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Monomial Bases for Free Post-Lie Algebras

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Abstract. Many attempts have been made to describe bases for the post-Lie algebra, using many types of Lie bases. Here, we try to describe special kind of post-Lie bases using those bases described in the pre-Lie (respectively Lie) contexts, founded before in [8, 9]. In addition, we try to understand the effect of the second operation of the post-Lie structure on the form of these bases, using some cases of the generating set, and translate it in terms of rooted trees.

1. Introduction

First appearance of post-Lie algebras was in 2007, introduced by Valette in [1]. After that, many authors have been written in this context with diverse fields, for example: H. Munthe-Kass, D. Manchon, K. Ebrahimi-Fard, A. Lundervold, C. Curry, B. Owren and others more [2, 5, 7]. Many algebraic properties for Lie, pre-Lie, Rota Baxter, and others algebras have been extended to the post-Lie in algebraic, differential, geometrical and numerical domains. In this work, we study some post-Lie properties and relate it with these satisfied by the other algebras. Bases for free post-Lie algebras are described in several ways. H. Munthe-Kass and A. Lundervold are presented post-Lie bases with tree version (in planar case) [5], using a strategy similar to that one used by F. Chapoton and M. Livernet in [4].

This paper contains two main sections. Section 2 consists of two subsections 2.1, 2.2. Some preliminaries on post-Lie structures and some of their properties have been studied. Using preceding work in Lie (respectively pre-Lie) algebras [8, 9], some connection are founded between these algebras with the post-Lie by generalizing certain identities with nice conditions. A monomial of generators r's in a post-Lie algebra is a parenthesize of these elements combined with the two binary post-Lie operations, for example, a monomial of a post-Lie element r (with respect to two operations $[\,.\,,.\,], \rhd$) is:

$$r \rhd r, r \rhd (r \rhd r), (r \rhd r) \rhd r, [r,\ r \rhd r] \tag{1}$$

Finally, (Monomial) post-Lie bases for$\mathcal{L}(E)$'s algebras are calculated, using special cases of the generating set E.

2. Post-Lie Algebras

Let $\mathcal{L}$ be a vector space endowed with two bilinear operations$[.,.], \rhd: \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$, such that $[.,.]$ is a Lie bracket and the product $\rhd$ satisfies:

$$x \rhd [y,z] = [x \rhd y, z] + [y, x \rhd z], \tag{2}$$
$$[x,y] \rhd z = a_{\rhd}(x,y,z) - a_{\rhd}(y,x,z), \tag{3}$$

for any $x, y, z \in \mathcal{L}$, where $a_{\rhd}(x,y,z) = x \rhd (y \rhd z) - (x \rhd y) \rhd z$. A triple $(L, [.,.], \rhd)$ is called post-Lie algebra. One can verify that every Lie algebra has a natural post-Lie structure, by setting the second product $\rhd$ to be the Lie bracket itself. In above definition, if $[.,.]$ is commutative then the post-Lie identities are reduced to pre-Lie property. Next, some post-Lie properties have been presented by H. Munthe-Kass and A. Lundervold.

Proposition 1. [5] *The post-Lie* $[.,.]$, $\rhd$ *operations introduced new products of* $\mathcal{L}$ *as:*

$$[\![x,y]\!] := x \rhd y - y \rhd x + [x,y], \tag{4}$$

$$x \blacktriangleright y := x \rhd y + [x,y], \qquad (for\ all\ x, y \in \mathcal{L}) \tag{5}$$

Then the bracket $[\![.,.]\!]$ *is Lie, and the triple* $(\mathcal{L}, -[.,.], \blacktriangleright)$ *is post-Lie.*
Proof. See [Propositions 2.5, 2.6, 5].

2.1. Free post-Lie Algebras

A tree presentation in the pre-Lie case has been introduced by Chapoton and Livernet in [4]. Munthe-Kass and Lundervold, in [5], followed a similar tree strategy with the post-Lie (in the planar case), we review here their joint work.

Define a magma to be a set E endowed with $*$ binary operation, which has not any property. For any (non-empty) set E, the free magma over E collects words obtained by combining letters, in E, each with other by the concatenation. A concrete presentation of the free magma over a set E that ones defined by the rooted trees: take the set T_{pl}^{E} of all E-decorated planar rooted trees, and let $\searrow$ be the (left) Butcher product defined on T_{pl}^{E} as:

$$\sigma \searrow \tau = B_{+}(\sigma\, \tau_1\, \tau_2 \ldots \tau_k), \tag{6}$$

for each $\sigma, \tau_2, \ldots, \tau_k \in T_{pl}^{E}$, such that $\tau = B_{+}(\tau_1\, \tau_2 \ldots \tau_k)$, where B_{+} is the operator which grafts a monomial $\tau_1\, \tau_2 \ldots \tau_k$ on a choosing vertex called the root, decorated by some r in E to obtain a new tree. For example (in the undecorated context):

Denote by $\mathcal{T}_{pl}^{E}$ the linear span of the set T_{pl}^{E}. This space has two structures of magmatic algebras constructed by $\circ\!\!\searrow$ and $\searrow$, where $\searrow$ is defined by:

$$\sigma \searrow \tau = \sum_{v \; vertex \; of \; \tau} \sigma \searrow_{v} \tau \, ,$$

where $\sigma \searrow_{v} \tau$ is the tree produced by the left attracting of the tree σ, on a vertex v in τ [6, 9]. Below, undecorated example:

Call $\mathcal{L}(\mathcal{T}_{pl}^{E})$ the free Lie algebra spanned by $\mathcal{T}_{pl}^{E}$, more details exist in [3]. The left Butcher and left grafting products by $\circ\!\!\searrow$, $\searrow$ supply $\mathcal{L}(\mathcal{T}_{pl}^{E})$ by two structures of free post-Lie algebras, as in following theorem and its corollary.

Theorem 2. [5] *$\mathcal{L}(\mathcal{T}_{pl}^{E})$ equipped with the extension of the left grafting $\searrow$ defined on $\mathcal{T}_{pl}^{E}$ by:*

$$\sigma \searrow [\tau, \tau'] = [\sigma \searrow \tau, \tau'] + [\tau, \sigma \searrow \tau'] \tag{7}$$
$$[\sigma, \tau] \searrow \tau' = a_{\searrow}(\sigma, \tau, \tau') - a_{\searrow}(\tau, \sigma, \tau') \, , \tag{8}$$

for all $\sigma, \tau, \tau' \in \mathcal{L}(\mathcal{T}_{pl}^{E})$, is the free post-Lie algebra.

Proof. See [Proposition 3.1, Theorem 3.2, 5].

Corollary 3. *The left Butcher grafting $\circ\!\!\searrow$ can be extended to post-Lie product on $\mathcal{L}(\mathcal{T}_{pl}^{E})$.*

From the (unpublished) work for Dominique Manchon and Ebrahimi-Fard Kuruch, detailed in [8, 9], about the magmatic algebra isomorphism Ψ defined between the two magmatic algebras$(\mathcal{T}_{pl}^{E}, \circ\!\!\searrow)$ and $(\mathcal{T}_{pl}^{E}, \searrow)$ by:

$$\Psi: \; (\mathcal{T}_{pl}^{E}, \circ\!\!\searrow) \longrightarrow (\mathcal{T}_{pl}^{E}, \searrow)$$
$$\Psi(\bullet) = \bullet, \text{ and for any } \sigma, \tau \in \mathcal{T}_{pl}^{E}, \Psi(\sigma \circ\!\!\searrow \tau) = \Psi(\sigma) \searrow \Psi(\tau) \tag{9}$$

Now, we can find an extension of Ψ mapping, as in the following proposition.

Proposition 4. *There is an isomorphism between the corresponding free post-Lie algebras* $\left(\mathcal{L}(\mathcal{T}_{pl}^{E}), [\,.\,,.\,], \circ\!\!\searrow\right)$ and $\left(\mathcal{L}(\mathcal{T}_{pl}^{E}), [\,.\,,.\,], \searrow\right)$.

Proof. Using the isomorphismΨdefined in (9) above, we can define an extension of Ψ, call it $\widetilde{\Psi}$, described in Figure 1 below, such that for any $\sigma_1, \sigma_2 \in \mathcal{T}_{pl}^{E}$, and for $\sigma = \sigma_1 \circ\!\!\searrow \sigma_2, \tau \in \mathcal{L}(\mathcal{T}_{pl}^{E}), \widetilde{\Psi}$ satisfies:

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$$\begin{tikzcd}
(\mathcal{T}_E^{pl}, \circ\!\!\searrow) \arrow[r, "\Psi"] \arrow[d, hook, "i"] & (\mathcal{T}_E^{pl}, \searrow) \arrow[d, hook, "i"] \\
(\mathcal{L}(\mathcal{T}_E^{pl}), [\,.\,,\,.\,], \circ\!\!\searrow) \arrow[r, "\widetilde{\Psi}"] & (\mathcal{L}(\mathcal{T}_E^{pl}), [\,.\,,\,.\,], \searrow)
\end{tikzcd}$$

Figure 1. Post-Lie Algebra Isomorphism

$$\widetilde{\Psi}([\sigma,\tau]) = [\Psi(\sigma), \Psi(\tau)],$$

$$\widetilde{\Psi}(\sigma) = \Psi(\sigma) = \Psi(\sigma_1) \searrow \Psi(\sigma_2).$$

One can note that $\widetilde{\Psi}$ is isomorphism of post-Lie algebras.

2.2. From free Lie to free post-Lie algebras

Let $\mathcal{L}(E)$ be the free Lie algebra graded by a generating set E as:

$$E = \sqcup_{n\in N} E_n,$$

where E_n is the subset of all elements $r_1^{(n)}, r_2^{(n)}, \dots, r_{d_n}^{(n)}$ of E of degree n, such that its cardinal number is $\#E_n = d_n, \forall n \in N$. Then $\mathcal{L}(E)$ is written as:

$$\mathcal{L}(E) := \bigoplus_{\mathrm{n}\in\mathrm{N}} \mathcal{L}_n, \tag{10}$$

$\mathcal{L}_n$ is the homogeneous component of $\mathcal{L}$ which contains all elements of degree n, such that $\mathrm{E_n}$ is a subset of $\mathcal{L}_n$[3]. Supplying the free Lie algebra $\mathcal{L}(E)$ by a binary operation $\rhd$ that satisfies the post-Lie properties indicated in (2) and (3).

Here, we present important results that it linked between the freeness properties of the Lie and post-Lie algebras respectively.

Lemma 5. *There is a post-Lie homomorphism between* $\left(\mathcal{L}(\mathcal{T}_{pl}^E), [\,.\,,.\,], \searrow\right), (\mathcal{L}(E), [\,.\,,.\,], \rhd)$.

Proof. Since the magmatic algebras $\left(\mathcal{T}_{pl}^E, \circ\!\!\searrow\right), \left(\mathcal{T}_{pl}^E, \searrow\right)$ respectively have isometric structures of free post-Lie algebras described in Proposition 4 above, and by the freeness property of post-Lie algebras explained (for more details about the freeness property of post-Lie algebras see [Theorem 3.2, 5]) by the Figure 2 below, we define the mapping $\widetilde{\Psi}$ by:

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$$\begin{tikzcd} \overline{E} \arrow[r, hook, "i"] \arrow[dr, "f"'] & (\mathcal{L}(\mathcal{T}_{pl}^E), [\,.\,,.\,], \circ\!\searrow) \arrow[r, "\widetilde{\Psi}"] & (\mathcal{L}(\mathcal{T}_{pl}^E), [\,.\,,.\,], \searrow) \arrow[dl, "\widetilde{\Phi}"] \\ & (\mathcal{L}(E), [\,.\,,.\,], \rhd) & \end{tikzcd}$$

Figure 2. Free Post-Lie Algebra Homomorphism

where $\bar{E}$ is the set of all generators $\overset{r}{\bullet}$ in T_{pl}^E decorated by the elements of E, and $\widetilde{\Phi}$ is defined by:

$$\widetilde{\Phi}(\overset{r}{\bullet}) = \Phi(\overset{r}{\bullet}) = r, \forall r \in E, \text{ where } |\overset{r}{\bullet}| = |r|,$$

$$\widetilde{\Phi}([\sigma, \tau]) = [\Phi(\sigma), \Phi(\tau)], \forall \sigma, \tau \in \mathcal{L}(\mathcal{T}_{pl}^E),$$

$$\widetilde{\Phi}(\sigma_1 \searrow (\sigma_2 \searrow (\cdots \searrow (\sigma_k \searrow \overset{r}{\bullet}) \cdots))) = x_1 \rhd (x_2 \rhd (\cdots \rhd (x_k \rhd r) \cdots)),$$

for $\Phi(\sigma_i) = x_i$, and $|\sigma_i| = |x_i|$, $\forall i = 1, 2, \dots, k$, where Φ is the Lie isomorphism explained in Proposition 6 below. By construction, $\widetilde{\Phi}$ is a post-Lie homomorphism.

Proposition 6. *$\widetilde{\Phi}$ is an extension of the Lie isomorphism between the two free Lie algebras$\big(\mathcal{L}(\mathcal{T}_{pl}^E), [\,.\,,.\,]\big)$ and $(\mathcal{L}(E), [\,.\,,.\,])$.*

Proof. The free Lie algebra $\mathcal{L}(\mathcal{T}_{pl}^E)$ constructed by the planar rooted trees by taking two-sided ideal I of $\mathcal{T}_{pl}^E$ spanned by the Jacobi property, for the left grafting $\searrow$, and all forms below:

$$|\sigma|\ \sigma \searrow \tau + |\tau| \tau \searrow \sigma, \ \ for\ all\ \sigma, \tau \in T_{pl}^E,$$

$\mathcal{L}(\mathcal{T}_{pl}^E) = \mathcal{T}_{pl}^E / I$ has a structure of free Lie algebra, see [Proposition 4.2, 8]. This free Lie algebra is isomorphic uniquely to $\mathcal{L}(E)$, by Φ showed in Figure 3 below (more details about this isomorphism are explained in [Corollary 4.7, 8]):

$$\begin{tikzcd} \overline{E} \arrow[r, hook, "i"] \arrow[dr, "f"'] & (\mathcal{T}_{pl}^E, \searrow) \arrow[r, hook, "i"] & (\mathcal{L}(\mathcal{T}_{pl}^E), [\,.\,,.\,]) \arrow[dl, "\Phi"] \\ & (\mathcal{L}(E), [\,.\,,.\,]) & \end{tikzcd}$$

Figure 3. Free Lie Algebra Isomorphism

$\widetilde{\Phi}$ is an extension of Φ, by its construction.

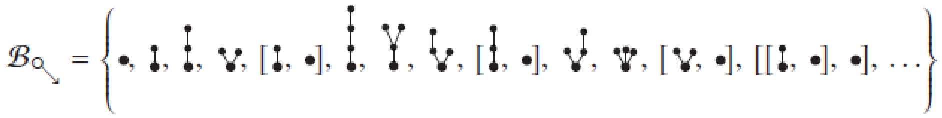
Theorem 7. *The mapping $\widetilde{\Phi}$, described in Lemma 5 above, is a unique post-Lie isomorphism.*

Proof. Indeed, any element x in $\mathcal{L}(E)$ is written, in a unique way, as a parenthesize of some generators of a's of E with the operation $\rhd$, and corresponding to this expression of x there is an element σ (a tree or linear combination of tress) is written, in the same way of x, as parenthesize of the generators $\bullet^{r}$'s with the left grafting $\searrow$, such that $\Phi(\sigma) = x$, hence Φ is a surjective. Moreover, since $Ker\,\Phi = \{\sigma \in \mathcal{L}(\mathcal{T}_{pl}^{E}) \mid \Phi(\sigma) = 0\} = \{\phi\}$, where ϕ is the empty tree. Indeed, for any $\sigma \in \mathcal{L}(\mathcal{T}_{pl}^{E})$, if $\Phi(\sigma) = 0$, then $|\sigma| = |0| = 0$ and then $\sigma = \phi$. Thus Φ is an injective mapping. The uniqueness of $\widetilde{\Phi}$ is induced by that one satisfied by Φ in Proposition 6 above.

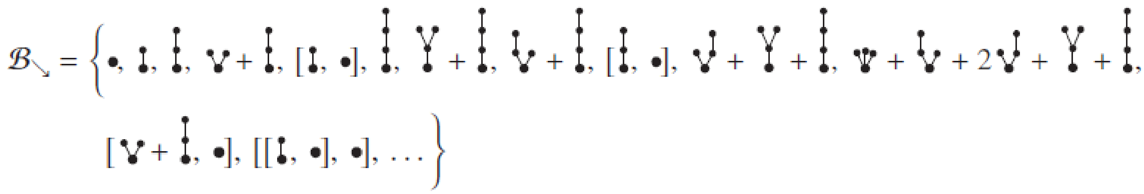
Remark 8. *The composition* $\widetilde{\Phi} \circ \widetilde{\Psi}$, *in Figure* 2, *is also a post-Lie algebra isomorphism.*

3. Monomial Bases For Free Post-Lie Algebras

The free post-Lie algebra $(\mathcal{L}(\mathcal{T}_{pl}^{E}), [\,.\,,.\,], \circ\!\!\searrow)$ has a special kind of bases called *Lyndon basis*. In the case of one generator $= \{\bullet\}$, this basis is described as [5]:



Hence, using the isomorphism $\widetilde{\Psi}$ described in Figures 1 and 2, one can define a basis for the free post-Lie algebra $(\mathcal{L}(\mathcal{T}_{pl}^{E}), [\,.\,,.\,], \searrow)$, by considering the image of the basis above by $\widetilde{\Psi}$. The first elements are:



These two bases, above, are considered as monomial bases for the free post-Lie algebras $\mathcal{L}(\mathcal{T}_{pl}^{E})$, in the case of one generator $\bullet$, with respect to the graftings $\circ\!\!\searrow$, $\searrow$ respectively.

Here, we calculate monomial bases for the free post-Lie algebra $\mathcal{L}(E)$, using the isomorphism $\widetilde{\Phi}$ described in Theorem 7 above, as in the following cases:

1. If $E = \{r\}$ is a singleton set, then the monomial basis for $\mathcal{L}(E)$ is:

$$\mathcal{B}_{\rhd} = \{r, r \rhd r, (r \rhd r) \rhd r, r \rhd (r \rhd r), [r \rhd r, r], ((r \rhd r) \rhd r) \rhd r, (r \rhd (r \rhd r)) \rhd r, (r \rhd r) \rhd (r \rhd r), [(r \rhd r) \rhd r, r], r \rhd ((r \rhd r) \rhd r), r \rhd (r \rhd (r \rhd r)), [r \rhd (r \rhd r), r], [[r \rhd r, r], r], \ldots\}$$

In this case, the dimension to each homogeneous subspace $\mathcal{L}_n$ is calculated by the following formula [Proposition 3.3, 5]:

$$\dim(\mathcal{L}_n) = \frac{1}{2n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{2d}{d} n, \qquad (11)$$

where μ is the Möbius function. The numerical sequence of the dimensions of $\mathcal{L}_n$, for $n = 1, 2, 3, 4, \ldots$, is the sequence $A022553 : 1, 1, 3, 8, 25, 75, 245, \ldots$ [10].

2. If $E = \{r_n : n \in N\}$, such that $|r_n| = n, \forall n \in N$, and r_n's are ordered as $r_1 < r_2 < \cdots < r_n < \cdots$. Then the first four elements of the monomial basis for $\mathcal{L}(E)$ are:

$$\mathcal{B}_{\rhd,\mathbf{1}} = \{r_1\},$$

$$\mathcal{B}_{\rhd,\mathbf{2}} = \{r_2, r_1 \rhd \mathrm{r}_1\},$$

$$\mathcal{B}_{\rhd,\mathbf{3}} = \{r_3, [r_1, r_2], r_1 \rhd r_2, r_2 \rhd r_1, [r_1, r_1 \rhd r_1], r_1 \rhd (r_1 \rhd r_1), (r_1 \rhd r_1) \rhd r_1\}$$

$$\begin{aligned}\mathcal{B}_{\rhd,4} = \{&r_4, [r_1, r_3], r_1 \rhd r_3, r_2 \rhd r_2, r_3 \rhd r_1, [r_1, r_1 \rhd r_2], [r_1, r_2 \rhd r_1], [r_2, r_1 \rhd r_1],\\ &\big[[r_1, r_2], r_1\big], r_1 \rhd (r_1 \rhd r_2), r_1 \rhd (r_2 \rhd r_1), r_2 \rhd (r_1 \rhd r_1), (r_1 \rhd r_1) \rhd r_2,\\ &(r_1 \rhd r_2) \rhd r_1, (r_2 \rhd r_1) \rhd r_1, \big[[r_1, r_1 \rhd r_1], r_1\big], [r_1, (r_1 \rhd r_1) \rhd r_1], [r_1, r_1 \rhd (r_1 \rhd r_1)],\\ &\big((r_1 \rhd r_1) \rhd r_1\big) \rhd r_1, \big(r_1 \rhd (r_1 \rhd r_1)\big) \rhd r_1, (r_1 \rhd r_1) \rhd (r_1 \rhd r_1), r_1 \rhd ((r_1 \rhd r_1) \rhd r_1),\\ &r_1 \rhd (r_1 \rhd (r_1 \rhd r_1))\}\end{aligned}$$

3. In general, if $E = \sqcup_{n \in N} E_n$, where E_n is the subset of all elements $r_1^{(n)}$, $r_2^{(n)}, \ldots, r_{d_n}^{(n)}$ of E of degree n, such that its cardinal number is $\# E_n = d_n$, $, \forall n \in N$. Then the monomial bases for the homogeneous components $\mathcal{L}_n$, up to order four, of the free post-Lie algebra $\mathcal{L}(E)$ are:

$$\mathcal{B}_{\rhd,\mathbf{1}} = \left\{r_i^{(1)} \in E_1, \forall\, i = 1, 2, \ldots, d_1\right\} = E_1,$$

$$\begin{aligned}\mathcal{B}_{\rhd,\mathbf{2}} = E_2 \sqcup &\left\{\left[r_i^{(1)}, r_j^{(1)}\right] \mid r_i^{(1)}, r_j^{(1)} \in E_1, \forall\, i, j = 1, 2, \ldots, d_1, such\ that\ i \lneqq j\right\}\\ &\sqcup \left\{r_i^{(1)} \rhd r_j^{(1)} \mid r_i^{(1)}, r_j^{(1)} \in E_1, \forall\, i, j = 1, 2, \ldots, d_1\right\},\end{aligned}$$

$$\begin{aligned}&\mathcal{B}_{\rhd,\mathbf{3}} = E_3 \sqcup \left\{\left[r_i^{(1)}, r_j^{(2)}\right] \mid r_i^{(1)} \in E_1, r_j^{(2)} \in E_2, \forall\, i = 1, 2, \ldots, d_1,\ \forall\, j = 1, 2, \ldots, d_2\right\}\\ &\sqcup \left\{\left[r_i^{(1)},\ r_j^{(1)} \rhd\ r_k^{(1)}\right],\ \left(r_i^{(1)} \rhd r_j^{(1)}\right) \rhd r_k^{(1)},\ r_i^{(1)} \rhd (r_j^{(1)} \rhd r_k^{(1)}) \mid r_i^{(1)},\ r_j^{(1)},\ r_k^{(1)} \in\right.\\ &\left.E_1, \forall\, i, j, k = 1, 2, \ldots, d_1\right\} \sqcup \left\{\left[\left[r_i^{(1)}, r_j^{(1)}\right],\ r_k^{(1)}\right] \mid \forall\, r_i^{(1)}, r_j^{(1)},\ r_k^{(1)} \in E_1, \forall\, i, j, k =\right.\\ &\left.1, 2, \ldots, d_1, such\ that\ i \lneqq j\right\},\end{aligned}$$

$$\mathcal{B}_{\triangleright,4} = E_4 \sqcup \{[r_i^{(1)}, r_j^{(3)}], r_i^{(1)} \triangleright r_j^{(3)}, r_k^{(2)} \triangleright r_l^{(2)}, r_j^{(3)} \triangleright r_i^{(1)} \mid r_i^{(1)} \in E_1, r_k^{(2)}, r_l^{(2)} \in E_2, r_j^{(3)} \in E_3, \forall i = 1,2,\ldots,d_1, \forall k,l = 1,2,\ldots,d_2, \forall j = 1,2,\ldots,d_3\} \sqcup \{[[r_i^{(1)}, r_j^{(1)}], r_k^{(2)}] \mid r_i^{(1)}, r_j^{(1)} \in E_1, r_k^{(2)} \in E_2, \forall i,j = 1,2,\ldots, d_1, \forall k = 1,2,\ldots,d_2, \text{ such that } i \lneqq j\} \sqcup \{[[r_i^{(1)}, r_j^{(2)}], r_k^{(1)}], [r_i^{(1)} \triangleright r_k^{(1)}, r_j^{(2)}], [r_i^{(1)} \triangleright r_j^{(2)}, r_k^{(1)}], [r_j^{(2)} \triangleright r_i^{(1)}, r_k^{(1)}], (r_i^{(1)} \triangleright r_j^{(2)}) \triangleright r_k^{(1)}, (r_i^{(1)} \triangleright r_k^{(1)}) \triangleright r_j^{(2)}, (r_j^{(2)} \triangleright r_i^{(1)}) \triangleright r_k^{(1)}, r_i^{(1)} \triangleright (r_k^{(1)} \triangleright r_j^{(2)}), r_i^{(1)} \triangleright (r_j^{(2)} \triangleright r_k^{(1)}), r_j^{(2)} \triangleright (r_i^{(1)} \triangleright r_k^{(1)}) \mid r_i^{(1)}, r_k^{(1)} \in E_1, r_j^{(2)} \in E_2, \forall i,k = 1,2,\ldots,d_1, j = 1,2,\ldots,d_2\} \sqcup \{[[[r_i^{(1)}, r_j^{(1)}], r_k^{(1)}], r_l^{(1)}] \mid r_i^{(1)}, r_j^{(1)}, r_k^{(1)}, r_l^{(1)} \in E_1, \forall i,j,k,l = 1,2,\ldots,d_1, \text{ such that } i \lneqq j\} \sqcup \{[[r_i^{(1)}, r_j^{(1)}], [r_k^{(1)}, r_l^{(1)}]] \mid r_i^{(1)}, r_j^{(1)}, r_k^{(1)}, r_l^{(1)} \in E_1, \forall i,j,k,l = 1,2,\ldots,d_1, \text{ such that } i \lneqq j, k \lneqq l, i+j \lneqq k+l\} \sqcup \{[r_i^{(1)} \triangleright r_j^{(1)}, r_k^{(1)} \triangleright r_l^{(1)}] \mid r_i^{(1)}, r_j^{(1)}, r_k^{(1)}, r_l^{(1)} \in E_1, \forall i,j,k,l = 1,2,\ldots,d_1, \text{ such that } r_i^{(1)} \triangleright r_j^{(1)} \neq r_k^{(1)} \triangleright r_l^{(1)}\} \sqcup \{[[r_i^{(1)}, r_j^{(1)}], r_k^{(1)} \triangleright r_l^{(1)}] \mid r_i^{(1)}, r_j^{(1)}, r_k^{(1)}, r_l^{(1)} \in E_1, \forall i,j,k,l = 1,2,\ldots,d_1, i \lneqq j\} \sqcup \{[[r_i^{(1)} \triangleright r_j^{(1)}, r_k^{(1)}], r_l^{(1)}], [(r_i^{(1)} \triangleright r_j^{(1)}) \triangleright r_k^{(1)}, r_l^{(1)}], [r_i^{(1)} \triangleright (r_j^{(1)} \triangleright r_k^{(1)}), r_l^{(1)}], ((r_i^{(1)} \triangleright r_j^{(1)}) \triangleright r_k^{(1)}) \triangleright r_l^{(1)}, (r_i^{(1)} \triangleright (r_j^{(1)} \triangleright r_k^{(1)})) \triangleright r_l^{(1)}, (r_i^{(1)} \triangleright r_j^{(1)}) \triangleright (r_k^{(1)} \triangleright r_l^{(1)}), r_i^{(1)} \triangleright ((r_j^{(1)} \triangleright r_k^{(1)}) \triangleright r_l^{(1)}), r_i^{(1)} \triangleright (r_j^{(1)} \triangleright (r_k^{(1)} \triangleright r_l^{(1)})) \mid r_i^{(1)}, r_j^{(1)}, r_k^{(1)}, r_l^{(1)} \in E_1, \forall i,j,k,l = 1,2,\ldots,d_1\}$$

The dimensions of the homogeneous component $\mathcal{L}_n$ subspaces, up to order four, described in (3) above, are:

$$dim(\mathcal{L}_1) = d_1, dim(\mathcal{L}_2) = d_2 + \frac{3}{2} d_1^2 - \frac{1}{2} d_1, \; dim(\mathcal{L}_3) = d_3 + \frac{7}{2} d_1^3 + 3d_1d_2 - \frac{1}{2} d_1^2$$

$$dim(\mathcal{L}_4) = d_4 + \frac{77}{8} d_1^4 + 3d_1d_3 + d_2^2 + \frac{21}{2} d_1^2 d_2 - \frac{5}{4} d_1^3 - \frac{5}{8} d_1^2 - \frac{1}{2} d_1 d_2 + \frac{1}{4} d_1.$$

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