\mathbf{p} - \mathbf{q}) \rightarrow \mathbf{k}$, and then obtain

$$\Xi(\omega, \mathbf{p}, \mathbf{q}) = 1 + n_{\text{imp}} u^2 \int_0^1 dx \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \left\{ \frac{1}{A_1^2 \omega^2 + v^2 k^2 + A_3^2 \Delta^2 + x(1-x)v^2(\mathbf{p} - \mathbf{q})^2} - \frac{2A_1^2 \omega^2 + 2x(1-x)v^2(\mathbf{p} - \mathbf{q})^2}{[A_1^2 \omega^2 + v^2 k^2 + A_3^2 \Delta^2 + x(1-x)v^2(\mathbf{p} - \mathbf{q})^2]^2} \right\}. \quad (35)$$

One can verify that $\Xi(\omega, \mathbf{p}, \mathbf{q})$ is the function of $|\mathbf{p} - \mathbf{q}|$, which means $\Xi(\omega, \mathbf{p}, \mathbf{q}) \equiv \Xi(\omega, |\mathbf{p} - \mathbf{q}|)$. Performing the integrations over k and x leads to the following vertex correction:

$$\begin{aligned} \Xi(\omega, |\mathbf{p} - \mathbf{q}|) = 1 + \frac{\gamma}{2} & \left\{ \ln \left(\frac{A_1^2 \omega^2 + v^2 \Delta^2 + A_3^2 \Delta^2}{A_1^2 \omega^2 + A_3^2 \Delta^2} \right) \right. \\ & + \frac{4A_1^2 \omega^2 + v^2 |\mathbf{p} - \mathbf{q}|^2}{v|\mathbf{p} - \mathbf{q}| \sqrt{J}} \ln \left(\frac{\sqrt{J} - v|\mathbf{p} - \mathbf{q}|}{\sqrt{J} + v|\mathbf{p} - \mathbf{q}|} \right) \\ & \left. - \frac{4A_1^2 \omega^2 + v^2 |\mathbf{p} - \mathbf{q}|^2}{v|\mathbf{p} - \mathbf{q}| \sqrt{J + 4v^2 \Delta^2}} \ln \left(\frac{\sqrt{J + 4v^2 \Delta^2} - v|\mathbf{p} - \mathbf{q}|}{\sqrt{J + 4v^2 \Delta^2} + v|\mathbf{p} - \mathbf{q}|} \right) \right\}, \end{aligned} \quad (36)$$

where γ is defined in equation (27), and

$$J = 4(A_1^2 \omega^2 + A_3^2 \Delta^2) + v^2 |\mathbf{p} - \mathbf{q}|^2. \quad (37)$$

After including the vortex correction, the equations for A_1 and A_3 now become

$$A_1 = 1 + n_{\text{imp}} u^2 A_1 \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{\Xi(\omega, k)}{A_1^2 \omega^2 + v^2 k^2 + A_3^2 \Delta^2} \quad (38)$$

$$A_3 = 1 + n_{\text{imp}} u^2 A_3 \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{\Xi(\omega, k)}{A_1^2 \omega^2 + v^2 k^2 + A_3^2 \Delta^2}. \quad (39)$$

We now employ the re-scaling transformations $\frac{k}{\Lambda} \rightarrow k$, $\frac{\omega}{v\Lambda} \rightarrow \omega$, and $\frac{\Delta}{v\Lambda} \rightarrow \Delta$, and making use of the relation $A_1 = A_3 = A$, obtain

$$A = 1 + \gamma A \int_0^1 dk k \frac{\Xi(\omega, k)}{A^2 \omega^2 + k^2 + A^2 \Delta^2}, \quad (40)$$

where the gap Δ is still given by equation (26).

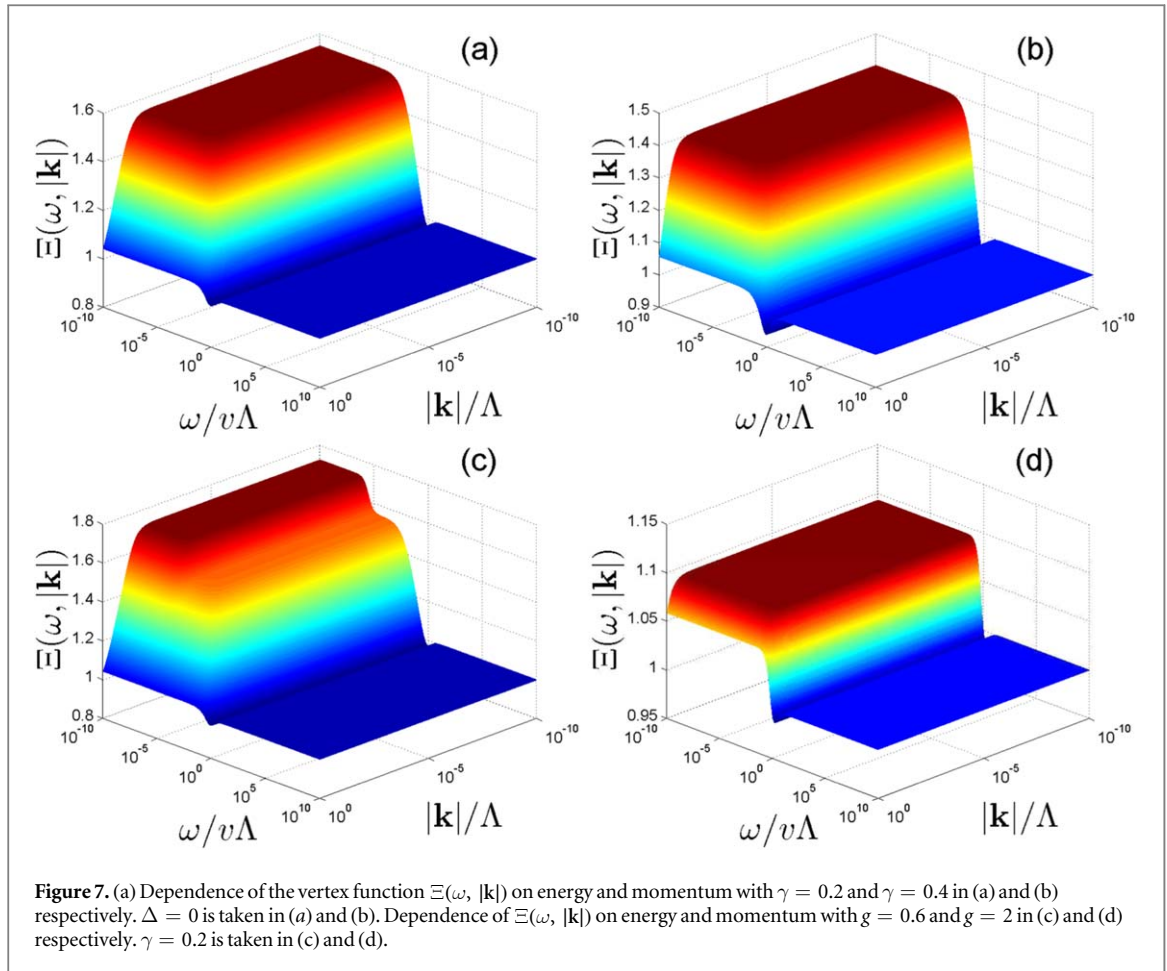
In the limit of $\Delta = 0$, the system stays in the non-SC phase. We present the ω -dependence of A and ωA in figures 2(a) and (b) respectively by the dashed lines. Comparing the solid curves and dashed curves, we can observe that the disorder scattering rate Γ is made larger by the inclusion of the vertex correction. Katanin [118] performed a functional RG analysis of this problem, and also found a larger scattering rate Γ and a larger $\rho(0)$ comparing to those obtained by using the self-consistent Born approximation (SCBA). Sinner and Ziegler [124] reported a $1/N$ -expansion study, which claims that higher order corrections lead to enhancement of disorder scattering rate and DOS.

Now we consider the SC phase where $\Delta \neq 0$. The g -dependence of gap Δ is displayed in figures 3(a) and (b) by the dashed lines. A clear result is that the zero-energy gap Δ is increased when the vertex correction is taken into account, which means that including vertex correction leads to further promotion of superconductivity. Dependence of Δ on γ with different values of g is presented in figures 4(a) and (b) by the dashed lines. We can find that the qualitative property of the disorder effect on superconductivity remains nearly the same after including the vertex correction.

In the case of $\Delta \neq 0$, we show the dependence of $A(\omega)$ on ω in the presence of vertex correction in figure 5(b). It is obvious that $A(\omega)$ approaches to a smaller constant comparing to the one shown in figure 5(a). The reason for this behavior is that the gap Δ becomes larger after including the vertex correction and thus leads to a stronger suppression effect for the disorder scattering rate.

The energy and momentum dependence of the vertex function $\Xi(\omega, k)$ are shown in figure 7. An apparent fact is that the vertex correction is important at low energies and small momenta, and can be nearly ignored only when the energy and momentum are sufficiently large. As the pairing interaction gets stronger, the vertex correction becomes less important [87].

The SM-SC QCP exists in a clean 2D DSM, but is eliminated when the system contains weak random chemical potential. Since there is always certain amount of impurity in the material, it seems extremely difficult to realize and probe the predicted quantum critical phenomena at such a QCP.



3. Superconductivity in 3D Dirac semimetal

In this section, we will investigate the fate of superconductivity formed by Cooper pairing of 3D Dirac fermions, which could emerge at low energies at the QCP between a normal band insulator and a 3D topological insulator. This type of 3D DSM has been observed in $\text{TiBiSe}_{2-x}\text{S}_x$ [125, 126] and $\text{Bi}_{2-x}\text{In}_x\text{Se}_3$ [127, 128] by fine tuning the doping level. Theoretical works [129, 130] predicted that a crystal-symmetry protected 3D DSM might be realized in such materials as A_3Bi ($\text{A} = \text{Na}, \text{K}, \text{Rb}$) and Cd_3As_2 . Shortly after this prediction, ARPES and quantum transport measurements have reported evidence of 3D DSM state in Na_3Bi and Cd_3As_2 [131–135].

Recent RG analysis revealed that weak attraction is irrelevant in 3D DSM, and that only sufficiently strong attraction can induce superconductivity [96] and there is also a QCP separating the SM and SC phases. In the non-SC phase, the physical effect caused by the random chemical potential is a subject of considerable interest [106, 119–123, 136, 137]. Recent studies based on SCBA, RG analysis, and exact numerical simulation all found that there is a QCP between the SM and CDM phases by adjusting the strength of random chemical potential [106, 119–123]. If the rare region effect is considered, it was found that arbitrarily weak disorder drives the system to enter into the CDM phase [136, 137]. In this paper, we do not consider the rare region effect, and focus on the fate of s -wave superconductivity.

Similar to 2D DSM, the 3D DSM hosts only discrete Fermi points, and thus the vertex correction may also be important. In addition, the zero-energy DOS vanishes in both cases, and might become finite if the system is turned by random chemical potential into a CDM. We now parallel the analysis performed in the last section, and include explicitly the vertex correction in the self-consistent equations for A and Δ . The disorder effect on superconductivity can be analogously analyzed.

3.1. Clean case

The mean-field Hamiltonian for 3D DSM is formally similar to that of 2D DSM, and will be not explicitly given here. We directly write down the gap equation obtained in the clean limit:

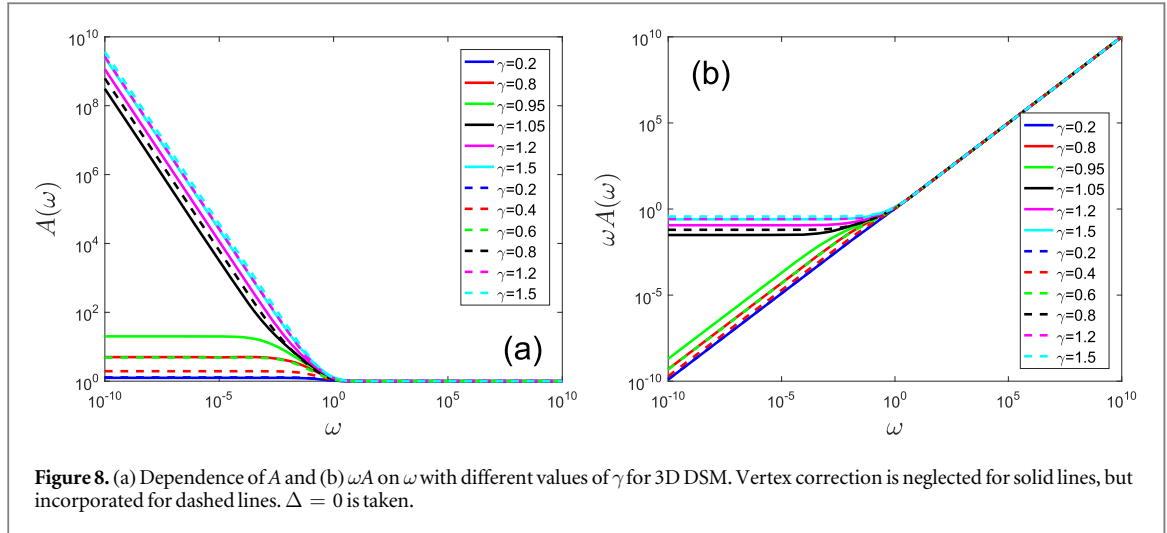


Figure 8. (a) Dependence of A and (b) ωA on ω with different values of γ for 3D DSM. Vertex correction is neglected for solid lines, but incorporated for dashed lines. $\Delta = 0$ is taken.

$$\Delta = 2g \int \frac{d\omega}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\Delta}{\omega^2 + v^2 k^2 + \Delta^2}, \quad (41)$$

where g is the strength parameter for pairing interaction. Integrating over momenta results in

$$1 = \frac{g}{2\pi^3 v^2} \int_{-\infty}^{+\infty} d\omega \left[\Lambda - \frac{1}{v} \sqrt{\omega^2 + \Delta^2} \arctan \left(\frac{v\Lambda}{\sqrt{\omega^2 + \Delta^2}} \right) \right], \quad (42)$$

where Λ is the momentum cutoff. The critical attraction strength can be determined by taking $\Delta = 0$, satisfying the following equation:

$$1 = \frac{g_{c0}}{2\pi^3 v^2} \int_{-\infty}^{+\infty} d\omega \left[\Lambda - \frac{|\omega|}{v} \arctan \left(\frac{v\Lambda}{|\omega|} \right) \right] = \frac{g_{c0}\Lambda^2}{4\pi^2 v}. \quad (43)$$

The critical value is $g_{c0} = 4\pi^2 v / \Lambda^2$.

3.2. Analysis by AG method without vertex correction

Under the original AG approximation, the self-consistent equations for A and Δ are given by

$$A = 1 + \gamma A \left[1 - \sqrt{A^2 \omega^2 + A^2 \Delta^2} \arctan \left(\frac{1}{\sqrt{A^2 \omega^2 + A^2 \Delta^2}} \right) \right], \quad (44)$$

$$1 = \frac{g}{g_{c0}} \frac{2}{\pi} \int_{-\infty}^{+\infty} d\omega A \left[1 - \sqrt{A^2 \omega^2 + A^2 \Delta^2} \arctan \left(\frac{1}{\sqrt{A^2 \omega^2 + A^2 \Delta^2}} \right) \right], \quad (45)$$

where

$$\gamma = \frac{n_{\text{imp}} u^2 \Lambda}{2\pi^2 v^2}. \quad (46)$$

In the derivation, we have employed the transformations: $\frac{k}{\Lambda} \rightarrow k$, $\frac{\omega}{v\Lambda} \rightarrow \omega$, and $\frac{\Delta}{v\Lambda} \rightarrow \Delta$.

Before analyzing the disorder effect on superconductivity, it is necessary to first consider the impact of weak disorder on the low-energy behavior of Dirac fermions in the non-SC phase. This is an interesting problem and has been studied recently by means of several different approaches [106, 119–123, 136, 137].

In the limit of $\Delta = 0$, the equation for A has the form

$$A = 1 + \gamma A \left[1 - A|\omega| \arctan \left(\frac{1}{A|\omega|} \right) \right]. \quad (47)$$

The solutions for this equation are shown in figures 8(a) and (b) by the solid lines. We find that $A(\omega)$ approaches a finite value in the limit $\omega \rightarrow 0$ if γ is smaller than a critical value γ_c . In contrast, if $\gamma > \gamma_c$, $A(\omega)$ is divergent in the limit $\omega \rightarrow 0$, yet satisfying

$$\lim \omega A(\omega) \rightarrow \Gamma, \quad (48)$$

where Γ takes a finite value. The constant Γ should be identified as the disorder scattering rate. A finite DOS is generated at the Fermi level, which tends to favor superconductivity. The dependence of Γ on γ is depicted in

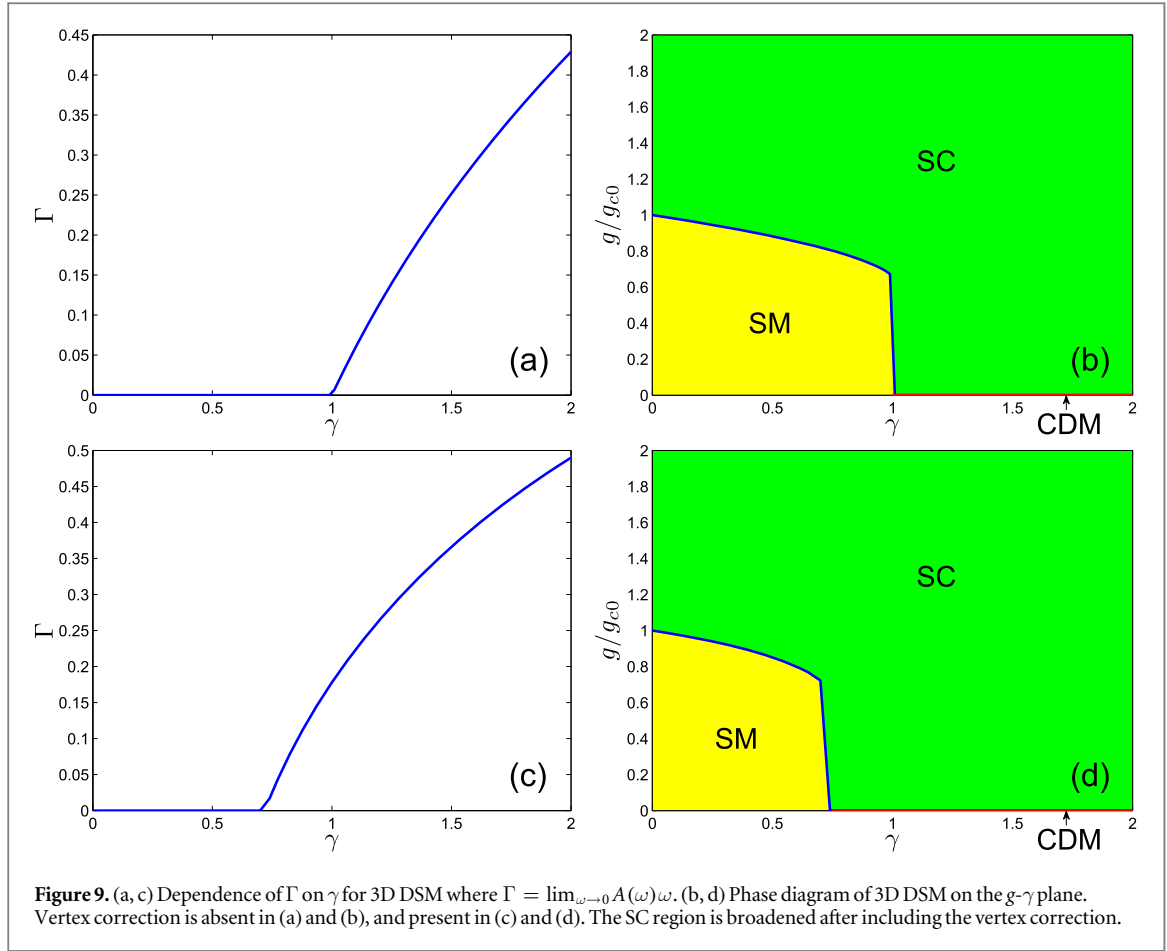


figure 9(a), which clearly shows that $\gamma_c = 1$. Making use of the approximation $\omega A(\omega) \rightarrow \Gamma$, we rewrite equation (47) in the form

$$1 = \gamma \left[1 - \Gamma \arctan\left(\frac{1}{\Gamma}\right) \right]. \quad (49)$$

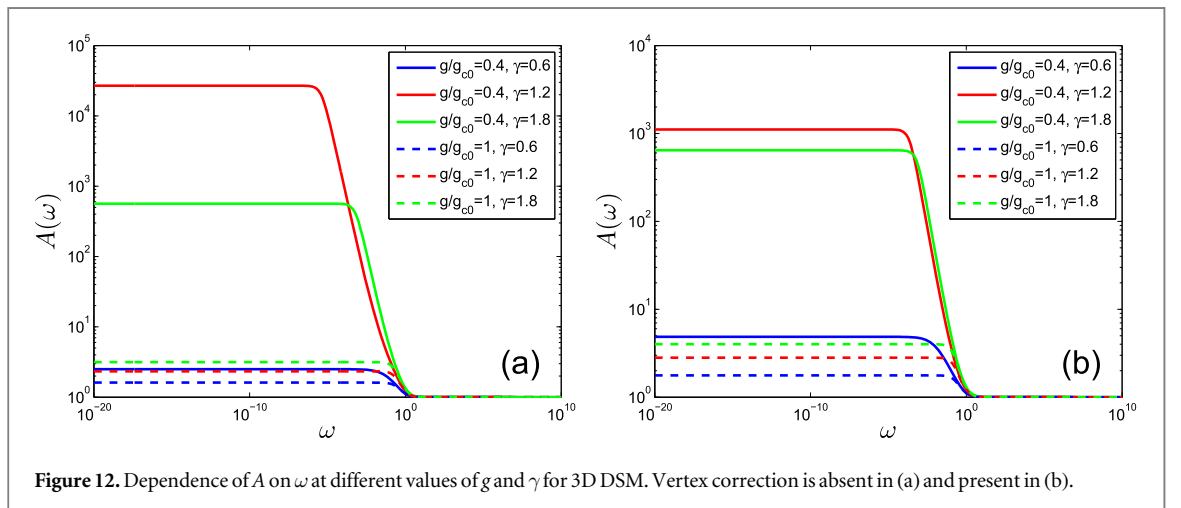
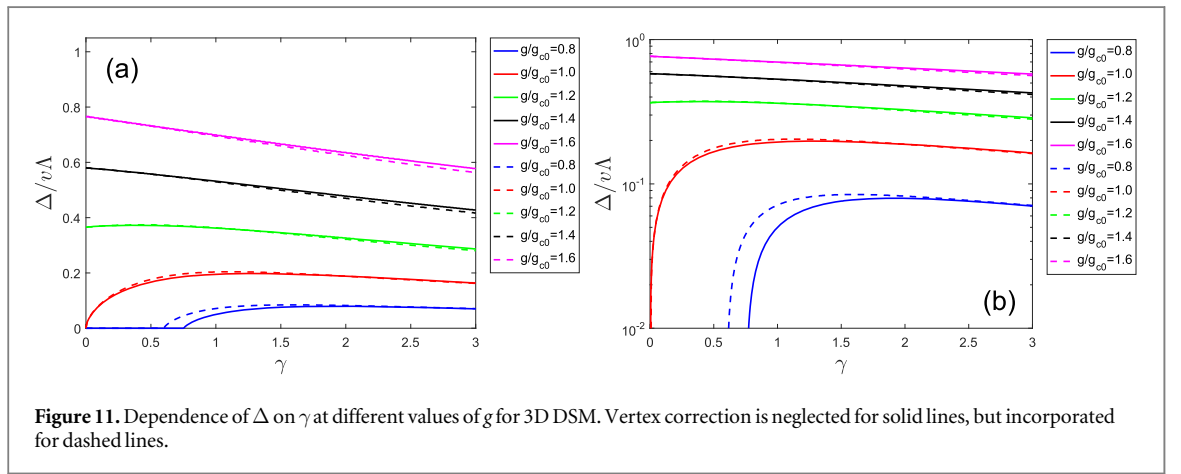
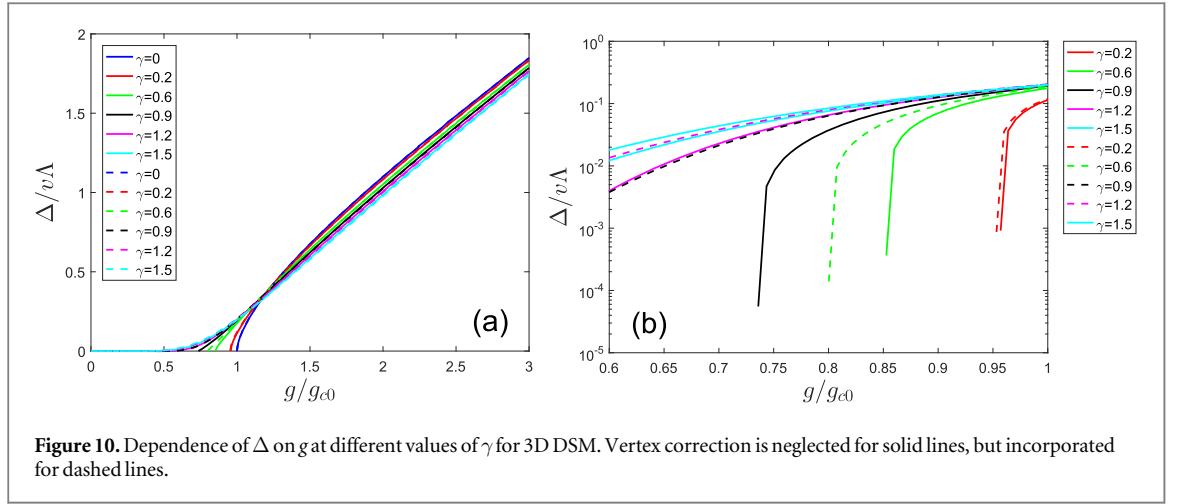
By setting $\Gamma = 0$, we find that $\gamma_c = 1$. According to the results presented in figures 8(a) and (b), and figures 9(a) and (b), the system undergoes a quantum phase transition from the SM phase to a CDM phase at $\gamma = \gamma_c$. This result is consistent with previous studies based on perturbative RG [119, 120], functional RG [122], and direct numerical calculation [121]. We point out here that we do not consider the effects of rare region for simplicity [87, 136, 137].

We then turn to solve the coupled equations (44) and (45), which will be used to analyze the impact of disorder on superconductivity. As can be seen from figure 9(b), the critical value g_c decreases as the disorder parameter γ grows, indicating that superconductivity is promoted. More concretely, in the range $0 < \gamma < 1$, g_c is made smaller than g_{c0} but remains finite. Thus, there is still a SC QCP and it is possible to observe the corresponding quantum critical phenomena. If $\gamma > 1$, we take the limit $\Delta \rightarrow 0$ for equation (45) and then obtain an equation for g_c :

$$1 = \frac{g_c}{g_{c0}} \frac{2}{\pi} \int_{-\infty}^{+\infty} d\omega A \left[1 - A|\omega| \arctan\left(\frac{1}{A|\omega|}\right) \right]. \quad (50)$$

At low energies, $A(\omega)$ behaves as $A(\omega) \sim \frac{\Gamma}{|\omega|}$. Accordingly, the integral in equation (50) is divergent, indicating that the critical value of attraction vanishes, i.e., $g_c \rightarrow 0$. It turns out that even arbitrarily weak attraction suffices to form Cooper pairs. The dependence of Δ on g at different values of γ can be found in figures 10(a) and (b) by the solid lines. The dependence of Δ on γ at different values of g is shown in figures 11(a) and (b) by the solid lines. We observe that the gap Δ displays a non-monotonic dependence on γ if g is relatively small: Δ is enhanced by weak disorder but gets suppressed by sufficiently strong disorder. However, the gap is always suppressed when g becomes relatively large. This behavior is qualitatively similar to 2D DSM.

The asymptotic behavior of $A(\omega)$ for different values of g and γ is shown in figure 12(a). We can find that $A(\omega)$ generally approach to some finite value determined by g and γ . In the SM phase, Δ is vanishing, and $A(\omega)$ is



saturated to finite value and $\omega A(\omega)$ vanishes in the lowest energy limit. In the SC phase, the nonzero gap Δ serves as a cutoff and prevents $A(\omega)$ from being divergent in the lowest energy limit.

3.3. Beyond AG approximation

Paralleling the analysis carried out in the case of 2D DSM, we now examine the role played by the vertex correction. After including the vertex correction into the equations of A and Δ , equation (44) becomes

$$A = 1 + \gamma A \int_0^1 dk \frac{k^2 \Xi(\omega, k)}{A\omega^2 + k^2 + A\Delta^2}, \quad (51)$$

but the gap equation (45) is not changed. Straightforward calculations lead us to the following expression for the vertex function Ξ :

$$\Xi(\omega, \mathbf{p}, \mathbf{q}) \equiv \Xi(\omega, |\mathbf{p} - \mathbf{q}|) = 1 + \gamma \left[2 + 2 \frac{A^2 \Delta^2 + 1}{v|\mathbf{p} - \mathbf{q}| \sqrt{J_1}} \ln \left| \frac{\sqrt{J_1} - v|\mathbf{p} - \mathbf{q}|}{\sqrt{J_1} + v|\mathbf{p} - \mathbf{q}|} \right| - \int_0^1 dx \frac{J_3}{\sqrt{J_2}} \arctan \left(\frac{1}{\sqrt{J_2}} \right) \right], \quad (52)$$

where

$$J_1 = 4(A^2 \omega^2 + 1 + A^2 \Delta^2) + v^2(\mathbf{p} - \mathbf{q})^2, \quad (53)$$

$$J_2 = A^2 \omega^2 + A^2 \Delta^2 + x(1 - x)v^2(\mathbf{p} - \mathbf{q})^2, \quad (54)$$

$$J_3 = 2A^2 \omega^2 + A^2 \Delta^2 + 2x(1 - x)v^2(\mathbf{p} - \mathbf{q})^2. \quad (55)$$

First, we assume $g = 0$ and analyze the influence of disorder in the normal phase of 3D DSM. The dependence of $A(\omega)$ and $\omega A(\omega)$ on ω are displayed in figures 8(a) and (b) by the dashed lines. We can find $A(\omega)$ is saturated to a finite value if γ is small. Whereas, $A(\omega)$ becomes divergent and $\omega A(\omega)$ approaches to a finite value provided γ is large enough. These results are qualitatively the same as those obtained by ignoring the vertex correction presented in figures 8(a) and (b) by the solid lines. However, the magnitude of γ_c becomes smaller as the vertex correction is taken into account, which can be easily observed from figure 9. We notice that, a recent functional RG analysis [122] incorporated the vertex correction and argued that the critical value of disorder strength is smaller than that obtained by using the SCBA. Ominato and Koshino [123] also emphasized the importance of the vertex correction in the estimate of disorder scattering rate.

Through numerical calculations of the equations (51), (52), and (45), we obtain the phase diagram shown in figure 9(d). According to this phase diagram, g_c becomes smaller as γ increases. Comparing figures 9(b) to (d), we observe that the area of the SC phase is broadened when the vertex correction is incorporated.

The dependence of Δ on g is given in figures 10(a) and (b) by the dashed lines, and the dependence of Δ on γ in figures 11(a) and (b) by the dashed lines. The gap Δ is enhanced by weak disorder and suppressed by sufficiently strong disorder when g is small, but is always suppressed by disorder if g takes a large value. These results are qualitatively the same as those obtained by ignoring the vertex correction. From figure 10, we can find that the enhancement effect of the SC gap induced by weak disorder for small g is even more significant when the vertex correction is considered.

According to figure 12, $A(\omega)$ is still saturated to certain finite constant. This is qualitatively the same as the case of neglecting the vertex correction. Quantitatively, $A(\omega)$ does acquire certain amount of modification after including the vertex correction.

The behavior of $\Xi(\omega, |\mathbf{k}|)$ is presented in figure 13. In figures 13(a) and (b), we take $\Delta = 0$ by assuming that $g = 0$. According to figures 13(a) and (b), Ξ is amplified when γ becomes larger. From figures 13(c) and (d), we see that in the SC phase, Ξ decreases with growing g , indicating that the vertex correction becomes less important once a finite gap is opened.

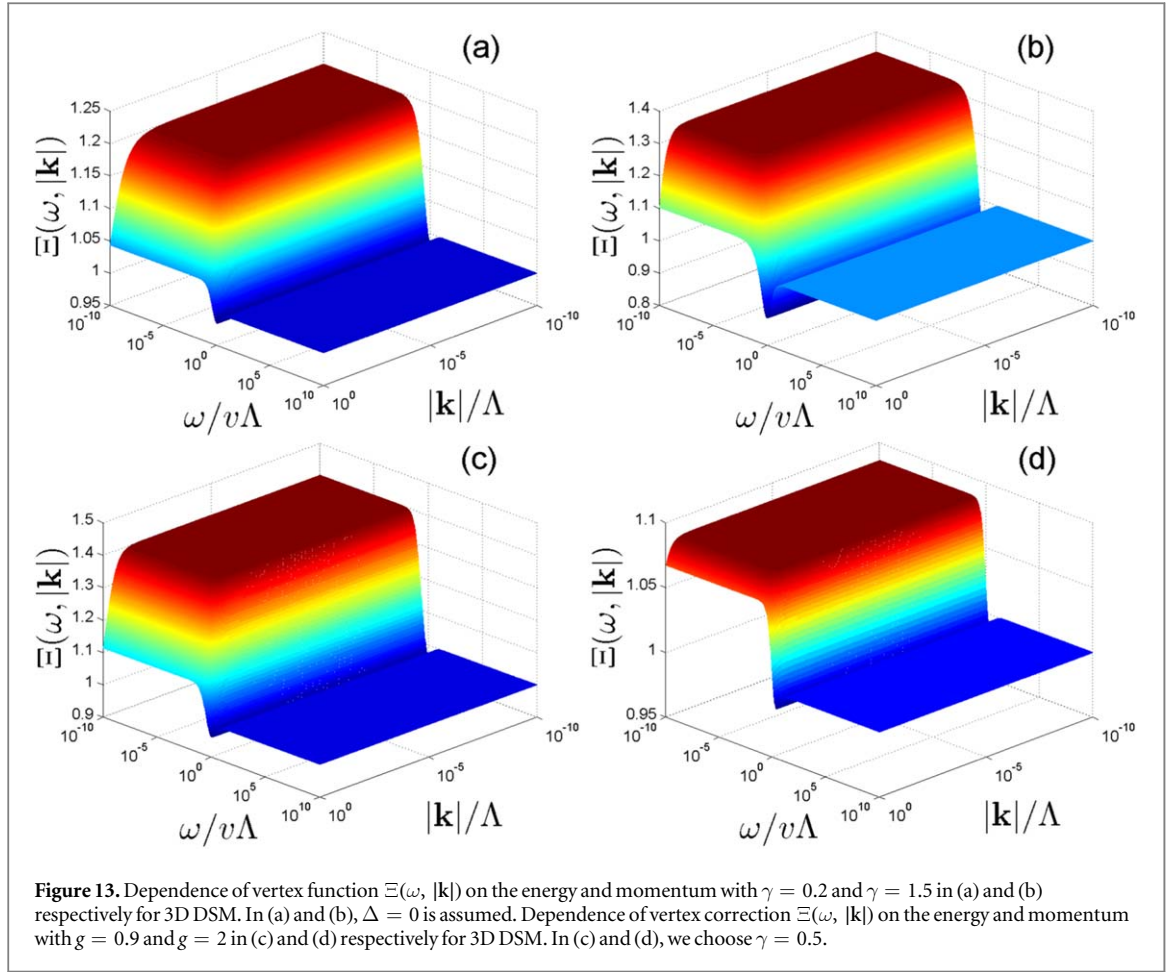
4. Superconductivity in 2D semi-Dirac semimetal

The dispersion for 2D semi-Dirac fermions is given by

$$E = \pm \sqrt{a^2 k_x^4 + v^2 k_y^2}. \quad (56)$$

This dispersion is in between the fermion dispersions of ordinary 2D metal and 2D DSM, thus the corresponding materials is usually called 2D semi-DSM. Such a type of fermions might emerge at the QCP between a 2D DSM and a band insulator upon merging two separate Dirac points to a single one.

Generation of semi-Dirac fermions through merging pairs of Dirac points is theoretically predicted to exist in deformed graphene [19–22], pressured organic compound α -(BEDT-TTF)₂I₃ [21–23], few-layer black phosphorus that is subject to a perpendicular electric field [24, 25] or doping [26], and also some artificial optical lattices [27, 28]. Experimentally, the merging of Dirac points and the appearance of semi-Dirac fermions were observed in ultracold Fermi gas of ⁴⁰K atoms arranged on a honeycomb lattice [29] and microwave cavities with graphene-like structure [30]. Kim *et al* [31] realized semi-DSMs in few-layer black phosphorus at critical surface doping with potassium. Moreover, robust semi-DSM state was predicted to emerge in TiO₂/VO₂ nanostructure under suitable conditions [32, 33]. It was suggested by first-principle calculations that semi-Dirac fermions are



the low-energy excitations of the strained puckered arsenene [35, 36]. In addition, semi-DSM state may also be realized at the QCP between normal insulator and topological insulator, and the QCP between normal insulator and 2D DSM in time-reversal invariant 2D noncentrosymmetric system [42].

Recently, Uchoa and Seo [43] studied the possibility of Cooper pairing in 2D semi-DSM by making a mean field analysis, and argued that *s*-wave superconductivity is favored. Close to the QCP between SM and SC phases, they found that the anisotropy of quasiparticles leads to a novel smectic state of SC stripes. Roy and Foster [56] investigated the influence of various short-range interactions on the low-energy behavior of 2D semi-DSM by making a RG analysis, and also discovered an *s*-wave superconductivity. In the non-interacting limit, recent SCBA [45] and RG [45, 46] studies showed that arbitrarily weak random chemical potential turns the 2D semi-DSM into a CDM state, analogous to what happens in 2D DSM.

Like 2D and 3D DSMs, the 2D semi-DSM also has a vanishing zero-energy DOS if the chemical potential is tuned exactly at the touching points. Consequently, the original AG approximation is no longer valid. In what follows, we will examine how weak random chemical potential affects the *s*-wave superconductivity in a 2D semi-DSM by means of the AG approach and its generalization. Once again, we will not give the mean-field Hamiltonian, and start our discussion directly from the gap equation.

4.1. Clean case

In the clean limit, the equation for the SC gap takes the form

$$\Delta = 2g \int \frac{d\omega}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\Delta}{\omega^2 + a^2 k_x^4 + v^2 k_y^2 + \Delta^2}, \quad (57)$$

which can be further written as

$$1 = g \frac{1}{\pi^2} \int d|k_x| d|k_y| \frac{1}{\sqrt{a^2 k_x^4 + v^2 k_y^2 + \Delta^2}}. \quad (58)$$

We employ the transformations

$$E = \sqrt{a^2 k_x^4 + v^2 k_y^2}, \quad \delta = \frac{a k_x^2}{v |k_y|}, \quad (59)$$

which are equivalent to

$$|k_x| = \frac{\sqrt{\delta} \sqrt{E}}{\sqrt{a} (1 + \delta^2)^{1/4}}, \quad |k_y| = \frac{E}{v \sqrt{1 + \delta^2}}. \quad (60)$$

The measures of integration satisfy the relation

$$d|k_x|d|k_y| = \left| \begin{vmatrix} \frac{\partial |k_x|}{\partial E} & \frac{\partial |k_x|}{\partial \delta} \\ \frac{\partial |k_y|}{\partial E} & \frac{\partial |k_y|}{\partial \delta} \end{vmatrix} \right| dE d\delta = \frac{\sqrt{E}}{2v\sqrt{a}\sqrt{\delta}(1 + \delta^2)^{3/4}} dE d\delta. \quad (61)$$

Adopting the transformations equations (60) and (61), the gap equation becomes

$$\begin{aligned} 1 &= g \frac{1}{2\pi^2 v \sqrt{a}} \int_0^{\Lambda_E} \frac{\sqrt{E}}{\sqrt{E^2 + \Delta^2}} dE \int_0^{+\infty} d\delta \frac{1}{\sqrt{\delta} (1 + \delta^2)^{3/4}} \\ &= g \frac{\Gamma(\frac{5}{4})}{\pi^{3/2} \Gamma(\frac{3}{4}) v \sqrt{a}} \int_0^{\Lambda_E} \frac{\sqrt{E}}{\sqrt{E^2 + \Delta^2}} dE, \end{aligned} \quad (62)$$

where Λ_E is a cutoff for the variable E . Taking the limit $\Delta \rightarrow 0$, we obtain the following critical value

$$g_{c0} = \frac{\pi^{3/2} \Gamma(\frac{3}{4}) v \sqrt{a}}{2 \Gamma(\frac{5}{4}) \sqrt{\Lambda_E}}, \quad (63)$$

beyond which a nonzero SC gap is opened. This g_{c0} is the QCP that separates the SM and SC phases.

4.2. Analysis by AG method without vertex correction

Including the disorder scattering, the self-consistent equations obtained under the original AG approximation are found to have the forms

$$A = 1 + \gamma A F(\omega, \Delta), \quad (64)$$

$$1 = \frac{g}{g_{c0}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega A(\omega) F(\omega, \Delta), \quad (65)$$

where

$$\gamma = \frac{\Gamma(\frac{5}{4}) n_{\text{imp}} u^2}{\pi^{3/2} \Gamma(\frac{3}{4}) v \sqrt{a} \sqrt{\Lambda_E}}, \quad (66)$$

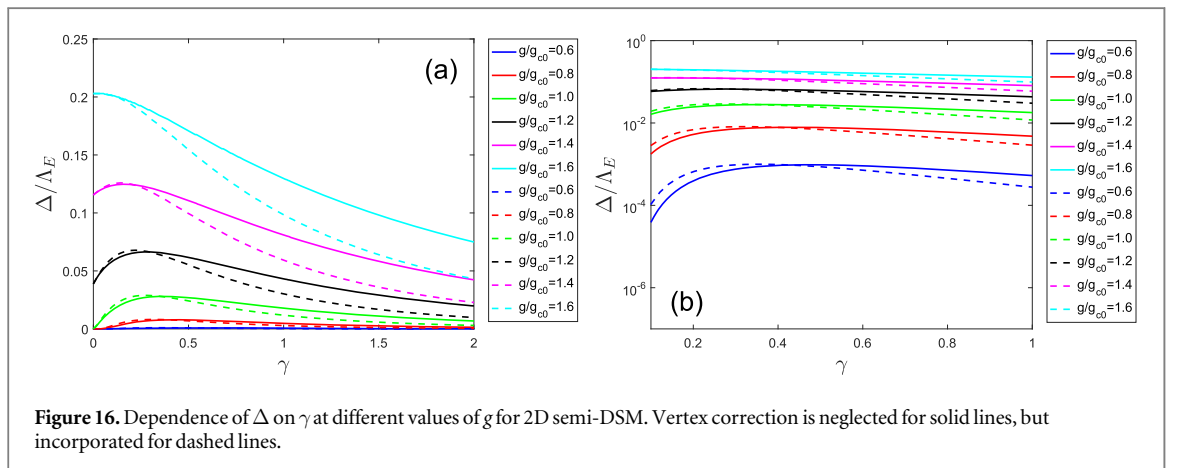
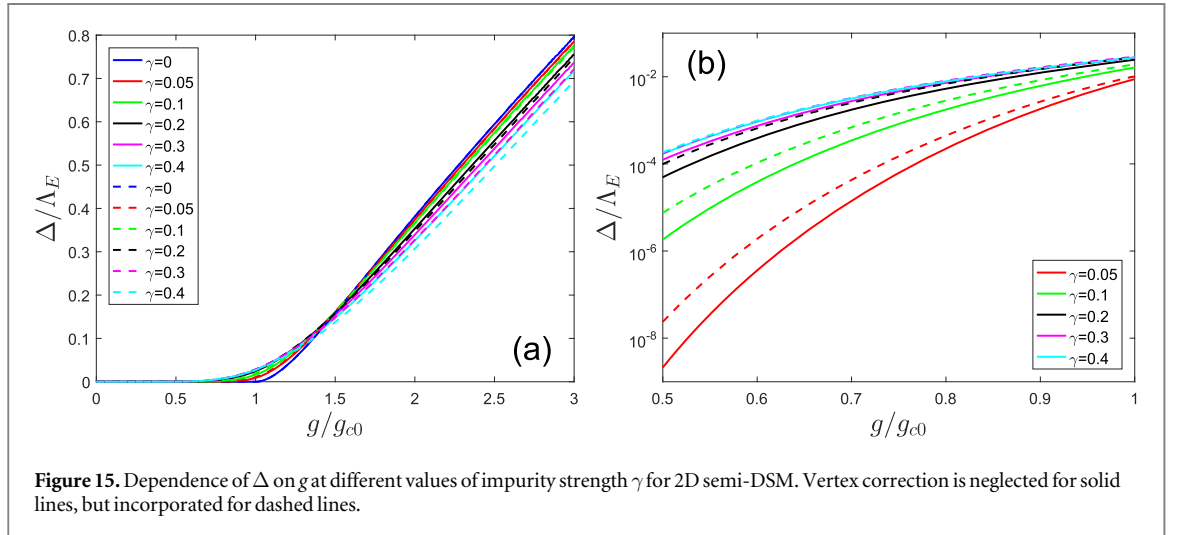
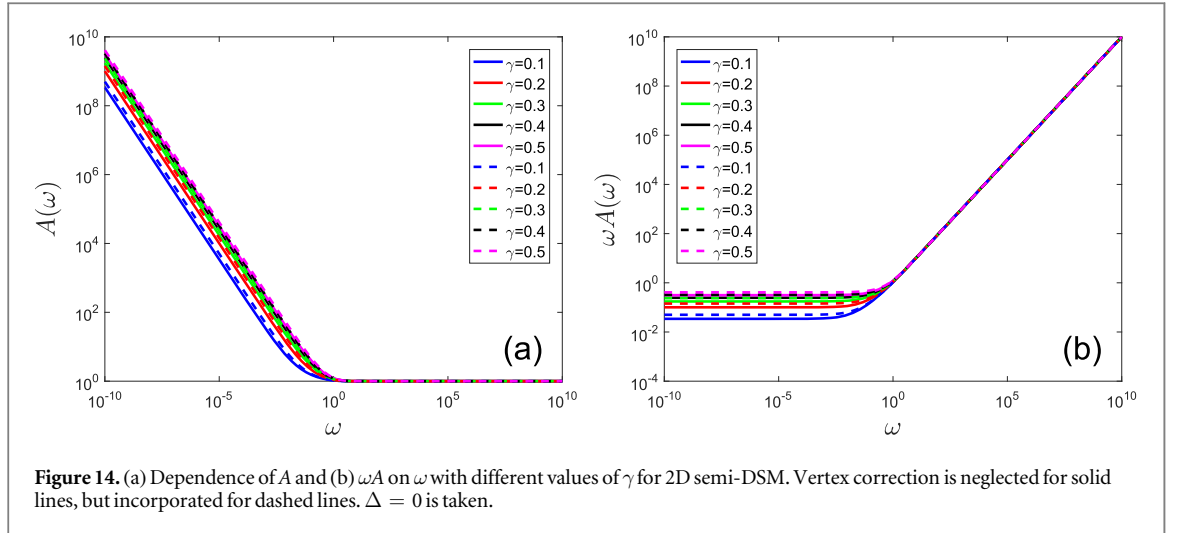
$$\begin{aligned} F(\omega, \Delta) &= \frac{1}{\sqrt{2} J_4^{1/4}} \left[\frac{1}{2} \ln \left(\frac{J_4^{1/2} - \sqrt{2} J_4^{1/4} + 1}{J_4^{1/2} + \sqrt{2} J_4^{1/4} + 1} \right) + \arctan \left(1 + \frac{\sqrt{2}}{J_4^{1/4}} \right) \right. \\ &\quad \left. - \arctan \left(1 - \frac{\sqrt{2}}{J_4^{1/4}} \right) \right], \end{aligned} \quad (67)$$

with $J_4 = A^2 \omega^2 + A^2 \Delta^2$. The scaling transformations $\frac{\omega}{\Lambda_E} \rightarrow \omega$ and $\frac{\Delta}{\Lambda_E} \rightarrow \Delta$ have been used.

By taking $\Delta = 0$, we obtain the solution of $A(\omega)$ in the normal state, and show the results in figures 14(a) and (b) by the solid lines. We find that $A(\omega)$ is divergent in the lowest energy limit, and $\omega A(\omega)$ approaches to a constant in 2D semi-DSM, similar to 2D DSM. In the limit $\Delta = 0$, the integrand in the equation (65) satisfies

$$A(\omega) F(\omega, 0) \propto \frac{\Gamma}{|\omega|} \quad (68)$$

in the low energy regime. It is easy to see that the integration in the equation (65) is divergent, which in turn means $g_c/g_{c0} \rightarrow 0$. Therefore, superconductivity is produced by arbitrarily weak pairing interaction once random chemical potential is considered. Dependence of Δ on g with different values of γ is presented in figures 15(a) and (b) by the solid lines, which clearly exhibits the promotion of superconductivity due to weak random chemical potential if g is small.



In the SC phase with $\Delta \neq 0$, the dependence of Δ on γ is depicted in figures 16(a) and (b) by the solid lines. For small values of g , the gap Δ increases as γ is growing, and then starts to decrease with growing γ when γ becomes sufficiently large. For larger values of g , Δ decreases monotonously as γ increases. We thus can see that the influence of random chemical potential on s -wave superconductivity in 2D semi-DSM is qualitatively the same as 2D DSM.

4.3. Analysis beyond AG approximation

In this subsection, we move to examine the impact of the vertex correction. After including the vertex correction, the equation for A is of the form

$$A = 1 + \gamma \frac{\Gamma(\frac{3}{4})}{\pi^{1/2}\Gamma(\frac{5}{4})} A \int_0^1 dk_x \int_0^1 dk_y \frac{\Xi(\omega, |k_x|, |k_y|)}{A^2\omega^2 + k_x^4 + k_y^2 + A^2\Delta^2}, \quad (69)$$

where the vertex function Ξ is given by

$$\begin{aligned} \Xi(\omega, |p_x - q_x|, |p_y - q_y|) = & 1 + \gamma \frac{\pi^{1/2}\Gamma(\frac{3}{4})}{8\Gamma(\frac{5}{4})} \int_0^1 dx \\ & \times \left[\int_0^1 dk_x \frac{(k_x + |p_x - q_x|)^2 k_x^2 + x(k_x + |p_x - q_x|)^4 + (1-x)k_x^4 + 2A^2\Delta^2}{[A^2\omega^2 + x(k_x + |p_x - q_x|)^4 + (1-x)k_x^4 + A^2\Delta^2 + x(1-x)|p_y - q_y|^2]^{3/2}} \right. \\ & \left. + \int_0^1 dk_x \frac{(k_x - |p_x - q_x|)^2 k_x^2 + x(k_x - |p_x - q_x|)^4 + (1-x)k_x^4 + 2A^2\Delta^2}{[A^2\omega^2 + x(k_x - |p_x - q_x|)^4 + (1-x)k_x^4 + A^2\Delta^2 + x(1-x)|p_y - q_y|^2]^{3/2}} \right]. \quad (70) \end{aligned}$$

We have used the transformations

$$\frac{k_x}{\Lambda_x} \rightarrow k_x, \quad \frac{k_y}{\Lambda_y} \rightarrow k_y, \quad \frac{\omega}{\Lambda_E} \rightarrow \omega, \quad \frac{\Delta}{\Lambda_E} \rightarrow \Delta, \quad \frac{v^2(p_y - q_y)^2}{\Lambda_E^2} \rightarrow (p_y - q_y)^2, \quad (71)$$

and also made the assumption that $a\Lambda_x^2 = v\Lambda_y = \Lambda_E$. The equation for the gap Δ is still given by equation (65).

After including the vertex correction, we solve the self-consistent equations, and show the dependence of Δ on g at different values of γ in figures 15(a) and (b) by the dashed lines. The relation between Δ and γ at different values of g is displayed in figures 16(a) and (b) by the dashed lines. We observe from figures (15) and (16) that including the vertex correction does not change the qualitative property of the influence of random chemical potential on superconductivity in 2D semi-DSM, but it leads to obvious quantitative increase of the SC gap in presence of weak attraction and weak disorder. All these results are similar to that obtained in the case of 2D DSM.

5. Superconductivity in Bilayer graphene

In bilayer graphene with Bernal AB stacking, in which two layers of carbon atoms are rotated by 60° , the Fermi surface is also composed of discrete points [1, 4]. There are also other sorts of configuration of bilayer graphene, such as AA stacking that match the A sublattices of two layers. Here, we focus on bilayer graphene with Bernal configuration, which is the most frequently studied case. In such a system, the fermions display the following dispersion [1, 4]

$$E = \pm a|k|^2, \quad (72)$$

where a is a constant. For Bernal bilayer graphene, the Berry phase around the touching points is trivial, in distinct to monolayer graphene that exhibits a non-trivial Berry phase. This difference can be clearly observed in quantum Hall effect experiments [138]. Due to its special dispersion and dimensionality, the bilayer graphene has a finite zero-energy DOS [4]: $\rho(0) \propto 1/a$. As a result, an infinitesimal short-range interaction is able to drive some type of phase-transition instability [4, 139, 140].

5.1. Clean case

In the clean limit, the gap equation for s -wave superconductivity in bilayer graphene is given by

$$\Delta = 2g \int \frac{d\omega}{2\pi} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\Delta}{\omega^2 + a^2k^4 + \Delta^2}, \quad (73)$$

where g is the strength parameter of pairing interaction. Performing the integrations of energy and momenta, we obtain

$$1 = \frac{g}{4\pi a} \ln \left(\frac{\sqrt{a^2\Delta^4 + \Delta^2} + a\Delta^2}{\Delta} \right), \quad (74)$$

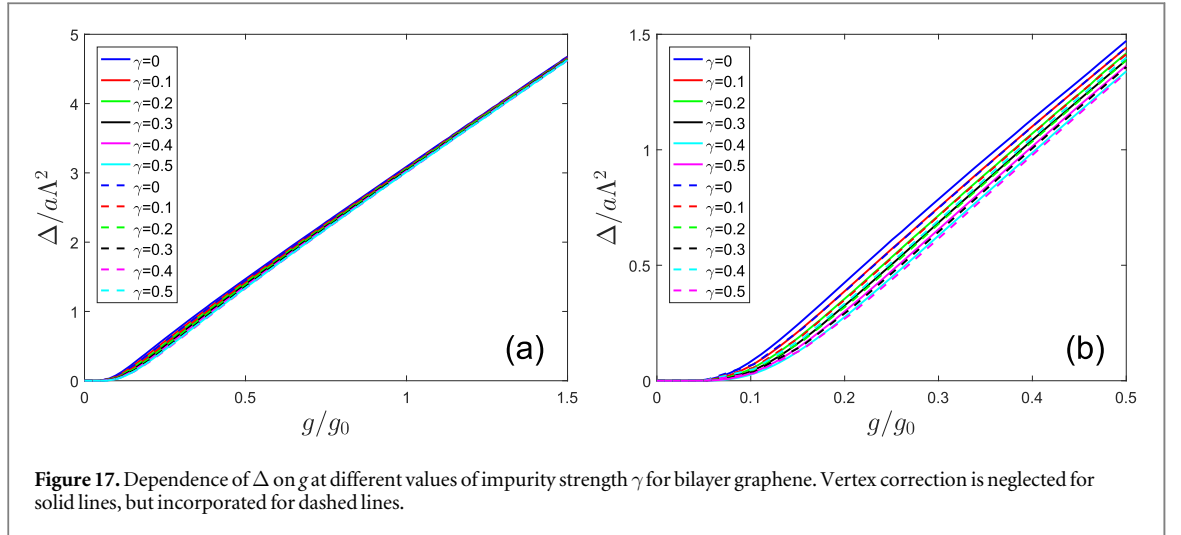


Figure 17. Dependence of Δ on g at different values of impurity strength γ for bilayer graphene. Vertex correction is neglected for solid lines, but incorporated for dashed lines.

where Λ is the cutoff of the momentum. In the limit $\Delta \rightarrow 0$, g approaches a critical value g_{c0} that satisfies

$$1 = \lim_{\Delta \rightarrow 0} \frac{g_{c0}}{4\pi a} \ln \left(\frac{2a\Lambda^2}{\Delta} \right). \quad (75)$$

It is obvious that $g_{c0} \rightarrow 0$, so arbitrarily weak attraction leads to superconductivity in bilayer graphene. This behavior is markedly different from the aforementioned SMs with vanishing DOS at the Fermi level.

5.2. Analysis by AG method without vertex correction

In the absence of vertex correction, the self-consistent equations for the renormalization factor $A(\omega)$ and the SC gap Δ can be written as

$$A = 1 + \gamma \frac{a\Lambda^2}{\sqrt{\omega^2 + \Delta^2}} \arctan \left(\frac{a\Lambda^2}{\sqrt{J_4}} \right), \quad (76)$$

$$1 = \frac{g}{g_0} \int d\omega \frac{1}{\sqrt{\omega^2 + \Delta^2}} \arctan \left(\frac{a\Lambda^2}{\sqrt{J_4}} \right), \quad (77)$$

where

$$\gamma = \frac{n_{\text{imp}} u^2}{4\pi a^2 \Lambda^2}, \quad g_0 = 4\pi^2 a, \quad (78)$$

and $J_4 = A^2 \omega^2 + A^2 \Delta^2$. Here, g_0 is chosen as the unit of attraction strength g .

Solving equations (76) and (77), we obtain the dependence of Δ on g at different values of γ , which is shown in figures 17(a) and (b) by the solid lines. The SC gap Δ is quantitatively suppressed by random chemical potential. The relation between Δ and γ at different values of g is displayed in figure 18 by the solid lines. The gap Δ is suppressed monotonously by increasing γ , but the suppression effect is not significant. Thus, the s -wave superconductivity seems to be robust against weak disorder in bilayer graphene. For bilayer graphene, there is not any evidence of disorder-induced promotion of superconductivity, which occurs in 2D DSM, 3D DSM, and 2D semi-DSM. Such difference originates from the fact that the zero-energy DOS $\rho(0)$ is nonzero in bilayer graphene, but vanishes in the other three types of SM.

5.3. Analysis beyond AG approximation

After including the vertex correction, we re-write the self-consistent equations for the function A and the vertex Ξ as follows

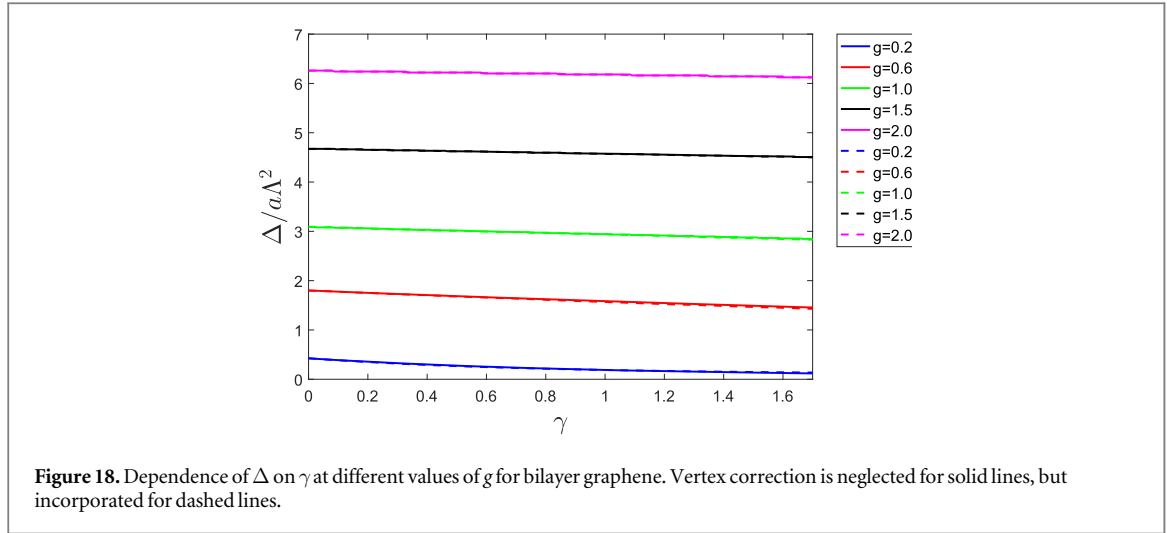


Figure 18. Dependence of Δ on γ at different values of g for bilayer graphene. Vertex correction is neglected for solid lines, but incorporated for dashed lines.

$$\begin{aligned}
 A &= 1 + 2\gamma A \int_0^1 dk k \frac{\Xi(\omega, |\mathbf{k}|)}{A^2\omega^2 + k^4 + A^2\Delta^2}, \\
 \Xi(\omega, |\mathbf{p} - \mathbf{q}|) &= 1 + \gamma \frac{1}{\pi} \int_0^1 dk k \int_0^{2\pi} d\varphi \\
 &\times [-A^2\omega^2 + (k^2 + 2k|\mathbf{p} - \mathbf{q}| \cos \varphi + |\mathbf{p} - \mathbf{q}|^2)k^2 + A^2\Delta^2] \\
 &\times [A^2\omega^2 + (k^4 + 4k^3|\mathbf{p} - \mathbf{q}| \cos \varphi + 4k^2|\mathbf{p} - \mathbf{q}|^2 \cos^2 \varphi + 2k^2|\mathbf{p} - \mathbf{q}|^2 \\
 &+ 4k|\mathbf{p} - \mathbf{q}|^3 \cos \varphi + |\mathbf{p} - \mathbf{q}|^4) + A^2\Delta^2]^{-1} [A^2\omega^2 + k^4 + A^2\Delta^2]^{-1}.
 \end{aligned} \tag{79}$$

The gap still satisfies the equation (77). Including the vertex correction, the relation between Δ and g for different values of γ are shown in figures 17(a) and (b) by the dashed lines. The dependence of Δ on γ for different values of g is depicted in figure 18 by the dashed lines. We can find that vertex correction does not lead to qualitative change of the results. Quantitatively, the suppression of gap by disorder becomes only slightly stronger once the vertex correction is considered, which indicates that the vertex correction can be nearly neglected.

6. Remarks on related issues

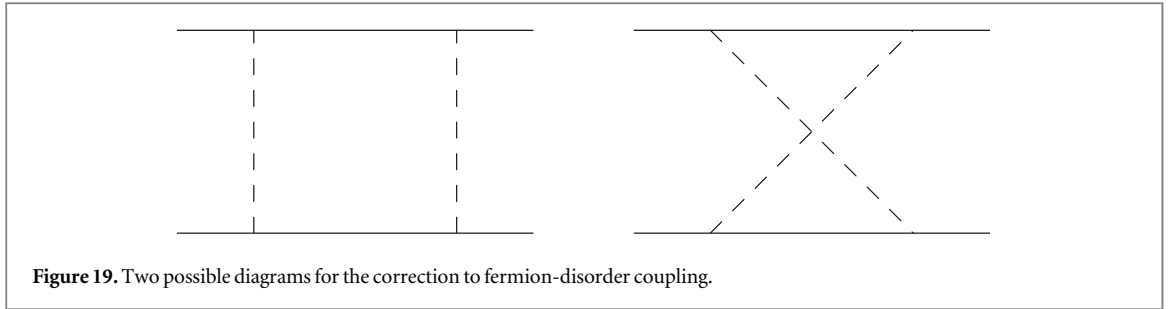
In this section, we discuss several related issues.

6.1. Truncation of DSEs

As analyzed in the Introduction, disorder scattering and Cooper pairing can affect each other substantially, and thus should be treated self-consistently. The non-perturbative DSEs approach provides an ideal formalism for such a self-consistent treatment. In the past several decades, this approach has been widely applied to investigate the non-perturbative phenomenon of dynamical symmetry breaking in high-energy physics [141, 142], the dynamical chiral symmetry breaking in three-dimensional quantum electrodynamics (QED₃) [143–146] which is effective model in several important condensed matter systems, the excitonic insulating transition in various SM materials [41, 67, 147–156]. Moreover, the DSEs approach has been applied to study the interplay of non-Fermi liquid behavior and SC pairing in various strongly correlated systems [157–170]. The most noticeable advantage of DSEs approach is that it provides a non-perturbative framework to quantitatively calculate various physical quantities, such as the Landau damping rate, disorder scattering rate, and SC gap *etc.*, by incorporating several types of interactions in a self-consistent manner.

In the [171], Zhang *et al* studied the impact of nonmagnetic short-range disorder on some ordered states by solving a special set of self-consistent equations, which is similar to the AG method. Their main finding is that, the fully gapped state is suppressed by disorder more significantly than the nematic state, and disorder may induce a quantum phase transition between a fully gapped ordered state and a nematic state. Such results might account for the discrepancy between experiments of bilayer graphene. The DSEs method was also applied to investigate the interplay of disorder scattering and Coulomb interaction in 3D DSM [156].

In the application of DSEs method to the superconductivity in disordered systems, one needs to construct and solve a set of non-linear integral equations for the gap Δ and disorder scattering rate Γ . In its most generic form, the DSEs are exact and contain all the physical processes. However, solving the complete set of DSEs is impossible. In practice, it is always necessary to employ certain truncation, which is implemented by retaining the most important contribution and discarding some higher order contributions. All the existing DSEs studies



[41, 67, 141–170] adopt the following strategy: consider the lowest order truncation to capture the key physical picture; include the higher order corrections step by step to examine the validity of the conclusion obtained by the lowest order calculation. The AG method [112] can be regarded as the lowest order truncation of DSEs of disorder scattering and Cooper pairing. Under the original AG approximation, the fermion self-energy is calculated at the leading order, with all vertex corrections entirely ignored. We emphasize here that, although the AG method neglects most higher order corrections, it is non-perturbative in nature and can treat the mutual influence between disorder scattering and Cooper pairing equally. When the AG method is applied to disordered SM materials, the vertex correction may no longer be simply ignored. In this paper, we find that, including vertex correction does not alter the qualitative conclusion obtained by the original AG method, but leads to considerable enhancement of SC gap size in 2D DSM, 3D DSM, and 2D semi-DSM in weak-attraction regime.

To examine the validity of our conclusion, one might endeavor to include even higher order corrections into the self-consistent equations studied in this paper. This work is interesting, but out of the scope of the present paper. The main difficulty is that, the self-consistent DSEs become very complicated and are hard to solve numerically. Here we only make a brief remark on the possible influence of such corrections. From the results presented in the last several sections, we can infer that the vertex correction becomes progressively less important as the SC gap grows. For large SC gap, the vertex correction can be safely ignored. Actually, the SC gap provides an infrared cutoff, which regularizes the infrared behavior and as such suppresses higher order corrections. When the SC gap is not large, the higher order corrections omitted in our analysis might more or less modify our results quantitatively. We leave this project for future research.

Apart from including higher order corrections by brute force, our conclusion, which states that random chemical potential promotes superconductivity in 2D DSM, 3D DSM, and 2D semi-DSM, could also be verified by experiments. The SC gap can be detected in scanning tunneling microscopy (STM) measurements. One might prepare a series of different SM materials to test whether it is easier to achieve superconductivity in more disordered materials. We expect that future experiments would be performed to clarify the reliability of our conclusion.

6.2. Contribution of two special diagrams

One might think that the two Feynman diagrams, shown in figure 19, should be taken into account in the calculation of the corrections to fermion-disorder coupling. We now explain why these two diagrams are neglected.

In our work, we consider only one type of disorder, namely random chemical potential. For 2D DSM and 3D DSM that contain only one type of disorder, the contributions from these two Feynman diagrams cancel, which is discussed in [120]. For 2D semi-DSM and bilayer graphene, these two diagrams can dynamically generate other types of disorder, such as random gauge potential and random mass [45]. To simplify the problem, we truncate the DSEs by neglecting the dynamically generated disorder. An important point is that random chemical potential always dominates the dynamically generated random gauge potential and random mass. Including the dynamically generated disorders would further enhance random chemical potential and then induce a larger DOS $\rho(0)$ in the normal state. For 2D semi-DSM, according to our results, a larger disorder-induced DOS $\rho(0)$ would lead to stronger enhancement superconductivity in presence of weak pairing interaction. In the case of bilayer graphene that has a finite $\rho(0)$ even in the clean limit, increasing the strength of random chemical potential also does not change our basic conclusion that superconductivity is slightly suppressed.

6.3. Rare region effect

In this article, we calculate the magnitude of SC gap, which can be measured by STM experiments [172, 173]. We do not consider the rare region effect here. In [87], Nandkishore *et al* studied the enhancement of superconductivity by the rare region effect of disorder in 2D DSM [87]. They showed that local

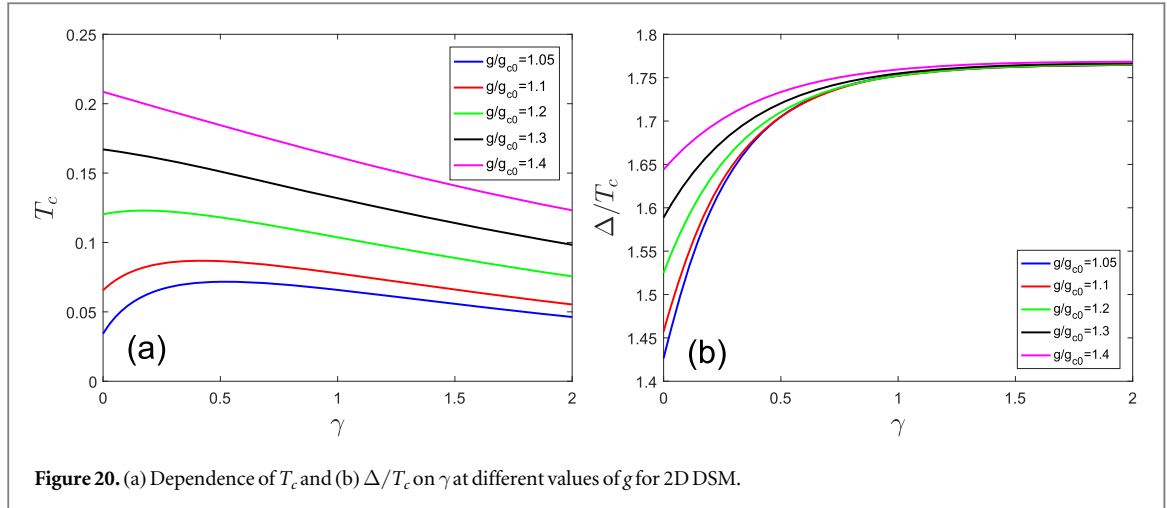


Figure 20. (a) Dependence of T_c and (b) Δ/T_c on γ at different values of g for 2D DSM.

superconductivity is substantially enhanced in the regions with stronger disorder and larger local DOS. In case the Josephson coupling between locally SC regions is strong enough to establish global phase coherence, the system becomes SC globally. After examining the rare region effect, they obtained an obviously larger value of T_c comparing to the one calculated by using the mean-field analysis (AG method).

Nandkishore *et al* later studied the rare region effect in 3D DSM [136], and concluded that arbitrarily weak random chemical potential can induce a finite $\rho(0)$ due to rare region effect. They also considered the interplay of random chemical potential and superconductivity in 3D DSM. Superconductivity is triggered only when the strength of attraction exceeds a threshold in the clean system. However, local pairing can be triggered in rare regions where the local DOS is nonzero, which drives the system into a SC state once phase coherence between the islands is established by Josephson coupling. These results suggest that superconductivity is promoted by random chemical potential in 3D DSM, which is qualitatively consistent with our conclusion presented in section 3.

6.4. The ratio Δ/T_c

How RSP affects the ratio Δ/T_c is an interesting question. In this subsection, we show the results obtained by AG method for 2D DSM. The results for other SMs could be calculated similarly. For 2D DSM, the self-consistent equations for T_c are given by

$$A = 1 + \frac{\gamma}{2} A \ln \left(1 + \frac{1}{A^2 (2n+1)^2 \pi^2 T_c^2} \right), \quad (80)$$

$$1 = \frac{g}{g_{c0}} T_c \sum_n A \ln \left(1 + \frac{1}{A^2 (2n+1)^2 \pi^2 T_c^2} \right). \quad (81)$$

The relation between T_c and γ is displayed in figure 20(a). Comparing figure 20(a) with figure 11, we can find that relation between T_c and γ has similar characteristic with relation between Δ on γ . Dependence of Δ/T_c on γ is shown in figure 20(b). According to figure 20(b), Δ/T_c increases with growing of γ .

6.5. Anderson localization

DSMs have obvious different behaviors under the influence of RSP comparing with traditional metals. For 2D conventional metals, arbitrarily weak RSP drives the system to Anderson insulator (AI) [174]. However, for 2D DSM with a single Dirac cone, the system is robust against Anderson localization under RSP [105, 175–177].

For 3D conventional metals, there are only two phases under the influence of RSP, namely CDM and AI [174]. However, for 3D DSM, there are three phases under the influence of RSP, namely SM, CDM, and AI [178, 179]. We should notice that the critical value $\gamma_c = 1$ is corresponding to the QCP from SM to CDM [119, 120, 178, 179]. The transition from CDM to AI occurs at another critical value γ_c^{AI} which is much larger than γ_c [178, 179]. In a wide range value of γ around γ_c , the system is still in SM or CDM phase, but not AI phase. Thus, in this case, the wave functions are not localized, but still extended. Then, our calculation is valid for a wide range of disorder strength around $\gamma_c = 1$. If the disorder strength is very large, i.e. $\gamma > \gamma_c^{\text{AI}}$, Anderson localization appears, and our calculation becomes invalid.

7. Summary and conclusion

In summary, we have studied the influence of random chemical potential on the fate of *s*-wave superconductivity in 2D DSM by using the AG diagrammatic approach along with its proper generalization. It is found that an arbitrarily weak attraction suffices to trigger Cooper pairing instability. In the case of weak attraction, the magnitude of SC gap first increases with growing disorder strength and then decreases once the disorder strength exceeds a critical value. For relatively strong attraction, the gap decreases monotonously as disorder gets stronger. To obtain a quantitatively more reliable result, we have gone beyond the original AG approximation, and taken into account the vertex correction to the fermion-disorder coupling. Our finding is that, including vertex correction does not change the qualitative behavior of superconductivity in 2D DSM with random chemical potential. However, for weak pairing interaction and weak disorder, the disorder-induced enhancement of SC gap size becomes more significant in the presence of vertex correction. Therefore, the conclusion that superconductivity in 2D DSM is promoted by random chemical potential [87] is robust.

We then have applied the AG method and its generalization to investigate the fate of *s*-wave superconductivity in other analogous materials, including 3D DSM, 2D semi-DSM, and bilayer graphene. For 3D DSM, we have found that the critical pairing interaction strength g_c is reduced to a smaller value by weak random chemical potential, and thus there is still a QCP separating the SM and SC phases. This QCP provides an ideal platform to study the rich quantum critical phenomena. Nevertheless, when random chemical potential becomes sufficiently strong, the critical value g_c vanishes, and superconductivity is achieved no matter how weak the pairing interaction. In both cases, we see that superconductivity is promoted by random chemical potential.

For 2D semi-DSM, the disorder effect on the *s*-wave superconductivity is nearly the same as that of 2D DSM. In particular, superconductivity can be induced by arbitrarily weak attraction if the system contains random chemical potential. When the vertex correction is considered, the promotion of superconductivity by disorder in the weak attraction regime also becomes more obvious. However, the vertex correction does not modify the qualitative results obtained by using the original AG approximation.

Comparing to the above three types of SMs, the bilayer graphene is spectacular since its zero-energy DOS takes a finite value at the Fermi level. As the disorder strength increases, the magnitude of the SC gap decreases, yet at a very low speed. Apparently, such behavior is in sharp contrast to that of 2D DSM, 3D DSM, and 2D semi-DSM, where the zero-energy DOS vanishes in the clean limit and acquires a finite value only when the system contains random chemical potential.

Recently, Ozfidan *et al* studied the influence of magnetic impurity on superconductivity in 2D DSM by using the AG method [180], and found a gapless helical SC state. It would be interesting in the future to investigate whether the vertex correction makes an important contribution to the physical effects of magnetic disorder on superconductivity.

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