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Pu-visible Submodule with their most prominent characteristics

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Abstract:- In this article, the concept of pu-visible submodules of an S-module Y is introduced which is a generalization of the concept of visible submodule. Some characterizations and properties are proven. Also the results of pu-visible submodule by a module Y will be presented by giving many results and properties that explain and discuss that concept, one of the most important of them show that the submodule of Y is pu-visible when the residual of it is pu-visible ideal of S and the opposite is achieved. In addition to many results that have been presented.

Keywords:- Pu-visible submodules; Faithful module; Multiplication module; Finitely generated module; Fully cancellation module.

1- Introduction

In this article, S will be a commutative ring with identity, and Y is a module identified by ring S. In [1] the concept of visible submodule is presented by Mahmood S. and Buthyna N, where a proper submodule L of an S-module Y (for short, $L < Y$) is equal to BL for every nonzero ideal B of S, that is $L = BL$. Here the concept of pu-visible submodules will be presented, which is considered one of the generalizations of the concept of visible submodules. In this article, the most important properties that characterize this concept have been demonstrated and researched, and a study has been presented on the residual of this type of submodule. The article contains two items. The first under the heading pu-visible submodules, where the basic definition, characteristics and main features have been introduced. The second item, the residual of pu-visible submodule have been studied. Many important results have been obtained that the reader can follow in this item.

2. pu-visible Submodules

We introduce in this section the concept of pu-visible submodule which is a generalization of the visible submodule. Many basic properties about this concept have been established.

2.1. Definition

Let Y be an S-module and C be a proper submodule of Y. Then C is said to be pu-visible submodule whenever $C = BC$ for every nonzero pure ideal B of S.

An ideal B of a ring S is called pu-visible if it is pu-visible S-submodule of Y



2.2. Examples and Remarks

1-Every proper submodule of a Z -module Z is pu-visible, notice if we take for example the submodule $\langle 4 \rangle$ of a module Z , then $\langle 4 \rangle = \langle 1 \rangle \langle 4 \rangle$ where $\langle 1 \rangle$ is the only nonzero pure ideal of Z .

2-Consider the module Z_n over a ring Z , n is positive integer. Each proper submodule of Z_n is pu-visible. In particular, the submodule $\langle \bar{2} \rangle$ of a Z -module Z_{12} is pu-visible, since $\langle \bar{2} \rangle = \langle 1 \rangle \langle \bar{2} \rangle$ where $\langle 1 \rangle$ is the only nonzero pure ideal of Z .

3- For each prime number p , the zero submodule is the only pu-visible submodule of Z_p as a Z_p -module.

4- The two submodules $\langle \bar{2} \rangle$ and $\langle \bar{3} \rangle$ of the Z_6 -module Z_6 is not pu-visible, since $\langle \bar{2} \rangle, \langle \bar{3} \rangle$ are nonzero pure ideals of a ring Z_6 , but $\langle \bar{2} \rangle \neq \langle \bar{3} \rangle$ and $\langle \bar{3} \rangle \neq \langle \bar{2} \rangle \langle \bar{3} \rangle$.

5- All nonzero proper submodules of $Z_3 \oplus Z_6$ as a Z_6 -module is not pu-visible. since $\langle \bar{2} \rangle$ and $\langle \bar{3} \rangle$ are nonzero pure ideals of Z_6 , but

$$Z_3 \oplus \langle \bar{2} \rangle \neq \langle \bar{3} \rangle . (Z_3 \oplus \langle \bar{2} \rangle)$$

$$Z_3 \oplus \langle \bar{3} \rangle \neq \langle \bar{2} \rangle . (Z_3 \oplus \langle \bar{3} \rangle)$$

$$Z_3 \oplus \langle \bar{0} \rangle \neq \langle \bar{3} \rangle . (Z_3 \oplus \langle \bar{0} \rangle)$$

$$\langle \bar{0} \rangle \oplus \langle \bar{2} \rangle \neq \langle \bar{3} \rangle . (\langle \bar{0} \rangle \oplus \langle \bar{2} \rangle)$$

$$\langle \bar{0} \rangle \oplus \langle \bar{3} \rangle \neq \langle \bar{2} \rangle . (\langle \bar{0} \rangle \oplus \langle \bar{3} \rangle)$$

$$\langle \bar{0} \rangle \oplus Z_6 \neq \langle \bar{3} \rangle . (\langle \bar{0} \rangle \oplus Z_6) \text{ or } \langle \bar{0} \rangle \oplus Z_6 \neq \langle \bar{2} \rangle . (\langle \bar{0} \rangle \oplus Z_6)$$

6- For each positive integer n , all proper submodules of a Z -module $Z_n \oplus Z$

and a Z -module $Z_n \oplus Z_n$ are pu-visible submodules.

7- All proper submodule of Z_{p^∞} as a Z -module are pu-visible.

8- Let C be a pu-visible submodule of Y . Then for every proper submodule L of Y is pu-visible whenever $\alpha: C \rightarrow L$ is an epimorphism.

Proof:

Let $\alpha: C \rightarrow L$ be an epimorphism. Then $\alpha(C) = L$, but C is pu-visible submodule of Y , therefore for every nonzero ideal B of S we have $C = BC$ which implies that $L = \alpha(C) = \alpha(BC) = B\alpha(C) = BL$ and hence L is pu-visible submodule.

9- Let Y_1, Y_2 be two S -modules and let $F: Y_1 \rightarrow Y_2$ be an S -homomorphism. The next is achieved:

(i) If C is pu-visible submodule of Y_1 implies $F(C)$ is pu-visible submodule of Y_2

(ii) If H is pu-visible submodule of Y_2 implies $F^{-1}(H)$ is also pu-visible submodule of Y_1 .

Proof:

(i) Since C is pu-visible submodule of Y_1 , then $C = BC$ for every nonzero pure ideal B of S . Therefore $F(C) = F(BC) = BF(C)$.

(ii) By our hypothesis λ is pu-visible of Y_2 . Hence $B F^{-1}(H) = F^{-1}(BH)$ Pure ideal B of S , we are making $= BH = F^{-1}(H)$.

2.3 Corollary

Let Y be an S -module and C be pu-visible submodule of Y . If $L \leq C$, then CL is pu-visible of Y/L .

Proof: Directly from Examples and Remarks ((2.2),(8)).

The following proposition prove that the sum of two pu-visible sub module is always pu-visible submodule.

2.4 Proposition

Let Y be an S -module and $C, L \leq Y$. If C, L are pu-visible submodules, then $C+L$ is pu-visible submodule.

Proof :

Let B be a nonzero pure ideal of S and $C, L \leq Y$. Then $B(C+L) =$

$BC+BL = C+L$ (since C and L are pu-visible), therefore $C+L$ is pu-visible submodule of Y .

2.5 Remark

As a generalization of proposition (2.4), we have if $\{G_i\}_{i=1}^n$ is a finite collection of submodules of an S -module Y and G_i is pu-visible submodule for all i , then the sum of all these submodules is pu-visible submodule of Y .

The next proposition study the hereditary of pu-visible submodules.

2.6 Proposition

Let Y be an S -module and C be a pu-visible submodule of Y . Then every submodule of C is also pu-visible.

Proof:

Let C be a pu-visible submodule of an S -module Y and let $L < C$. That is $L < C = BC$ for every nonzero pure ideal B of S (since C is pu-visible) which implies that $L + BC = BC$ (1). Also, from the above inclusion, we get $BL \subseteq BC$ and hence $BL + BC = BC$ (2).

From (1) and (2) we get $BC + BL = BC + L$ and hence

$BL = L$. Therefore L is pu-visible submodule.

Now we give some results of proposition(2.6)

2.7 Corollary

Let Y be an S -module and $L_1, L_2 \leq Y$, if either L_1 or L_2 is pu-visible submodule of Y , then $L_1 \cap L_2$ is also pu-visible submodule.

Proof:

It is clear that $L_1 \cap L_2 \leq L_1$, but L_1 is pu-visible or L_2 is pu-visible, then by proposition (2.6) $L_1 \cap L_2$ is also pu-visible submodule.

As a directly result of corollary (2.7) we give the following generalization.

2.8. Corollary

Let $\{L_i\}_{i=1}^n$ be a family of submodules of an S-module Y such that at least one of them is pu-visible, then $\bigcap_{i=1}^n L_i$ is pu-visible submodule.

Proof:

It's clear that $\bigcap_{i=1}^n L_i \leq L_i \quad \forall i$. Then the result is directly from proposition(2.6).

The converse of proposition (2.6) need not to be true for example:

The module Z_6 as Z_6 -module. since $(\bar{0})$ content in each submodule of any S-module Y and $(\bar{2})$ is pu-visible submodule. But a submodule $(\bar{2})$ of

module Z_6 is not pu-visible by Examples and Remarks(2.2).

"An ideal B of a ring S is called idempotent if $B^2=B$ [2]".

2.9. Remark

Every pure ideal of a ring S is an idempotent ideal. Notice if B is pure ideal, then $B^2 = B$. $B = B \cap B = B$. Hence B is an idempotent.

2.10. Definition

An S-module Y is called fully cancellation module if for each ideal B of S and for each submodule L_1, L_2 of Y such that $BL_1 = BL_2$ implies $L_1 = L_2$ [3]."

It is possible to obtain the correctness of the opposite of proposition

(2.6) under a certain condition, which is illustrated by the following result.

2.11. Proposition

Let Y be a fully cancellation S-module and L be a pu-visible submodule of Y. If C is a proper submodule of Y containing L, then C is a pu-visible submodule of Y.

Proof:

Let B be a nonzero pure ideal of S. To prove that $BC = C$, we have, then $BL \leq BC$ which implies that $BL + BC = BC$ (1). $L \leq C$

But B is pure ideal hence B is an idempotent by remark(2.9)

Therefore $BL + B^2C = BC$ (2).

From (1) and (2) we get $BL + B^2C = BL + BC$, hence $B^2C = BC$.

But Y is fully cancellation module, then $BC = C$. Thus C is pu-visible submodule.

We see that under a certain condition placed on the module Y can be achieve the opposite. First we will need the following concepts to prove our result.

Now, we can give some characterizations for the submodule to be pu-visible through the following proposition.

2.12. Proposition

Let L be a proper submodule of a fully cancellation S -module Y .

Then the following are equivalent:

- 1- L is pu-visible submodule of Y .
- 2- $L = BL$ for each nonzero finitely generated pure ideal B of S .
- 3- $L = (d)L$ for each nonzero principal pure ideal (d) of S .
- 4- $L = dL$ for each nonzero element $d \in S$.

Proof:

(1) \rightarrow (2) : (2) holds directly from (1)

(2) \rightarrow (3) : Take $B = (d)$, $d \in S$

(3) \rightarrow (4) : Clearly, since $d \in \langle d \rangle$, then $dL = (d)L = L$

(4) \rightarrow (1) : Suppose that $L = dL$ for each $d \in S$. Let B be a nonzero

pure ideal of S . Then $BL = BdL$ which implies $BBL = BdL$ (since every pure ideal is an idempotent), but Y is fully cancellation module, then

$BL = dL = L$. That is L is a pu-visible submodule.

"Recall that an S -module Y is called strongly cancellation module if for each ideal B_1 and B_2 of S such that $B_1L = B_2L$, then $B_1 = B_2$ for every submodule L of Y [4]."

2.13. Proposition

Let L be a pu-visible submodule of Y and Y be a strongly cancellation S -module, since L is pu-visible, then $\text{anns}(BL) = \text{anns}(L)$, for every nonzero pure ideal B of S .

Proof:

Let $b \in \text{ann}(B)$. Then $bB = 0$ and hence $bBL = 0$ which implies that $b \in \text{ann}(LB)$. Therefore $\text{ann}(B) \subseteq \text{ann}(BL)$. Now, let $d \in \text{ann}(BL)$. Then $dBL = 0$ since B is a nonzero pure ideal of S and L is pu-visible submodule, then $dL = 0$, hence $dL = 0L$. But Y is strongly cancellation module. Then $d = 0$. Thus $dB = 0$ and hence $d \in \text{ann}(B)$. Therefore $\text{ann}(BL) \subseteq \text{ann}(B)$ and hence $\text{ann}(BL) = \text{ann}(B)$.

Before submitting our corollary we need to recall the following definition

"2.14. Definition

An ideal B of a ring S is called cancellation ideal if $A_1B = A_2B$ then $A_1 = A_2$ where A_1 and A_2 are two ideals of S . [5]"

2.15. Corollary

Let Y be a strongly cancellation module over a ring S and L is a pu-visible submodule of Y . Then every nonzero pure ideal B of S is cancellation.

Proof:

Let $A_1B = A_2B$ where A_1, A_2 are two ideals of S , B is pure ideal of S and $B \neq 0$. Then $A_1 - A_2 \subseteq \text{ann}(B)$. But by proposition(2.15). We have $\text{ann}(B) = \text{ann}(BL)$, L is pu-visible submodule of Y , then $A_1 - A_2 \subseteq \text{ann}(BL)$ and hence $A_1BL = A_2BL$. Since Y is strongly cancellation module and BL is a submodule of Y , therefore $A_1 = A_2$. Thus B is cancellation ideal.

2.16. Proposition

Let $\{P_\alpha\}$ be a nonempty collection of pu-visible submodule of an S -module Y . Then for each nonzero pure ideal B of S , $B(\cap_\alpha P_\alpha) = \cap_\alpha BP_\alpha$

Proof:

It is clear that $\cap_\alpha P_\alpha \subseteq P_\alpha$ for each α , but P_α is pu-visible submodule for each α and hence $BP_\alpha = P_\alpha$ for each α , also by Proposition(2.6) we get $\cap_\alpha P_\alpha$ is pu-visible submodule of Y . Now $B(\cap_\alpha P_\alpha) = \cap_\alpha BP_\alpha = \cap_\alpha P_\alpha$.

3. The Residual of pu-visible Sub modules

In this section, the residual of pu-visible submodules has been discussed with many important results.

Before submitting out first case let us recall the following.

3.1. Definition

An S -module Y is called faithful, if $\text{ann}_s(Y) = 0$, where $\text{ann}_s(Y) = \{r \in S : ry = 0 \ \forall y \in Y\}$ [6].

3.2. Definition

An S -module Y is called cyclic if and only if there exists $x \in Y$ such that $Y = Sx[2]$.

We also need to review the following:

“-If L is a submodule of an S -module Y , the annihilator of Y/L is denoted by $(L:Y)$ and it is defined by $(L:Y) = \{r \in S : rY \subseteq L\}$ [7].”

“-If L is a submodule of an S -module Y and B is an ideal of S , then $(L:B) = \{y \in Y : By \subseteq L\}$ is a submodule of Y containing L [7].”

“-Let Y be a multiplication S -module and L, P be submodules of Y . The residual of L by P in Y is $(L:P) = \{r \in S : rP \subseteq L \text{ for every } t \in P\}$ we will call $(0:L)$ annihilator of L in Y [8].”

3.3. Proposition

Let Y be a multiplication cancellation S -module. Then every proper L of Y is pu-visible submodule if and only if $(L:Y)$ is pu-visible ideal of S .

Proof:

\Rightarrow) Let L be a pu-visible submodule, to prove that $(L:Y)$ is pu-visible ideal. Let $r \in (L:Y)$. Then $rY \subseteq L$, implies $(r)Y \subseteq BL$ (since L is pu-visible submodule). Then $(r)Y \subseteq B(L:Y)Y$ (since Y is multiplication module). But Y is cancellation module. Therefore $(r) \subseteq B(L:Y)$ and hence $r \in B(L:Y)$. Then $(L:Y) \subseteq B(L:Y)$

Conversely $B(L:Y) \subseteq (L:Y)$. Therefore $(L:Y) = B(L:Y)$ for every nonzero pure ideal B of S . Let $(L:Y)$ is pu-visible ideal of S . Let $r \in L$, hence $(r) \subseteq L$ and hence $((r):Y) \subseteq (L:Y) = B(L:Y)$ for every nonzero pure ideal B of S . And hence $((r):Y)Y \subseteq B(L:Y)Y$ which implies that $(r) \subseteq BL$ (since Y is multiplication). Therefore $r \in BL$, and hence $L \subseteq BL$, also it is know that $BL \subseteq B$. Then from two above inclusions, we have $L = BL$ that is L is pu-visible submodule. This end the Proof.

From Proposition (3.3), we obtain the following corollary

3.4. Corollary

Let L be a proper submodule of a finitely generated faithful multiplication S -module Y . Then L is pu-visible submodule if and only if $(L:Y)$ is pu-visible ideal of S .

Proof:

From [9, Proposition (3.1), P.52], we get Y is cancellation and by Proposition (3.3) we obtain the result.

3.5. Proposition

Let Y be a cancellation multiplication module over S , and L be a pu-visible submodule of Y . Then for each nonzero pure ideal B of S , result from this $B(L:Y) = (BL:Y)$.

Proof:

Suppose that L is pu-visible submodule of Y , hence $L = BL$ for every nonzero pure ideal B of S and by Proposition (3.3) we have $(L:Y)$ is pu-visible sub ideal of S , hence $(L:Y) = B(L:Y)$ for every nonzero pure ideal B of S . Therefore $(BL:Y) = (L:Y) = B(L:Y)$. Thus $(BL:Y) = B(L:Y)$. Which complete the Proof.

Anther Proof for Proposition (3.3) which is depending on Proposition (3.5) as a result of which.

3.6. Corollary

If Y is a F.G faithful multiplication over S . Then $(L:Y)$ is pu-visible ideal if and only if L is pu-visible submodule of Y .

Proof:

\Leftarrow) Suppose that L is pu-visible submodule of Y , then for every nonzero pure ideal B of S , we write $L = BL$, therefore $(L:Y) = (BL:Y)$ and by Proposition (2.6) we obtain $(L:Y) = B(L:Y)$. Thus we get the result.

\Rightarrow) Let $(L:Y)$ is pu-visible, then for every nonzero pure ideal B of S . We have $(L:Y) = B(L:Y)$. And by Proposition (2.6), we obtain $(L:Y) = (BL:Y)$. Therefore $(L:Y)Y = (BL:Y)Y$ and hence $L = BL$. Thus L is pu-visible submodule.

3.7. Proposition

Let Y be a F. G faithful multiplication S -module and B be a proper ideal of S . Then the following hold:

(1) B is pu-visible ideal of S if and only if BY is pu-visible ideal of S (2) If L is pu-visible submodule of Y , then $\text{ann}_S(L) = \text{ann}_S(L:Y)$.

Proof:

(1) \Rightarrow) Let B be a pu-visible ideal of S , then $AB=B$ for every nonzero pure ideal A of S and hence $ABY=BY$. Therefore BY is pu-visible submodule.

\Leftarrow) Let BY be a pu-visible submodule of Y , then $ABY=BY$. But by (if Y is F. G faithful multiplication then Y is cancellation module). Therefore $AB=B$ and hence B is pu-visible ideal of S .

(2) Let $r \in \text{ann}_s(L:Y)$. Then $r(L:Y)=0$ which implies $rL=r(L:Y)Y=0$, therefore $r \in \text{ann}_s(L)$. Now, let $r \in L$. Then $0=rL=r(L:Y)Y$ implies that $r(L:Y)Y=0Y$ (If Y is F. G faithful multiplication then Y is cancellation). Therefore $r(L:Y)=0$, hence $r \in \text{ann}_s(L:Y)$.

3.8. Definition

A submodule L of an S -module Y is multiplication of Y if and only if $P \cap L = (P:L)L$ for every submodule P of Y [8].

The following Proposition introduce the necessary conditions for a pu-visible Sub module to be multiplication.

A ring R is called (von-Neumann) if for each element $a \in S$, there exists an element $t \in S$ such that $a = ata$ ($a = a^2$ if S is commutative) [2].

3.9. Proposition

If Y F.G faithful multiplication module over a regular ring S . If L a pu-visible submodule, then L is multiplication.

Proof:

Let P be any submodule of Y . Then $P = (P:Y)Y$ (since Y is multiplication module), we have L is pu-visible submodule of Y , then we get $L = BL$ for every nonzero ideal B of S (since S is a regular ring). Hence $L \cap P \subseteq BL = (P:L)L$ (choose $B = (P:L)$). Therefore $L \cap P \subseteq (P:L)L$. Now it is clear that $(P:L)L \subseteq Y$, then $(P:Y)(P:L)L \subseteq (P:Y)Y = P$. Which implies that $(P:Y)(P:L)L \cap L \subseteq P \cap L$. And hence $(P:Y)(P:L)L \subseteq P \cap L$. But $(P:Y)L = L$ (since L is pu-visible submodule and S is regular ring), hence $(P:L)L \subseteq P \cap L$. Therefore $(P:L)L = P \cap L$, that is multiplication submodule of Y .

In Proposition above the condition that a ring S is regular ring is very necessary and cannot be dropped, the following example shows:

Let Z be a module over Z such that Z is F. G faithful multiplication and a ring Z is not regular.

But $L = \langle 2 \rangle$ is pu-visible Submodule, but non multiplication Submodule since $\exists \langle 2 \rangle$ such that $\langle 2 \rangle \cap \langle 2 \rangle = \langle 2 \rangle$

$$(\langle 2 \rangle : Z) \langle 2 \rangle = \langle 2 \rangle. \langle 2 \rangle = \langle 4 \rangle \neq \langle 2 \rangle.$$

Recall the following concepts:-

- A submodule C of an S -module Y is named idempotent submodule if and only if $C = (C:Y)C$ [10, p.62].
- A submodule C of a module Y over a ring S is called multiplication if and only if $K \cap C = (K:C)C$ for every submodule K of Y [10].

3.10. Proposition

Every pu-visible submodule of a module Y over a regular ring is an idempotent.

3.11. Proposition

Let Y be a finitely generated faithful multiplication module over a regular ring S , L is proper submodule of Y . Then all will be equivalent:

- 1) L is pu-visible submodule of Y .
- 2) L is multiplication and is idempotent in Y .
- 3) L is multiplication and $P=(L:Y)P$ for each submodule P of L .
- 4) L is multiplication and $(p:L)L=(P:Y)L$ for each submodule P of Y .
- 5) $Sr=(L:Y)r$ for each $r \in L$.
- 6) $S=(L:Y)+\text{ann}_s(r)$ for each $r \in L$.

Proof:

(1) \Rightarrow (2) From proposition(2.9) and proposition(2.10.).

(2) \Rightarrow (3) Let $P \subseteq L$ and L pu-visible submodule, but P is pu-visible submodule by proposition(2.6). Hence $P=BP$ for every nonzero ideal over a regular ring S , hence we take $B=(L:Y)$. Therefore $P=(L:Y)P$.

(3) \Rightarrow (4) From (3), we obtain L is multiplication submodule. We have L is pu-visible submodule, then $L=BL$ for each nonzero ideal B over a regular ring S , hence we take $B=(P:Y)$. Also we can choose B another ideal of S , that is $B=(P:Y)$. Therefore $(P:L)L=L(P:L)L$.

(4) \Rightarrow (5) By L is multiplication, then for $r \in L$, we have $Sr=(L:Y)r$.

(5) \Rightarrow (6) By (5), we have, for each $r \in L$ $\exists m \in (L:Y) \exists r=mr$ Therefore $L=(m)L$ and hence $(m)=S$ (since L is cancellation module as a result we get it from the fact that Y is F.G faithful multiplication module). Hence $(L:Y)=S$ which implies $S+\text{ann}_s(r)=(L:Y)+\text{ann}_s(r)$ and hence $S=(L:Y)+\text{ann}_s(r)$.

(6) \Rightarrow (1) By (6), we get $SL=(L:Y)L+\text{ann}_s(r)L$ for each $r \in L$. Therefore $L=(L:Y)L$. But Y is multiplication module, then $L=BY$ for some ideal B of S . which implies $L=(BY:Y)L$ (since Y is cancellation). Therefore $L=BL$ and hence L is pu-visible submodule of Y .

4. References

- [1] M.S.Fiadh and B.N. Shihab, 2018, *Journal of Engineering and Applied Science*, "Visible submodules of a module X over a ring R ", vol.13, no. 24, pp. 10349-10355.
- [2] D. M. Burton, 1971, *Univ. of New Hampshire*, "Abstract and Linear Algebra".
- [3] I. M. Hadi, A. A. Elewi, 2014, *Journal of Al-Nahrain Univ* "Fully Cancellation and Naturaly Cancellation Module", vol. 17, no.3, pp. 178-184.
- [4] A. A. Elewi, 2016, *Iraqi Journal of Science*, "Strongly Cancellation Modules", vol.57, no.1A, pp.218-222.
- [5] A. M. Mijbass, 1992, M. Sc. Thesis, *Baghdad University* "On Cancellation Modules",
- [6] C. Faith, 1973, *Springer-Verlage, Berlin, Heidelberg, New York*, "Ring Modules and Categories I".
- [7] G.Desale and W.K. Nicholson, 1981, *J.Algebra* "Endoprimitive rings", vol.70, pp.584-560.

- [8] A. G. Naoum, M. A. K. Hasan, 1986, *Arch. Der Math* , "The Residual of Finitely Generated Multiplication Modules", vol. 46 ,pp.225-230,
- [9] J. R. Nezhad and H. M. Nadere, 2009, *Intemational Math. Forum*, "On Prime and Semeprime Submodules of Multiplication Modules", vol.4,no.26,pp. 1257-1266.
- [10] M.M.Ali,J.D.Smith,2004, *Beitrage Zur Algebra and Gemrtie Contributions to Algebra* ,, "PureSubmodule of multiplication modules", vol.45,no.1,pp. 61-74.