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The conjugate profile of the circular teeth of a spur gear. Part II: Problem solution

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Abstract. The paper aims finding the conjugate profile of a circular profile tooth from a gear mechanism with fixed parallel axes and constant transmission ratio. There are presented three manners of attaining the goal. The first two methods are based on the general methodology of gear wheel toothing by envelope machining presented in the first part of the work, considering elementary mechanisms with fixed axes and then a planetary one. The last method is based on applying the Reuleaux’s fundamental law of gearing and has the advantage of obtaining both the conjugate profile of the gear with circular profile and the line of action which is useful in the study of gearing phenomena.

1. Introduction

The main problems concerning the methodology of the synthesis of a mechanism with imposed law of motion of the driven element were presented in the first part of the paper. From the two major types of mechanisms – linkages and mechanisms with higher pair, the last are the ones offering an exact solution for any necessary law of motion. The gear mechanisms are a particular case of cam mechanisms for which the requirement is that the signal obtained at the driven element should be proportional to the input signal [1-6]. At their turn, the gear mechanisms are part of complex engineering machines, as an intermediate loop between actuating elements, from which the most common are electric [7] hydraulic [8] or pneumatic [9], and the effectors. It is obvious that each of these three elements- motor, kinematical transmission and effector, will affect the manner the mechanical energy supplied by the actuating element [9], is exploited. The possibility of using gears with circular arc tooth profile makes feasible [10] the replacement of sliding friction with the rolling friction between the flanks, which is more beneficial from the point of view of energetic losses and wear [11]. A foremost issue is the flank shape of a toothed wheel when there is known the position of the axes, the transmission ratio and the shape of the flank of the conjugate wheel. For planar gears, in the case of gears with parallel axes the problem consists in determining the profile of a wheel when the profile of the conjugate wheel is identified. For planar gears there are three manners of solving the problem:

- a) both wheels are regarded as mobile and the pure rolling condition is fulfilled; in the plane of one wheel, successive positions of the other wheel are represented;
- b) the wheel of unknown profile is regarded as a fixed sun gear and conferring planetary motion to the other gear by obeying the fundamental gear law.

In both cases the profile of the conjugate gear results as an envelope of successive positions of the profile of the given gear.



c) based on the fundamental law of gearing, the Reuleaux method [12] which allows for finding both the profile of the conjugate gear and the line of action, is applied.

Next, the three methods are illustrated and necessary observations are completed.

2. Determination of the conjugate profile using the motion of an ordinary gear mechanism

The scheme of a spur gear train with the radii of pitch circles $r_{w1,2}$ is presented in figure 1. The profile of gear 1 is made by z_1 angularly equidistant circles of r radius, with the centre on the circle of R radius, concentric to the pitch circle. The coordinates of a point M from the contour of the bolt in the coordinate system (1) are:

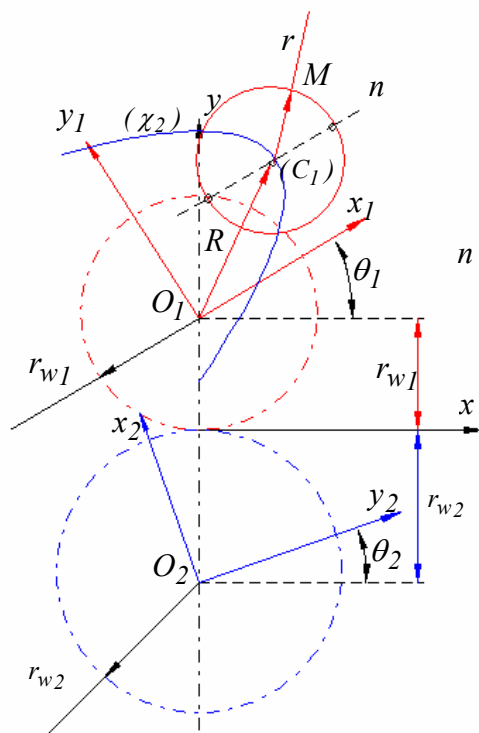


Figure 1. Spur gear scheme.

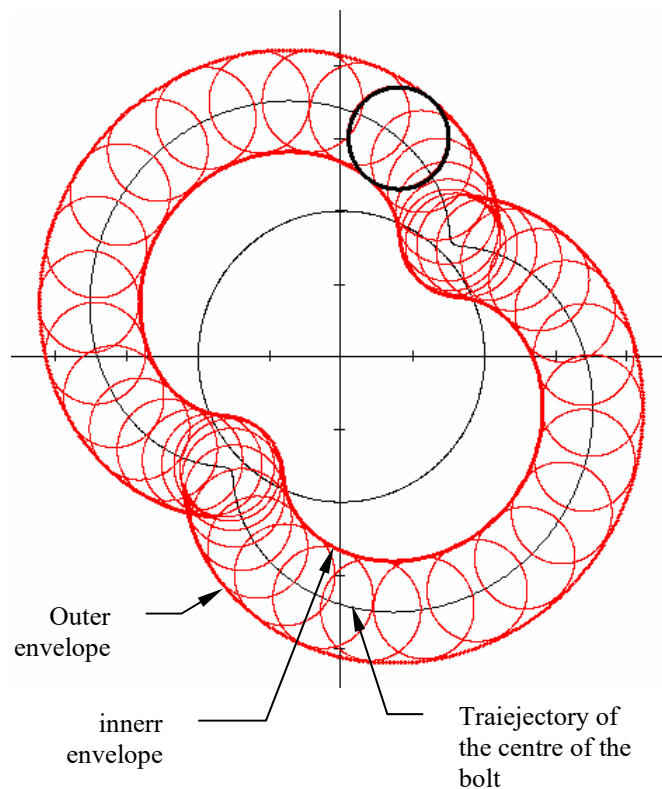


Figure 2. The trajectory of the centre of the bolt and its envelope in the plane attached to the conjugate gear.

$$\begin{aligned} x_1 &= R + r \cos \varphi \\ y_1 &= r \sin \varphi \end{aligned} \quad (1)$$

In order to obtain the expression of the coordinates of the M point in the frame attached to the gear 2, the method of homogenous plane operators proposed by McCarthy is used. Therefore, the coordinates of a point from the system 1 are transformed with the matrix relation:

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta_{12} & -\sin \theta_{12} & x_{12} \\ \sin \theta_{12} & \cos \theta_{12} & y_{12} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \quad (2)$$

where x_{12} , y_{12} are the coordinates of the origin of the frame (2) in the frame (1) and θ_2 is the angle of rotation of the axes of the system (2) with respect to the axes of the system (1). The tangible form of the matrix equation of coordinate transformation from system (1) to system (2) is:

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(-\theta_2) & -\sin(-\theta_2) & 0 \\ \sin(-\theta_2) & \cos(-\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r_{w1} + r_{w2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} \quad (3)$$

In figure 1, both gears are considered to rotate in positive sense. The pure rolling condition is:

$$v_{C1} - v_{C2} = \dot{\theta}_1 r_{w1} - (-\dot{\theta}_2) r_{w2} = 0 \quad (4)$$

From equation (4), with zero integration constants, it results:

$$\theta_2 = -\theta_1 \quad (5)$$

After the calculus is made and by convenient grouping the next equations are obtained:

$$\begin{cases} x_2(\theta_1) = x_1 \cos(1 + r_{w1}/r_{w2})\theta_1 - y_1 \sin(1 + r_{w1}/r_{w2})\theta_1 - (r_{w1} + r_{w2}) \sin(\theta_1 r_{w1}/r_{w2}) \\ y_2(\theta_1) = x_1 \sin(1 + r_{w1}/r_{w2})\theta_1 + y_1 \cos(1 + r_{w1}/r_{w2})\theta_1 + (r_{w1} + r_{w2}) \cos(\theta_1 r_{w1}/r_{w2}) \end{cases} \quad (6)$$

The equations (6) represent a family of epicycloids, generated using the base circle c_{w2} and the roulette c_{w1} which can be normal, elongated (prolate) and contracted (shortened, curtate) depending on the lying position of the point $M(x_1, y_1)$: on the circle, outside or inside the circle c_{w1} . The envelopes of the curves (6) represent the conjugate profile of the circular profile of the gear. In figure 2 there are presented a series of successive positions of one of the teeth of gear 1 during the rolling motion with respect the gear 2. It is noticed that the envelope has two branches, interior and exterior. The equations of these branches can be obtained applying the general methodology form differential geometry [13]. For the present case, a more convenient manner is to use the remark of Handra Luca: when the flank is a circle, the envelope is made of two curves equidistant placed at distance R on both sides of the curve χ_2 described by the circle γ with respect to the coordinate system attached to the gear 2. In order to obtain the equations of the curve χ_2 , the values of x_1 and y_1 are particularized for:

$$r = 0 \quad (7)$$

The resultant parametric equation of the curve χ_2 is:

$$\begin{cases} \xi_2(\theta_1) = R \cos(1 + r_{w1}/r_{w2})\theta_1 - (r_{w1} + r_{w2}) \sin(\theta_1 r_{w1}/r_{w2}) \\ \eta_2(\theta_1) = R \sin(1 + r_{w1}/r_{w2})\theta_1 + (r_{w1} + r_{w2}) \cos(\theta_1 r_{w1}/r_{w2}) \end{cases} \quad (8)$$

The two envelopes have the next equations:

$$\begin{aligned} X_{2,2}(\theta_1) &= \xi_2(\theta_1) \pm r \frac{\xi'_2(\theta_1)}{\sqrt{\xi'^2_2(\theta_1) + \eta'^2_2(\theta_1)}} \\ Y_{2,2}(\theta_1) &= \eta_2(\theta_1) \mp r \frac{\eta'_2(\theta_1)}{\sqrt{\xi'^2_2(\theta_1) + \eta'^2_2(\theta_1)}} \end{aligned} \quad (9)$$

In figure 2 there are presented a series of successive positions of one of the circular profiles of the gear 1, the trajectory of the centre and the two envelopes obtained using the equations (9). When varying the constructive parameters of the gear mechanism one can notice that a decisive part upon the shape of the conjugate tooth is played by the ratio between R , the radius where the centres of the bolts are placed and the radius r_{w1} . To illustrate this statement, the envelopes and the curve of the centres for three possible mentioned cases are presented in figure 3.

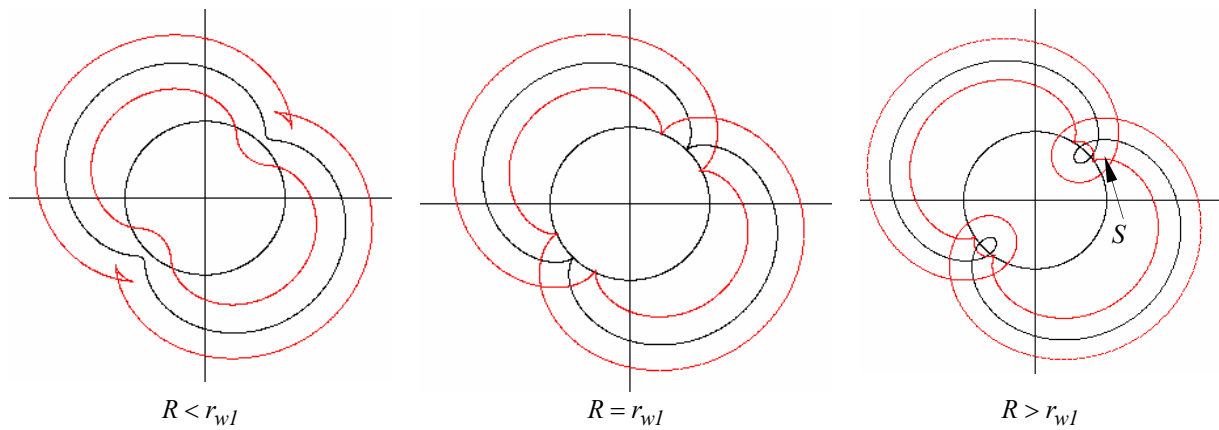


Figure 3. The effect of the ratio R/r_{wl} upon the flank of the conjugate gear.

From figure 3 it can be observed that for $R > r_{wl}$ the external envelope intersects the interior envelope in the point S , and the interference phenomenon occurs.

Another phenomenon that can be highlighted is the continuity of gearing. To this end, a gear mechanism with $z_2 = 6$ and $z_1 = 3$ is presented for three successive positions.

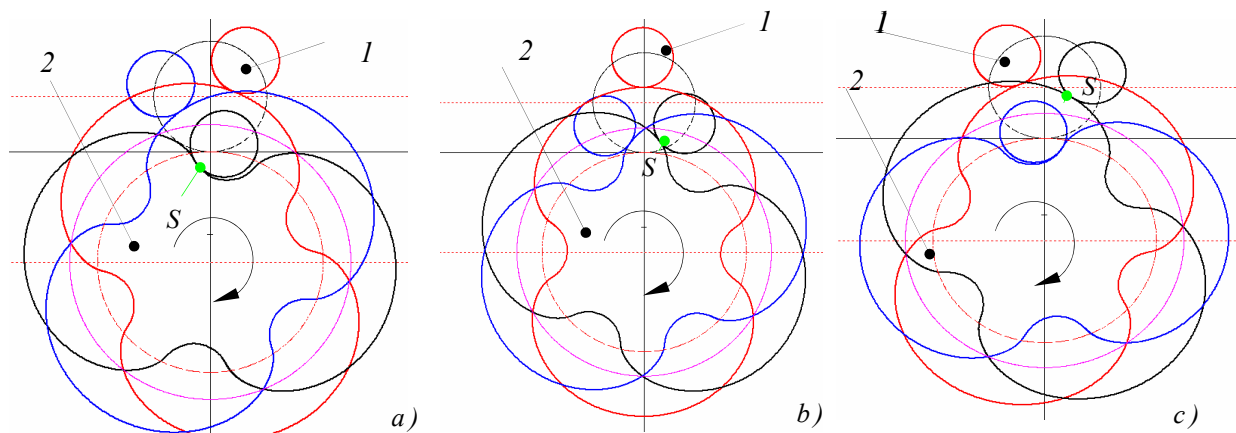


Figure 4. Three successive position of the gear mechanism.

From figure 4 it can be seen that the motion is initially transmitted from gear 2 to gear 1, the contact point S between the flanks which are transmitting the motion moves away from the centre of the gear figure 4a and 4b and thus, in figure 4c, the contact point is outside the addendum circle of gear 2 and the following tooth makes contact in an inactive region of the flank.

3. Determination of the conjugate profile using a planetary gear mechanism

Now it is considered that the gear with bolts performs a planetary motion with respect to the conjugate gear which is immobile. The centre of the gear 1 is on the rigid arm O_2O_1 which is the carrier. The motion between the two gears obeys the pure rolling condition between pitch circles $c_{wl,2}$. The schematics for the calculus and the geometrical-kinematical parameters of the mechanism are presented in figure 5. The position vector of the point M in the coordinate system of the gear 2 is given by:

$$\overline{O_2M} = \overline{O_2O_1} + \overline{O_1M} \quad (10)$$

or, expressing by means of the versors of the two frames:

$$\overline{O_2M} = x_{O1}i_2 + y_{O1}j_2 + x_I i_1 + y_I j_1 \quad (11)$$

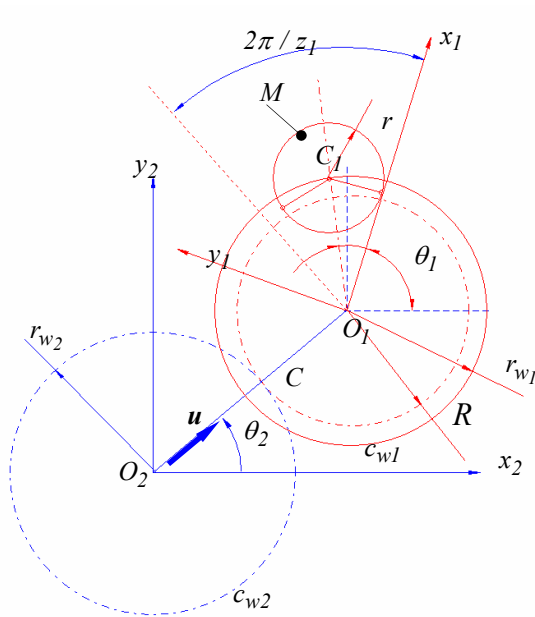


Figure 5. Planetary gear mechanism.

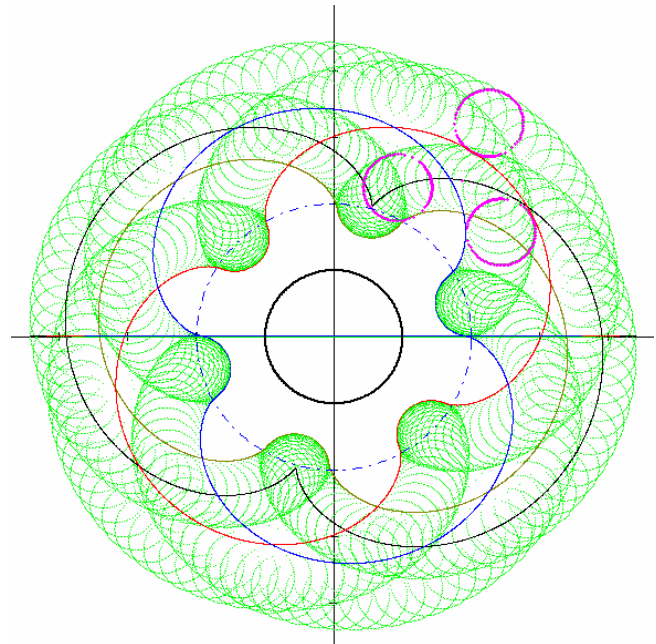


Figure 6. The epicycloids family described by the points from the profile of the gear.

The projections of the equation (11) on the axes of the coordinate system 2 are:

$$\begin{aligned} x_2 &= (r_{w1} + r_{w2}) \cos \theta_2 + x_I \cos \theta_1 - y_I \sin \theta_1 \\ y_2 &= (r_{w1} + r_{w2}) \sin \theta_2 + x_I \sin \theta_1 + y_I \cos \theta_1 \end{aligned} \quad (12)$$

The pure rolling condition requires that the velocity of the C^I point from the circle c_{w1} is:

$$\begin{aligned} \mathbf{v}_{C^I} &= \mathbf{v}_{O1} + \omega_1 \times \overline{O1C} = \omega_2 \times \overline{O2O1} = \dot{\theta}_2 \times (r_{w1} + r_{w2}) \mathbf{u} + \dot{\theta}_1 \times r_{w1} \mathbf{u} = \\ &= (\mathbf{k} \times \mathbf{u}) [\dot{\theta}_2 (r_{w1} + r_{w2}) + \dot{\theta}_1 r_{w1}] = \mathbf{0} \end{aligned} \quad (13)$$

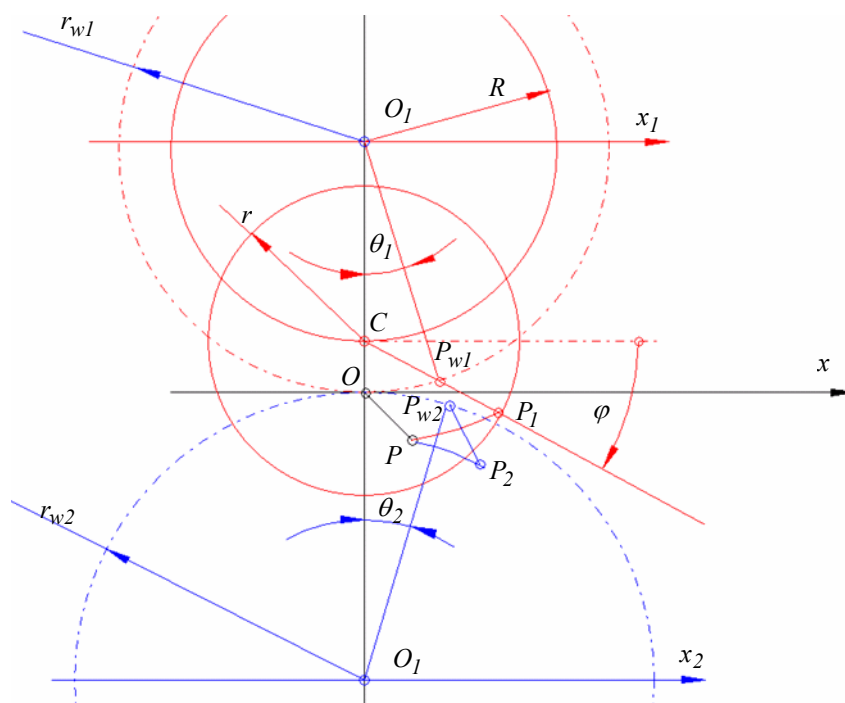
After integration of the condition (13) it results:

$$\theta_1 = -(1 + r_{w2}/r_{w1}) \theta_2 \quad (14)$$

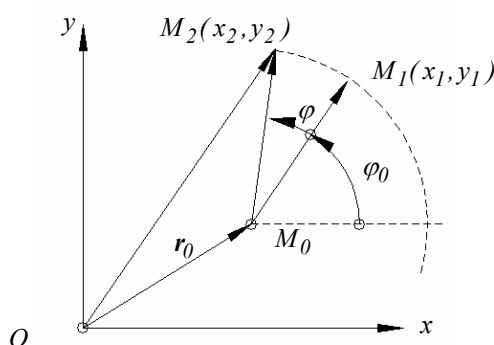
Replacing the equation (14) into relations (12), the parametric equations of the point M in the frame of the sun gear 2 are obtained, equations representing an epicycloid. When the point M is compelled to describe a circle of r radius, with the centre at a distance R from O_1 in the plane of the planetary gear, the points of this circle will generate in the plane of sun gear a family of epicycloids whose envelope should be the aimed flank of the sun gear. Figure 4 illustrates graphically the procedure described above and the internal envelopes of the successive positions of the three teeth of the planetary gear.

4. Determination of the conjugate profile using the fundamental law of gearing

Franz Reuleaux [12] stated the fundamental law of gearing which allows for finding both the conjugate profile and the line of action for a planar gearing when the pitch circles and the profile of one of the gears are known. The fundamental law of gearing affirms that for a gear mechanism to transmit constant angular velocity ratio, the necessary and sufficient condition is that common normal in the contact point passes through a fixed point, placed on the line of centres. The fixed point is



The fixed coordinate system O_{xy} has the O_x axis normal to the line of centres and the O_y axis is coincident to the line of centres; the mobile coordinate system $O_1x_1y_1$ is attached to the gear 1 which has the teeth with circular profile and the frame $O_2x_2y_2$ is attached to the driven gear, with the conjugate profile to be found. Using the notations from figure 8, the relations of coordinate changing of a point when it rotates about a fixed point are found:


$$\begin{aligned} x_2 &= x_0 + \rho \cos(\varphi_0 + \varphi) \\ y_2 &= y_0 + \rho \sin(\varphi_0 + \varphi) \end{aligned} \quad (15)$$

6

$$\begin{aligned}\rho \cos \varphi_0 &= x - x_0 \\ \rho \sin \varphi_0 &= y - y_0\end{aligned}\quad (16)$$

the final form of the transformation relations is obtained

$$\begin{aligned}x_2 &= x_0 + (x_1 - x_0) \cos \varphi - (y_1 - y_0) \sin \varphi \\ y_2 &= y_0 + (x_1 - x_0) \sin \varphi + (y_1 - y_0) \cos \varphi\end{aligned}\quad (17)$$

A point P_I from the circular profile of the flank of gear 1 is considered. According to figure 1, the next relations are written

$$\begin{cases} x_{PI} = r \cos \varphi \\ y_{PI} = r_{wI} - R - r \sin \varphi \end{cases}\quad (18)$$

The equation of the normal OP_I , to the circular profile, passing through the point P_I , is

$$\begin{aligned}y_{nI}(\lambda, \varphi) &= \lambda \cos \varphi \\ y_{nI}(\lambda, \varphi) &= r_{wI} - R - \lambda \sin \varphi\end{aligned}\quad (19)$$

The coordinates of the point P_{wI} are obtained by intersecting the normal OP_I with the pitch circle c_{wI} for the value λ_0 of the λ parameter:

$$\lambda_0(\varphi) = -R \sin \varphi + \sqrt{r_{wI}^2 - R^2 \cos^2 \varphi}\quad (20)$$

According to the fundamental law of gearing, the P_I point will become a contact point when the segment $P_{wI}P_I$ passes through the point C , the pitch point. To this end, the segment $P_{wI}P$ must revolve about the centre O_I with an angle θ_I in order to take the PC position, when it becomes the common normal for both profiles and P is a point from the line of action

$$\theta_I(\varphi) = \arctan[\lambda_0(\varphi) \cos \varphi / (R + \lambda_0(\varphi) \sin \varphi)]\quad (21)$$

The coordinates of the point P (from the line of action) are obtained:

$$\begin{cases} x_P(\varphi) = x_{PI}(\varphi) \cos \theta_I(\varphi) - (y_{PI}(\varphi) - r_{wI}) \sin \theta_I(\varphi) \\ y_P(\varphi) = r_{wI} + x_{PI}(\varphi) \sin \theta_I(\varphi) + (y_{PI}(\varphi) - r_{wI}) \cos \theta_I(\varphi) \end{cases}\quad (22)$$

The position of the point P_2 from the conjugate gear profile is obtained considering the fact that the arcs $P_{wI}C$ and $P_{w2}C$ from the two pitch circles must walked in the same period. Now, a rotation of angle θ_2 but contrary to the rotation of the gear 1, is applied to the point P

$$\theta_2(\varphi) = r_{wI} \theta_I(\varphi) / r_{w2}\quad (23)$$

and the position of the point P_2 is obtained:

$$\begin{cases} x_{P_2}(\varphi) = x_P(\varphi) \cos \theta_2(\varphi) + (y_P(\varphi) + r_{w2}) \sin \theta_2(\varphi) \\ y_{P_2}(\varphi) = -r_{w2} - x_P(\varphi) \sin \theta_2(\varphi) + (y_P(\varphi) + r_{w2}) \cos \theta_2(\varphi) \end{cases}\quad (24)$$

The relations (22) represent the parametric equations of the line of action and the equations (24) are the parametric equations of the conjugate profile. In figure 9a there are represented the circular profile, the conjugate profile and the line of action. Figure 9b presents: a detail concerning the line of action, a succession of positions of the circular profile and the corresponding conjugate profiles together with the contact points, obviously situated on the line of action.

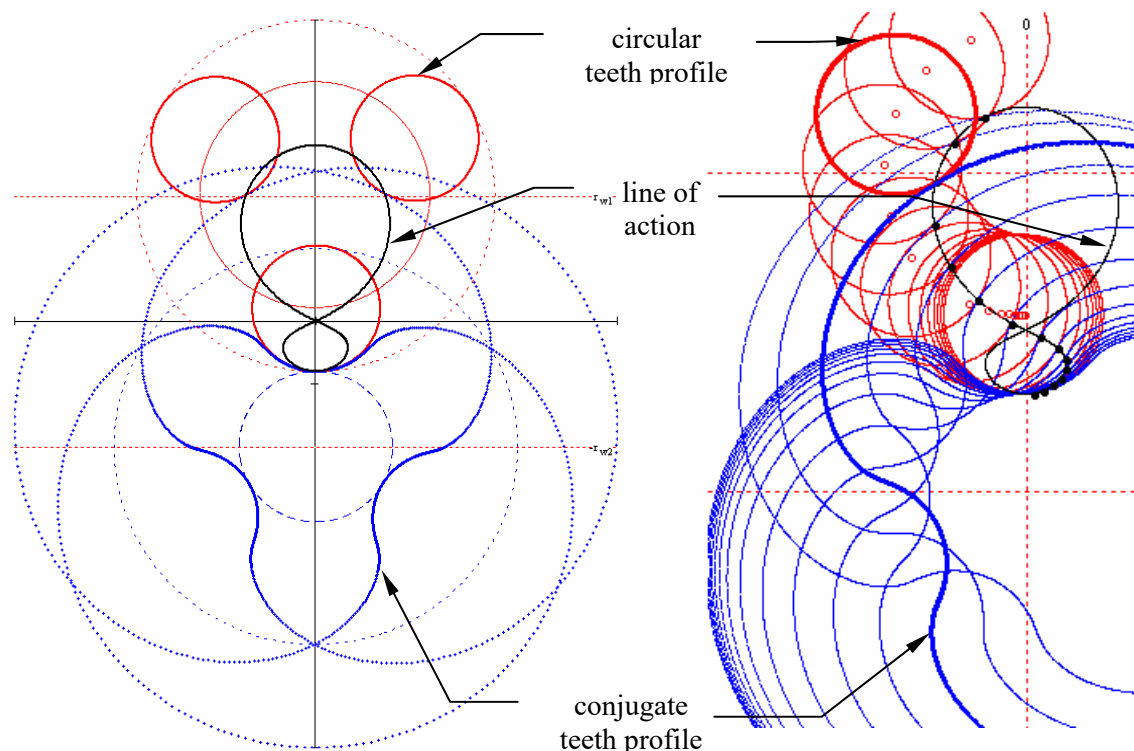


Figure 9. The line of action and conjugate teeth profile.

5. Conclusions

In the second part of the paper it is presented the manner of obtaining the conjugate profile of a spur gear with circular arc shape of the teeth. Three alternatives for accomplishing the conjugate profile are provided. The first two methods determine for the beginning, the trajectory described by the centre of a bolt in a plane fixed to the plane of the conjugate gear. The profile of the conjugate gear is achieved based on the remark that due to particular circular shape of the tooth, the conjugate profile will be completed by two curves equidistant to the trajectory of the centre of the bolt in the plane of the conjugate gear. The trajectory outlined by the centre of one of the circular teeth in the plane attached to the conjugate gear is obtained through two ways: using the motion of an ordinary gear mechanism and using a planetary gear mechanism. In the last case, the Reuleaux's fundamental law of gearing is employed for finding the profile of the conjugate gear. The method has as main advantages both finding the conjugate profile of the circular teeth and the line of action, which is necessary in the study of phenomena characteristic to gearing: continuity of gearing, contact ratio, undercutting and so on.

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