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Direct Sum and Projectivity of SemiHollow-Lifting Modules

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Abstract

Let $\mathbb{R}$ be a ring with identity and let T be a unitary left Module over $\mathbb{R}$. In this paper, we give some cases when a direct sum of hollow Modules is semihollow-lifting, Also; we give a condition under which a direct sum of two Modules is semihollow-lifting,

Keywords: Semhollow lifting Modules**,** projective Modules.

1. Introduction

A Submodule S of an $\mathbb{R}$-Module T is small Submodule of T ($S \ll T$) if for every Submodule D of T such that $T = S + D$ implies $D = T$[1]. A Submodule H of an $\mathbb{R}$-Module T is semismall of T ($H \ll_S T$) if H = 0 or $H/F \ll T/F$ for all nonzero Submodule F of H[2]. Let T be an $\mathbb{R}$-Module and H, F be Submodules of T such that $F \subset H \subset T$. F is called semicoessential Submodule of H in T ($F \subseteq_{sce}$ H in T) if $\frac{H}{F} \ll_S \frac{T}{F}$[3]. An $\mathbb{R}$-Module T is semihollow-lifting if for every Submodule H of T with $\frac{T}{H}$ hollow, there exists a Submodule F of T such that $T = F \oplus F^*$ and $F \subseteq_{sce}$ H in T[4].

Let T_1 and T_2 be $\mathbb{R}$-Modules, recall that T_1 is said to be T_2-projective if for every Submodule F of T_2, any homomorphism $g: T_1 \to \frac{T_2}{F}$ can be lifted to a homomorphism w: $T_1 \to T_2$. i.e. if $\pi: T_2 \to \frac{T_2}{F}$ is the natural epimorphism, then there exists an homomorphism $w: T_1 \to T_2$ such that $\pi \circ w = g$[5].

$$\begin{tikzcd} & T_1 \arrow[dl, dashed, "w"'] \arrow[d, "g"] & \\ T_2 \arrow[r, "\pi"'] & \frac{T_2}{F} \arrow[r] & 0 \end{tikzcd}$$

T_1 and T_2 are relatively projective if T_1 is T_2-projective and T_2 is T_1-projective.

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Example1[5] Consider $T_1 = Z$ as Z-Module and $T_2 = Z_{p^\infty}$ as Z-Module, thus T_1 is relatively T_2-projective.

Now, we prove the following proposition.

Proposition2 If T is a semihollow-lifting $\mathbb{R}$-Module and for every decomposition $T = U \oplus V$, U and V are relatively projective. Then for every Submodules X and Y of T with $\frac{T}{X}$ hollow and $T = X+Y$, there exists an idempotent $e \in \mathrm{End}(T)$, such that $e(T) \subseteq X$, $(I-e)(T) \subseteq Y$ and $(I-e)(X) \ll_s (I-e)(T)$.

Proof: Let X and Y be Submodules of T such that $T = X+Y$ and $\frac{T}{X}$ hollow. Since T is semihollow-lifting, thus there exists a Submodule E of X such that $T = E \oplus V$, for some $V \subseteq T$ and $X \cap V \ll_s V$. By modular law, $X = X \cap T = X \cap (E \oplus V) = E \oplus (X \cap V)$, hence $T = X+Y = E+(X \cap V)+Y$. But $X \cap V \ll_s V \subseteq T$, therefore $T = E+Y$. By our assumption V is E-projective, thus by [6, Lemma 5], there exists $D \subseteq Y$ such that $T = D \oplus E$. Now, consider the projection map $\pi: T \rightarrow E$ and the inclusion map $i: E \rightarrow T$ with respect to decomposition $T = D \oplus E$. Let $p = i \circ \pi: T \rightarrow T$. Clearly $p \in \mathrm{End}(T)$ is an idempotent and $p(T) \subseteq X$. Claim that $(I-p)(T) = D$, let $t \in T$ thus $t = h + d$, where $h \in E$ and $d \in D$, $(I-p)(t) = I(t) - p(t) = t - (i \circ \pi)(t) = h+d - \pi(h+d) = h+d-h = d \in D$. Thus $(I-p)(T) \subseteq D$. Let $d \in D$ this implies that $p(d) = 0$. Then, $(I-p)(d) = d - p(d) = d$, and hence $d \in (I-p)(T)$. Then $D \subseteq (I-p)(T)$. But $D \subseteq Y$, therefore $(I-p)(T) \subseteq Y$. Claim that $(I-p)(X) = X \cap (I-p)(T) = X \cap D$. To see that. Let $d \in (I-p)(X)$, thus there is $m \in X$ such that $d = (I-p)(m) = m - p(m)$. Then $d \in X$ and $d \in (I-p)(T)$. So $d \in X \cap (I-p)(T)$. Hence, $(I-p)(X) \subseteq X \cap (I-p)(T)$. Let $u \in X \cap (I-p)(T)$, thus $u \in X$ and $u \in (I-p)(T)$. There is $q \in T$ such that $u = (I-p)(q) = q - p(q)$.Then $u+p(q) = q \in X$, thus $u \in (I-p)(X)$. It is easy to show that $X \cap D \cong X \cap V$. But $X \cap V \ll_s V \cong D$, therefore $(I-p)(X) \ll_s (I-p)(T)$.

Note: Direct sum of two semihollow-lifting Modules need not be a semihollow-lifting Module[4,Examples3].

Let T_1 and T_2 be $\mathbb{R}$-Modules, T_1 is semismall T_2-projective (nearly T_2-projective) if for every homomorphism $g: T_1 \rightarrow \frac{T_2}{A}$, where A is a Submodule of T_2 and $\mathrm{Im}\, g \ll_s \frac{T_2}{A}$ ($\mathrm{Im}\, g \neq \frac{T_2}{A}$), can be lifted to a homomorphism $h: T_1 \rightarrow T_2$.

$$\begin{tikzcd}
& T_1 \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\
T_2 \arrow[r, "\pi"'] & \frac{T_2}{A} \arrow[r] & 0
\end{tikzcd}$$

Recall that a decomposition $T = \oplus_{i \in I} T_i$ is complement direct summands if for every direct summand F of T there exists a subset $J \subseteq I$ such that $T = F \oplus (\oplus_{i \in J} T_i)$[7, p.125].

The following proposition gives a condition under which a direct sum of semihollow-lifting Modules is semihollow-lifting.

Proposition3 Let $T = T_1 \oplus T_2$ such that T_1 and T_2 are semihollow-lifting Modules. if T_1 and T_2 are relatively projective, thus T is semihollow-lifting.

Proof: Let S be a Submodule of T such that T/S is hollow. Thus $T = T_1+S$ or $T = T_2+ S$. Assume that $T = T_1+S$ (In case $T = T_2+S$ being analogous). Thus $T_1/S\cap T_1$ is hollow. But T_2 is T_1-projective, there exists a Direct summand of T contained in S such that $T = T_1 \oplus D$[8, 41.14]. Thus $S = (T_1 \cap S) \oplus D$. But T_1 is semihollow-lifting, there exists a direct summand W of T_1 such that $W \leq S \cap T_1$ and $S\cap T_1/W \ll_s T_1/W$. Then $W \oplus D$ is a direct summand of T and $W \oplus D \leq (S \cap T_1)\oplus D$. Assume U be a Submodule of T with $W \oplus D \leq U$ and $(S\cap T_1)\oplus D/W\oplus D + U/W\oplus D = T/W\oplus D$. Thus $(S \cap T_1) + D + U = T$. So $(S\cap T_1) + U = T$. $S\cap T_1/W \ll_s T_1/W$ thus $U = T$. Then $W \oplus D$ is a semicoessential submodule of $(S\cap T_1)\oplus D = S$ in T.

Now, the following propositions give some cases when a direct sum of semihollow Modules is semihollow-lifting.

Proposition4 Assume $T = \oplus_{i\in I}T_i$, where all T_i are hollow and $\oplus_{i\in I}T_i$ complement direct summands. If T is semihollow-lifting, thus $\oplus_{i\neq j}T_i$ is nearly T_j-projective.

Proof: Let W any proper Submodule of T_j and the homomorphism $g: \oplus_{i\neq j}T_i \to \frac{T_j}{W}$ with $Img \neq \frac{T_2}{W}$ and the natural epimorphism $\pi: T_j \to \frac{T_j}{W}$. Define $V =\{a+b \mid a \in \oplus_{i\neq j}T_i,\ b \in T_j$ and $g(a) = -\pi(b)\}$. We claim that $T = V+T_j$. Clearly $V+T_j \subseteq T$. Let $t \in T$, thus $t = a+b$, where $a \in \oplus_{i\neq j} T_i$ and $b \in T_j$. Therefore, $g(a) \in \frac{T_j}{W}$. Since π is onto, there exists $b^* \in T_j$ such that $\pi(b^*) = g(a)$, therefore $g(a) = -\pi(-b^*)$.Then $t = a+b = a+ b^* - b^*+b$, where $a+b^* \in V$ and $-b^*+b \in T_j$, then $t \in V+T_j$ and $T \subseteq V+T_j$. Then $T = V+T_j$, $W \subseteq V$. Now, $\frac{T}{V} = \frac{V+T_j}{V}$, thus by second isomorphism theorem $\frac{V+T_j}{V} \cong \frac{T_j}{V\cap T_j}$. Since T_j is hollow, thus $\frac{T_j}{V\cap T_j}$ is hollow and then $\frac{T}{V}$ is hollow. Since T is semihollow-lifting, so there is a direct summand F of T such that $F \subseteq_{sce} V$ in T. Then by[3,Proposition7], $\frac{T}{F}$ is hollow. But the decomposition of T complement direct summands, so there is a subset $J \subseteq I$ such that $T = F\oplus(\oplus_{i\in J}T_i)$.Since $\frac{T}{F}$ is hollow, thus $\frac{T}{F}$ is indecomposable. Hence $T = F\oplus T_k$, for some $k \in I$. Now, $\frac{T}{F} = \frac{V+T_j}{F} = \frac{B}{F} + \frac{T_j+D}{F}$. Since $F \subseteq_{ce} V$ in T, thus $T = T_j + F$.Claim that $k = j$. If $k \neq j$ thus g is an epimorphism, to see that, let $x_j +W \in \frac{T_j}{W}$. Since π is onto then there exists $x_j \in T_j$ such that $\pi(x_j) = x_j +W$. Then $x_j \in T$, and $x_j= d+m_k$,where $d \in F$, $m_k \in T_k$.But $F \subseteq V$ therefore $d \in V$.Then $d = a+b$, where $a \in\oplus_{i\neq j}T_i$, $b \in T_j$ and $g(a) = -\pi(b)$ and hence $x_j = a+b+m_k$. So $x_j - b = x+m_k$. Since $k \neq j$ thus $T_k \subseteq \oplus_{i\neq j}T_i$ and hence $x_j - b = x+m_k \in \oplus_{i\neq j}T_i \cap T_j = 0$. Then $x_j = b$. Since $g(a) = -\pi(b)$, thus $g(-a) = \pi(b)$ and hence $g(-a) = \pi(x_j) = x_j +W$. Thus g is an epimorphism, which is a contradiction. Thus we get $k = j$ and hence $T = F\oplus T_j$. Now, let $\beta: F \oplus T_j \to T_j$ be the projection map, thus $\pi \circ \beta|_{(\oplus_{i\neq j}Ti)} = g$, to see that:

$$\begin{tikzcd}
 & & \bigoplus_{i\neq j}T_i \arrow[dl, dashed, "\beta|_{(\bigoplus_{i\neq j}Ti)}"'] \arrow[d, "g"] & \\
D\bigoplus T_j \arrow[r, "\beta"] & T_j \arrow[r, "\pi"'] & \frac{T_j}{W} \arrow[r] & 0
\end{tikzcd}$$

Let $z \in \bigoplus_{i\neq j}T_i$ thus $z \in F\bigoplus T_j$ and hence $z = d+m_j$, where $d \in F$, $m_j \in T_J$. Since $F \subseteq V$ thus $d \in V$ and hence $d = a+b$, where $a \in \bigoplus_{i\neq j}T_i$, $b \in T_j$. Thus we have $\pi \circ \beta|_{(}\bigoplus_{i\neq j}Ti)(z)= \pi \circ \beta|_{(}\bigoplus_{i\neq j}Ti)(d+m_j) = \pi(m_j)$. But $z = d+m_j = a+b+m_j$, where $a \in \bigoplus_{i\neq j}T_i$, $y \in T_j$ and $g(a) = -\pi(b)$, Therefore $z\text{-}a = b+m_j \in \bigoplus_{i\neq j}T_i \cap T_j = 0$. Then $z = a$ and $m_j = -b$. Now, $\pi \circ \beta|_{(}\bigoplus_{i\neq j}Ti)(z) = \pi(m_j) = \pi(-b) = -\pi(b) = g(a) = g(z)$. Hence $\pi \circ \beta|_{(}\bigoplus_{i\neq j}Ti) = g$. Then $\bigoplus_{i\neq j}T_i$ is nearly T_j-projective.

Proposition5 Let $T = \bigoplus_{i\in I}F_i$ be a direct sum of hollow Modules F_i such that the decomposition $\bigoplus_{i\in I}F_i$ is complement direct summands. If there is no epimorphism between F_i and F_j $(i \neq j)$ and T is semihollow-lifting, then $\bigoplus_{i\neq j}F_j$ is F_i-projective for each $i \in I$.

Proof: Assume W be a proper Submodule of T with $T = W+F_i$. Now, by second isomorphism theorem, $\frac{T}{W} = \frac{W+F_i}{W} \cong \frac{F_i}{W\cap F_i}$. Since F_i is hollow for all $i \in I$, thus $\frac{T}{W}$ is hollow. But T is semihollow-lifting, so there is a direct summand X of T such that $X \subseteq_{sce} W$ in T. Then by[3,Proposition7], $\frac{T}{X}$ is hollow. Now, $\frac{T}{X} = \frac{W+F_i}{X} = \frac{N}{X} + \frac{F_i+X}{X}$. This implies that $T = X+F_i$. Since the decomposition $\bigoplus_{i\in I}F_i$ complement direct summands, thus there exists a subset J of I such that $T = X\bigoplus(\bigoplus_{i\in J}F_i)$. But $\frac{T}{X}$ is hollow, so $\frac{T}{X}$ is indecomposable. Then $T = X\bigoplus F_k$, for some $k \in I$. Claim that $i = k$. If $i \neq k$, let $\pi: X\bigoplus F_k \to F_k$ be an epimorphism thus $\pi|Fi: H_i \to F_k$ is an epimorphism. To see that, let $f_k \in F_k$, thus $f_k \in T$, hence $f_k = x + f_i$, where $x \in X$ and $f_i \in F_i$. Thus $\pi(f_k) = \pi(x)+ \pi(f_i)$ and hence $\pi(f_k) = \pi(f_i)$. This implies that $\pi(f_i) = f_k$. Then there is an epimorphism between F_i and F_k with $(i \neq k)$ which is a contradiction. Therefore $i = k$, hence $T = X\bigoplus F_i$. Then by[6, Lemma 5], $\bigoplus_{i\neq j}F_j$ is F_i-projective for each $i \in I$.

Let T_1 and T_2 be $\mathbb{R}$-Modules, T_1 is h-semismall T_2-projective if every homomorphism $g:T_1\to \frac{T_2}{W}$,(where W is a submodule of T_2, $\frac{T_2}{W}$ is hollow and $\mathrm{Im}\, g \ll_s \frac{T_2}{W}$) can be lifted to a homomorphism $\varphi: T_1 \to T_2$.

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$$\begin{tikzcd} & T_1 \arrow[dl, dashed, "\varphi"'] \arrow[d, "g"] & \\ T_2 \arrow[r, "\pi"'] & \frac{T_2}{W} \arrow[r] & 0 \end{tikzcd}$$

Remark6 Let T_1 and T_2 be two $\mathbb{R}$-Modules then we have the implication:

T_1 is T_2 - projective $\Rightarrow$ T_1 is nearly T_2 – projective $\Rightarrow$ T_1 is semismall T_2 - projective $\Rightarrow$ T_1 is h-semismall T_2 - projective.

Proof: Clear.

Example7 Consider $T_1 = Z$ as Z-Module and $T_2 = Z_2$ as Z-Module, then T_1 is h-semismall T_2-projective.

The following lemma gives a characterization of h-semismall projectivity.

Lemma8 Let T_1 and T_2 be Modules and $T = T_1 \oplus T_2$. If T_1 is h-semismall T_2- projective then for every Submodule E of T such that $\frac{T}{E}$ is hollow and $T \neq T+E$, there exists a Submodule E^* of E such that $T = E^* \oplus T_2$.

Proof: Clear.

The following proposition gives conditions under which a direct sum of two Modules is semihollow-lifting.

Proposition9 Assume $T = T_1 \oplus T_2$ such that T_1 is h-semismall T_2-projective and T_2 is semihollow-lifting. If for every Submodule E of T such that $\frac{T}{E}$ is hollow, $T \neq T_1+E$. Then T is semihollow-lifting.

Proof: Let E be a Submodule of T such that $\frac{T}{E}$ is hollow. Thus by our assumption $T \neq T_1+E$. Now, $\frac{T}{E} = \frac{T_1 \oplus T_2}{E} = \frac{T_1+E}{E} + \frac{T_2+E}{E}$. But $\frac{T}{E}$ is hollow, therefore $E \subseteq_{sce} (T_1+E)$ in T. Then $T = T_2+E$. Since T_1 is h-semismall T_2-projective, thus by Lemma8, there exists a Submodule E^* of E such that $T = E^* \oplus T_2$. By second isomorphism theorem, $\frac{T}{E} = \frac{T_2+E}{E} \cong \frac{T_2}{E \cap T_2}$. Then $\frac{T_2}{E \cap T_2}$ is hollow. But T_2 is semihollow-lifting, thus there is a direct summand U of T_2 such that $U \subseteq_{sce} (E \cap T_2)$ in T_2. Since $U \subseteq T_2$ and T_2 is a direct summand of T, then U is a direct summand of T. By modular law, $E = E \cap T = E \cap (E^* \oplus T_2) = E^* \oplus (E \cap T_2)$. Since $U \subseteq E \cap T_2$ and $U \cap E^* = 0$, thus $U \oplus E^* \subseteq (E \cap T_2) \oplus E^*$ and hence $U \oplus E^* \subseteq E$. It is easy to show that $U \oplus E^*$ is a direct summand of T. We want to show that $U \oplus E^* \subseteq_{sce} E$ in T. Let $X \subseteq T$ and $\frac{E}{U \oplus E^*} + \frac{X}{U \oplus E^*} = \frac{T}{U \oplus E^*}$. Then $E+X = T$ and hence $E^* \oplus (E \cap T_2)+X = T$. But

$E^{*} \subseteq X$, therefore $(E \cap T_2)+X = T$. Now, $\frac{T}{U} = \frac{(E \cap T_2)+X}{U} = \frac{E \cap T_2}{U} + \frac{X}{U}$. Since $U \subseteq_{sce} (E \cap T_2)$ in T_2, thus $U \subseteq_{sce} (E \cap T_2)$ in T. Hence $\frac{T}{U} = \frac{X}{U}$ This implies that T = X and hence $U \oplus E^{*} \subseteq_{sce} E$ in T. Then T is semihollow-lifting.

An ℝ-Module T is said to have the (finite) exchange property if for any(finite) index set I, whenever $T \oplus N = \oplus_{i \in I} Ai$, for Modules N and Ai, then $T \oplus N = T \oplus (\oplus_{i \in I} Bi)$ for Submodules $Bi \subseteq Ai$[9].

Now, we consider some conditions for a Module T_1 to be h-semismall T_2-projective, when $T = T_1 \oplus T_2$ is semihollow-lifting.

Proposition10 Let $T = T_1 \oplus T_2$ be a semihollow-lifting Module. If T_1 has the finite exchange property and T_2 is hollow, thus T_1 is h-semismall T_2- projective.

Proof: Let W be a Submodule of T such that $\frac{T}{W}$ is hollow and $T \neq T_1 + W$. Since T is semihollow-lifting, thus there is a direct summand E of T such that $E \subseteq_{sce} W$ in T. Since $\frac{T}{W}$ is hollow, thus by[3,Proposition7], $\frac{T}{E}$ hollow. Now, $\frac{T}{E} = \frac{T_1 \oplus T_2}{E} = \frac{T1 + K}{E} + T = T_2 + E$. Assume $T = E \oplus E^{*}$, for some $E^{*} \subseteq T$. Since T_1 has the finite exchange property, thus $T_1 \oplus T_2 = T_1 \oplus X \oplus Y$, for some $X \subseteq E$ and $Y \subseteq E^{*}$. It is Clear that $T = T_1 + E + Y$ and $Y \cap E = B \cap E^{*} \cap E = 0$. So $\frac{T}{E} = \frac{T_1 + E}{E} + \frac{Y \oplus E}{E}$. Since $E \subseteq_{sce} (T_1 + E)$ in T, thus $T = Y \oplus E$. But $T = E \oplus E^{*}$ and $Y \subseteq E^{*}$ so, $E^{*} = Y$. Since $E^{*} \cap T_1 = Y \cap T_1 = 0$, thus $\frac{T}{T_1} = \frac{E \oplus E^{*}}{T1} = \frac{E + T_1}{T_1} + \frac{E^{*} \oplus T_1}{T_1}$. By the second isomorphism theorem, $\frac{T}{T1} \cong T_2$ thus $\frac{T}{T_1}$ is hollow. But $T \neq T_1 + E$ therefore $T_1 \subseteq_{sce} (E + T_1)$ in T and hence $T = E^{*} \oplus T_1$. Since $K^{*} = Y$, Thus by[10, lemma3.2], we get E has the finite exchange property. But $T = E \oplus E^{*} = T_1 \oplus T_2$, so there exists $Q \subseteq T_1$ and $F \subseteq T_2$ such that $T = E \oplus Q \oplus F$. It is Clear that $T = E + T_1 + F$. Now, $\frac{T}{E} = \frac{E + T1}{E} + \frac{D \oplus E}{E}$. Since $E \subseteq_{sce} (T_1 + E)$ in T thus $T = F \oplus E$. Also, since $F \cap T_1 = 0$, thus we have $\frac{T}{T1} = \frac{F \oplus E}{T1} = \frac{F \oplus T_1}{T_1} + \frac{E + T_1}{T1}$. Since $T_1 \subseteq_{sce} (E + T_1)$ in T, thus $T = F \oplus T_1$. But $T = T_1 \oplus T_2$ and $F \subseteq T_2$, therefore $F = T_2$ and hence $T = T_2 \oplus E$. Then T_1 is h-semismall T_2- projective.

Let $T = \oplus_{i \in I} T_i$ be a direct sum of Submodules T_i. Recall that the decomposition $T = \oplus_{i \in I} T_i$ is called exchange decomposition (or exchangeable) if for any direct summand N of T we have $T = N \oplus (\oplus_{i \in I} Ni)$ with $N_i \subseteq T_i$[11].

By [7, p.125], we have:

Remark11 Let $T = \oplus_{i \in I} T_i$ be a direct sum of Submodules T_i, then we have the implication:

T is complement direct summands $\Longrightarrow$ T is exchangeable

We end this section by the following Proposition.

Proposition12 If T is a semihollow-lifting Module with exchange decomposition $T = T_1 \oplus T_2$ and T_2 is a hollow Module. Then T_1 is h-semismall T_2- projective.

Proof: Assume T is a semihollow-lifting Module with exchange decomposition $T = T_1 \oplus T_2$. Suppose E be a Submodule of T such that $\frac{T}{E}$ is hollow and $T \neq T_1+E$. Now, $\frac{T}{E} = \frac{T_1 \oplus T_2}{E} = \frac{T_1+E}{E} + \frac{T_2+E}{E}$. Since $T \neq T_1 + E$ and $\frac{T}{E}$ is hollow, Thus $E \subseteq_{sce} (T_1 + E)$ in T and hence T = T+E. But T is semihollow-lifting, so there exists a direct summand D of T such that $D \subseteq_{sce} E$ in T. Since $\frac{T}{E}$ is hollow thus by[3,Proposition7], $\frac{T}{D}$ is hollow. Clearly $T \neq T_1 + D$. But, $\frac{T}{D} = \frac{T_1 \oplus T_2}{D} = \frac{T_1+D}{D} + \frac{T_2+D}{D}$, therefore $D \subseteq_{sce} (T_1 + D)$ in T and hence $T = T_2+D$. It is enough to prove that $T = T_2 \oplus D$. Since the decomposition $T = T_1 \oplus T_2$ is exchangeable and D is a direct summand of T, thus we have $T = D \oplus T_1^{*} \oplus T_2^{*}$ for Submodules $T_1^{*} \subseteq T_1$ and $T_2^{*} \subseteq T_2$. Hence $T = D+T_1+T_2^{*}$ and $T_2^{*} \cap T_1 = 0$. Since $T = T_1 \oplus T_2$, thus by the second isomorphism theorem, $\frac{T}{T1} \cong T_2$. But T_2 is hollow, thus $\frac{T}{T_1}$ is hollow. But $\frac{T}{T_1} = \frac{D+T1+T_2^{*}}{T1} = \frac{D+T_1}{T_1} + \frac{T_2^{*} \oplus T_1}{T_1}$, therefore $T_1 \subseteq_{sce} (D+T_1)$ in T and hence $T = T_2^{*} \oplus T_1$. Since $T = T_1 \oplus T_2$, thus $T_2 = T_2^{*}$. But $T = D \oplus T_1^{*} \oplus T_2^{*}$, so $T = D \oplus T_1^{*} \oplus T_2$. Since $T = T_2+D$. Thus $T = D \oplus T_2$, Then T_1 is h-semismall T_2- projective.

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