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Pre-Stability of Fixed Point

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Pre-Stability of Fixed Point

Abedal-Hamza Mahdi Haza[(1)], Ameer M. Sahi[(2)]

[1,2]Departement of Mathematics, Faculty of Computer Sciences and Mathematics, University of Kufa,Iraq.

[(1)] abdulhamza.hamza@uokufa.edu.iq [(2)] ameersahi89@gmail.com

Abstract. In this paper, we introduced certain types of stability of the fixed points in discrete dynamical systems which are pre-stability, pre-c-stability, and pre-ic-stability. We studied the relationships among these types of stability, also the relationships among these types of stability and certain types of stability which are stability, c-stability, and ic-stability

Keywords: pre-open , Fixed point, Pre-stable fixed point , Orbit , and dynamical System.

1. Introduction

A discrete dynamical system consists of a non-empty set X which is called the phase space and compositions $f^t, t \in N$ of a map $\mathrm{f}: \mathrm{X} \to \mathrm{X}$ where $\mathrm{f}^{\mathrm{t}} = \mathrm{f} \circ \mathrm{f} \circ \ldots \circ \mathrm{f}$ (n-times). These iterates form a group or semi group. A dynamical system could be a measure space and a function that preserves measure; a metric space with an isometry; or a topological space and a continuous function, etc. In this paper, we considered phase spaces which are topological spaces [3]. A strong concept of stability for dynamical system was first formulated by N.E. Zhukovskii [10] . He introduced in 1882 a strong orbital stability concept which is basisd on a reparametrisation of the time variable [11]. On the 12 October 1892 (by modern calendar) Alexander Mikhailoich Lyapunov defined his doctoral thesis the general problem of the stability of motion (at Moscow university) [13]. Lyapunov defined a fixed point x_0 to be stable if for every neighborhood U of x_0, there is a neighborhood $V \subseteq U$ such that every solution $x(t)$ starting in V $(x(t))$ remains in U for all $t \geq 0$. Otherwise, x_0 is unstable [12]. In 2014 Mohammed F. Al-Ali and A.M. Hamza introduced and studied new types of stability which are c-stability and ic-stability of the fixed points [9].

In this paper, τ^p, R, Z , N, $\overline{U^\circ}$ and $\overset{\circ}{\overline{U}}$ will denote the family of (p-o) sets, the set of real numbers, the set of integer numbers, the set of non-negative integers, the closure of the interior of U and the interior of the closure of U, respectively. For any non-empty set X, we denote by τ_u, τ_d, τ_{ind} and τ_c, the usual topology on R,the discrete topology, the indiscrete topology and the cofinite topology respectively. Finally we denote by A^c *and* $O(x)$, the complement of the set A and the orbit of . We used space, map, and DDS to refer to a topological space, continuous function and discrete dynamical system, respectively.

2. Preliminaries

2.1 Definition [3]

A DDS consists of a phase space X and iterates f^t, where t belong to N of a map $f: \mathrm{X} \to \mathrm{X}$, the nth iterate of f is the t-fold composition $f^t = f \circ f \circ \ldots \circ f$; we define f^0 to be the identity map. If f satisfy the invertible properties then $f^{-\mathrm{t}} = f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}$ (n times). Since $f^{\mathrm{t+m}} = f^{\mathrm{t}} \circ f^{\mathrm{m}}$, these iterates form a group if f is invertible, and semi group otherwise.

Although we have defined DDS in a completely abstract setting, where X is simply a set, in practice X usually has additional structure that is preserved by the map f. For example, (X, f) could be a measure space and a measure-preserving map; a space and a continuous map; a metric space and an isometry; or a smooth manifold and a differentiable map.

2.2 Definition [1]
Let (X, τ) be a space, and $f: X \to X$ be a function. A point $x \in X$ is said to be fixed point of f if $f(x) = x$.

2.3 Definition [1]
Let (X, τ) be a space, and $f: X \to X$ be a map. For all $x \in X$, the orbit of x under f is the set $\{x, f(x), f^2(x), \dots, f^n(x), \dots\}$, and it is denoted by $O(x)$, where $O(x) \subseteq X$.

2.4 Definition [5]
A subset A of a space X is called a pre-open (p-o) set if and only if $A \subseteq \overline{A}^{\circ}$. A is called a pre-closed if and only if A^c is (p-o) and it's easy to see that A is pre-closed if and only if $\overline{\mathring{A}} \subseteq A$.

2.5 Remark [6]
If A is a dense subset in *X*, Then it is a (p-o) set.

2.6 Theorem [7]

Let X be a space. If A is a (p-o) set in X, Then $A = U \cap B$, where U is an open set in X and B is a dense set in X.

2.7 Theorem [8]

The arbitrary union of (p-o) set is also (p-o) .

2.8 Definition [1]

Let (X, τ) be a space, $f: X \to X$ be a map , $x_0 \in X$ is called stable if for every open set $U \subseteq X$ containing x_0, there exists an open set $V \subseteq U$ containing x_0 such that $O(x) \subseteq U$, $\forall x \in V$.

Otherwise , x_0 is called unstable fixed point.

2.9 Theorem [9]

Let (X, τ) be a space, B_τ is a basis for τ, $f: X \to X$ be a map, and x_0 be a fixed point of f If x_0 is stable point with respect to B_τ, then x_0 is stable point with respect to τ.

2.10 Definition [9]

Let (X, τ) be a space, $f: X \to X$ be a map. A fixed point x_0 of f is called c-stable if for any open set U containing x_0, there exists an open set $V \subseteq U$ containing x_0 such that, $O(x) \subseteq \overline{U}$, $\forall x \in V$.

Otherwise, we say that x_0 is not c-stable fixed point.

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2.11 Theorem [9]

Let (X,τ) be a space, B_τ is a basis for τ, $f:\mathrm{X}\to\mathrm{X}$ be a map and x_0 is a fixed point of f. If x_0 is c-stable with respect to B_τ, then x_0 is c-stable with respect to τ.

2.12 Example

Consider the space $(\mathrm{R},\tau_\mathrm{u})$, and $f:\mathrm{R}\to\mathrm{R}$ be a function defined by $\mathrm{f(x)}=\frac{1}{3}x$. The DDS is $\left\{\left(\frac{1}{3}\right)^\mathrm{n} x\right\}_{\mathrm{n}\in\mathrm{N}}$, and 0 is the fixed point of f. $B_{\tau_u}=\{\,(a,b);\ a,b\in R\}$ is a basis for τ_u.

Let $\mathrm{U}=(x_0,x_1)\in\mathrm{B}_{\tau_\mathrm{u}}$, where $0\in\mathrm{U}$. Choose $\mathrm{V}=(-\mathrm{a},\mathrm{a})\in\mathrm{B}_{\tau_\mathrm{u}}$, where $0\in\mathrm{V}\subseteq\mathrm{U}$,

$\mathrm{a}=\min\{|x_0|,x_1\}$. Note that $\mathrm{O(x)}\subseteq\mathrm{V}\subseteq\overline{\mathrm{U}},\ \forall\mathrm{x}\in\mathrm{V}$. Then, 0 is c-stable.

2.13 Example

Consider the space (R,τ_u) and $f: R\to R$ is the function defined by $f(x)=-5x$.

$\mathrm{B}_{\tau_u}=\{\,(a,b);\ a,b\in R\}$ is a basis for τ_u.. The DDS is$\{(-5)^n x\}n\in N$, and 0 is the fixed point of f.

Let $U=(-1,1)\in B_{\tau_u}$. Note that, for any open subset $V$of U containing 0 , and for any $x\in V$, $O(x)\nsubseteq U$.

Hence, 0 is not c-stable fixed point.

2.14 Theorem [9]

Let (X,τ) be a space, $f: X\to X$ be a map and x_0 is a fixed point of f. If x_0 is stable, then it is c-stable.

2.15 Definition [9]

Let (X,τ) be a space, $f:\mathrm{X}\to\mathrm{X}$ be a map. $\mathrm{x}_0\in\mathrm{X}$ is called ic-stable if for every open set $\mathrm{U}\subseteq\mathrm{X}$ containing x_0, there exists an open set $\mathrm{V}\subseteq\mathrm{U}$ containing x_0 such that, $\mathrm{O(x)}\subseteq\overset{\circ}{\overline{\mathrm{U}}},\ \forall\mathrm{x}\in\mathrm{V}$.

Otherwise, we say that x_0 is not ic-stable fixed point.

2.16 Theorem [9]

Let (X,τ) be a space, B_τ is a basis for τ, $f:\mathrm{X}\to\mathrm{X}$ be a map and x_0 is a fixed point of f. If x_0 is ic-stable with respect to B_τ, then x_0 is ic-stable with respect to τ.

2.17 Theorem [9]

Let (X,τ) be a space, $f: X\to X$ be a map and x_0 is a fixed point of f. If

i- x_0 is stable, then it is ic-stable.

ii- x_0 is ic-stable, then it is c-stable.

3. Main Results

3.1 Definition

Let (X, τ) be a space in a DDS $\{f^n\}_{n\in N}$, and let x_0 be a fixed point of f. We say that x_0 is pre-stable if for any (p-o) set $U \subseteq X$ containing x_0, there exists a (p-o) set $V \subseteq U$ containing x_0 such that $O(x) \subseteq U;\ \forall x \in V$.

Otherwise, x_0 is called not pre-stable fixed point.

3.2 Example

Consider the space (R, τ), $\tau = \{R, \emptyset, Z, R\backslash Z\}$ and $f : R \to R$ is the function defined by

$$f(x) = \begin{cases} x^2, & x \in Z \\ x+1, & o.w \end{cases}.$$

The fixed points of f are 0 and 1.

0 is pre-stable:

Let U be any (p-o) set containing 0. Choose $V = \{0\} \subseteq U$. V is (p-o) subset of U containing 0 with $O(0) \subseteq U$.

So, 0 is pre-stable .

Similarly, 1 is pre-stable.

3.3 Example

Consider the space(R, τ_c), and $f : R \to R$ is the function defined by $f(x) = \frac{1}{5}x$. The fixed point of f is 0, and the DDS is $\{\left(\frac{1}{5}\right)^n x\}_{n\in N}$.

$U = \{0\} \cup [10,15]$ is a (p-o) set in (R, τ_c) containing 0. Let V be any (p-o) subsets of U containing 0. $O(x) \nsubseteq U,\ \forall x \in V$.

Hence, 0 is not pre-stable fixed point.

3.4 Remark

A stable fixed point needs not be pre-stable.(Example 3.5)

3.5 Example

Let (X, τ) be a space, $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, c\}\}$ and $f: X \to X$ is the function defined by, $f(a) = c$, $f(b) = b$ and $f(c) = a$.

The fixed point of f is b and the DDS is given by the following table.

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x	$f(x)$	$f^2(x)$	$f^3(x)$	...	$f^n(x)$	...
a	c	a	c	...	$f^n(a)$	...
b	b	b	b	...	b	...
c	a	c	a	...	$f^n(c)$	...

$\tau^{\rho} = \tau \cup \{\{a\},\{c\},\{a,b\},\{b,c\}\}$

b is stable fixed point but not pre-stable:

The only open set containing b is $U = X \in \tau$.

The only open subset of U that containing b is U itself , i.e. $V = U$.

$O(x) \subseteq U, \forall x \in V$. So, b is stable fixed point.

$U = \{a,b\}$ is a (p-o) set and $b \in U$.

The only (p-o) subset of U that containing b is U it self, i.e. $V = U$.

$O(a) = \{ a, c, a, c, \ldots\} \nsubseteq$ U.

So, b is not pre-stable.

Hence, stability $\nRightarrow$ pre-stability.

In the following theorem, we shall give a condition that make stability implies pre-stability and pre-stability implies stability.

3.6 Theorem

Let (X,τ) be a space, $\{f^n\}_{n\in N}$ be a DDS with a fixed point x_0 such that every open set containing x_0. Then x_0 is pre-stable if and only if it is stable.

$\Rightarrow$ **Proof:** Let x_0 be a pre-stable fixed point and U be any open set containing x_0. Then U is (p-o) set and $x_0 \in U$, so there exists (p-o) set V; $x_0 \in V \subseteq U$, and $O(x) \subseteq U,\ \forall x \in V$.

V° is open set containing x_0 and $V^\circ \subseteq V \subseteq U$ with $O(x) \subseteq U; \forall x \in V^\circ$.

Hence, x_0 is stable .

$\Leftarrow$ **Proof** : Let U be any (p-o) set containingx_0. Note that U° is open set. Since x_0 is stable, then there exists an open set V , $V \subseteq U^\circ$ such that $O(x) \subseteq U^\circ,\ \forall x \in V$. Now, V is (p-o) set with $O(x) \subseteq U^\circ \subseteq U,\ \forall x \in V$.

Hence, x_0 is pre-stable fixed point. □

3.7 Theorem

Let (X,τ) be a space. In any DDS with the topology $\tau = \{X, \emptyset, U, U^c\}$, $U \subset X$, every fixed point is a pre-stable .

Proof : In such space, every non-empty subset A of X is (p-o) :

$$\overline{A} = \begin{cases} U, & if\ A \subseteq U \\ U^c, & if\ A \subseteq U^c \\ X, & if\ A = A_1 \cup A_2,\ A_1 \subseteq U\ and\ A_2 \subseteq U^c \end{cases}.$$

So,

$$\overline{A}^{\circ} = \begin{cases} U, & if\ A \subseteq U \\ U^c, & if\ A \subseteq U^c \\ X, & if\ A = A_1 \cup A_2,\ A_1 \subseteq U\ and\ A_2 \subseteq U^c \end{cases}.$$

So, in such DDS, every fixed point x_0 is pre-stable, for $V = \{x_0\}$ is a (p-o) subset of any (p-o) set U containing x_0 and $O(x_0) \subseteq U$. Hence, x_0 is pre-stable.

3.8 Theorem

If the phase space X of a DDS has a basis of pairwise disjoint basic open sets, then every fixed point is pre-stable.

Proof: Let $B = \{A_\alpha\}_{\alpha \in \Lambda}$ be a basis for the topology of the phase space X in a DDS with $A_i \cap A_j = \emptyset, \forall\, i \neq j$.

Let $x_0 \in X$ be a fixed point. Then $x_0 \in A_{\alpha_0}$, $A_{\alpha_0} \in B$.

Now, let U be any (p-o) set containing x_0. Put $A = \{x_0\}$. Then $\overline{A}^{\circ} \subseteq A_{\alpha_0}$, so A is (p-o) set. We have, $x_0 \in A \subseteq U$ with $O(x_0) \subseteq U$.

So, x_0 is pre-stable fixed point. □

3.9 Theorem

If $\{f^n\}_{n \in N}$ is a DDS with $\tau = \{X, \emptyset, A\}$, $A \subseteq X$, then any fixed point in A is pre-stable.

Proof: Let $x_0 \in A$ be a fixed point and U be any (p-o) set containing x_0 . Then $V = \{x_0\}$ is (p-o) set containing x_0 and $V \subseteq U$ with $O(x_0) \subseteq U$.

Hence, x_0 is pre-stable fixed point. □

3.10 Definition

Let (X, τ) be a space in a DDS $\{f^n\}_{n \in N}$, and let x_0 be a fixed point of f. x_0 is called pre-c-stable if for any (p-o) set $U \subseteq X$ containing x_0, there exists a (p-o) set $V \subseteq U$ containing x_0 such that $O(x) \subseteq \overline{U}$; $\forall x \in V$.

Otherwise, x_0 is called not pre-c-stable fixed point.

3.11 Example

Consider the space (R, τ_{ind}),and $f : R \to R$ is the function defined by $f(x) = 4x - 1$. The fixed point of f is $\frac{1}{3}$, and the DDS is $\{4^n x - (4^{n-1} + 4^{n-2} + \cdots + 1\}_{n \in N}$.

Let U be any (p-o) sets in (R, τ_{ind}) contains $\frac{1}{3}$. $V = \{\frac{1}{3}\}$, is a (p-o) subset of U; $\frac{1}{3} \in V$. Note that, $O(x) \subseteq \overline{U} = R, \forall\, x \in V$.

Hence, $\frac{1}{3}$ is pre-c-stable

3.12 Example

Let (X,τ) be a space and $=\{1,2,3,4\}$, $\tau=\{X,\emptyset,\{4\},\{1,3\},\{2,4\},\{1,3,4\}\}$ and $f:X\to X$ is the function defined by, $f(1)=f(3)=4$ and $f(2)=f(4)=2$.

The fixed point of f is 2 and the DDS is given by the following table.

x	$f(x)$	$f^2(x)$	...	$f^n(x)$	...
1	4	2	...	2	...
2	2	2	...	2	...
3	4	2	...	2	...
4	2	2	...	2	...

$\tau^\rho=\tau\cup\{\{1\},\{3\},\{4\},\{1,2\},\{1,4\},\{3,4\},\{1,2,3\},\{2,3,4\}\}$

The (p-o) set $U=\{1,2\}$ is containing 2 . The only (p-o) subset of U containing 2 is $V=U$. $O(1)\nsubseteq\overline{U}$.

Hence, 2 is not pre-c-stable fixed point.

3.13 Theorem

Let (X,τ) be a space , $\{f^n\}_{n\in N}$ be a DDS with a fixed point $x_\circ$ such that every open set containing $x_\circ$. Then $x_\circ$ is pre-c-stable if and only if it is c-stable.

⇒Proof: Let $x_\circ$ be a pre-c-stable fixed point and U be any open set containing $x_\circ$. Then U is (p-o) set and $x_\circ\in U$. So, there exists (p-o) set V; $x_\circ\in V\subseteq U$, and

$O(x)\subseteq\overline{U},\ \forall x\in V$.

V° is open set containing x_0 with $V^\circ\subseteq V\subseteq U$.

So, $O(x)\subseteq\overline{U}$, $\forall x\in V^\circ$.

Hence, $x_\circ$ is c-stable .

⇐ **proof** : Let U be any (p-o) set containing x_0. Note that, U° is open set containing x_0. Since x_0 is c-stable, then there exists an open set V , $V\subseteq U^\circ$ such that $O(x)\subseteq\overline{U^\circ}$, $\forall x\in V$. Now , V is (p-o) set with $O(x)\subseteq\overline{U^\circ}\subseteq\overline{U}$, $\forall x\in V$.

Hence, x_0 is pre-c-stable fixed point.

□

3.14 Theorem

Let (X,τ) be a space, $\{f^n\}_{n\in N}$ be a DDS with a fixed point x_0. If x_0 pre-stable, then it is pre-c-stable.

Proof: Let U be a (p-o) set; $x_0\in U$. Since x_0 is pre-stable, then there exists (p-o) set V ; $x_0\in V\subseteq U$ such that, $O(x)\subseteq U,\forall x\in V$. So, $O(x)\subseteq\overline{U},\forall x\in V$.

Hence, x_0 pre-c-stable. □

The converse of above theorem isn't true generally .

3.15 Example

Consider the space (R, τ_c), and $f : R \to R$ be a function define by,

$f(x) = \begin{cases} 2x, & x < 1 \\ x + 2, & x \geq 1 \end{cases}$. The fixed point of f is 0.

0 is not pre-stable fixed point. Let $U = (-7, 5)$ is a (p-o) set in (R, τ_c) containing 0 .

Let V be any (p-o) sub set of U containing 0 .Then , $O(x) \nsubseteq U$, for some $x \in V$.

Hence, 0 is not pre-stable fixed point.

But 0 is pre-c-stable :

Let U be any (p-o) set of R containing 0 . $V = (-2, 5)$ is (p-o) set and $0 \in V \subseteq U$, Thus, $O(x) \subseteq \overline{U} = R$. Hence, 0 is pre-c-stable fixed point.

3.16 Theorem

If the phase space X of a DDS has a basis of pairwise disjoint basic open sets, then every fixed point is pre-c-stable.

Proof: it is clear [Theorem 3.8] and[Theorem 3.14]. □

3.17 Definition

Let (X, τ) be a space in a DDS $\{f^n\}_{n \in N}$, and let x_0 be a fixed point of f. x_0 is called pre-ic-stable if for any (p-o) set $U \subseteq X$ containing x_0, there exists a (p-o) set $V \subseteq U$ containing x_0, such that $O(x) \subseteq \overline{U}^{\circ}$; $\forall x \in V$.

Otherwise, x_0 is called not pre-ic-stable fixed point.

3.18 Example

Let (X, τ) be a space and $= \{1, 2, 3, 4\}$, $\tau = \{X, \emptyset, \{1,3\}, \{2,4\}\}$ and $f: X \to X$ be a function defined by, $f(1) = f(3) = 1$ and $f(2) = f(4) = 3$.

The fixed point of f is 1 and the DDS is given by the following table.

x	$f(x)$	$f^2(x)$	...	$f^n(x)$	...
1	1	1	...	1	...
2	3	1	...	1	...
3	1	1	...	1	...
4	3	1	...	1	...

$\tau^{\rho} = P(X)$

Let U be any (p-o) set containing 1. $V = \{1\}$ is a (p-o) set and $1 \in V \subseteq U$ with

$O(1) \subseteq U$. So, $O(1) \subseteq \overline{U}^{\circ}$.

Hence, 1 is pre-ic-stable fixed point.

3.19 Example

Let (X,τ) be a space and $X=\{a,b,c\}$, $\tau=\{X,\emptyset,\{a\},\{c\},\{a,b\},\{b,c\}\}$ and $f: X\to X$ be a function defined by, $f(a)=f(b)=b$, $f(c)=a$.

The fixed point of f is b and the DDS is given by the following table.

x	$f(x)$	$f^2(x)$	...	$f^n(x)$	...
a	b	b	...	b	...
b	b	b	...	b	...
c	a	b	...	b	...

$\tau^p=\tau$.

Let $\mathrm{U}=\{\mathrm{b},\mathrm{c}\}$. U is a (p-o) set with $b\in U$. The only (p-o) subset of U contains the fixed point b is U itself.

$O(c)=\{c,a,b,b,\dots\}$

So, $O(c)\nsubseteq \overline{U}^{\circ}$.

Hence, b is not pre-ic-stable fixed point.

3.20 Theorem

Let (X,τ) be a space, $\{f^n\}_{n\in N}$ be a DDS with a fixed point $x_\circ$ such that every open set containing $x_\circ$. Then $x_\circ$ is pre-ic-stable if and only if it is ic-stable.

$\Rightarrow$ ***Proof*:** Let $x_\circ$ be a pre-ic-stable fixed point and U be any open set containing $x_\circ$. Then U is a (p-o) set and $x_\circ\in U$. So, there exists (p-o) set V; $x_\circ\in V\subseteq U$, and

$O(x)\subseteq \overline{U}^{\circ}$, $\forall x\in V$.

V° is open set containing x_0 with $V^\circ\subseteq V\subseteq U$.

So, $O(x)\subseteq \overline{U}^{\circ}$, $\forall x\in V^\circ$.

Hence, $x_\circ$ is ic-stable fixed point.

$\Leftarrow$ ***proof*** : Let U be any (p-o) set containing x_0. Note that, U° is open set. Since x_0 is ic-stable, then there exists an open set V *containing* x_0, $V\subseteq U^\circ$ such that $O(x)\subseteq \overline{U^\circ}^{\circ}$, $\forall x\in V$. Now , V is (p-o) set with $O(x)\subseteq \overline{U^\circ}^{\circ}\subseteq \overline{U}^{\circ}$, $\forall x\in V$.

Hence, x_0 is pre-ic-stable fixed point. □

3.21 Theorem

Let (X,τ) be a space, $\{f^n\}_{n\in N}$ be a DDS with a fixed point x_0. If x_0 pre-stable, then it is pre-ic-stable.

Proof : Let U be a (p-o) set; $x_0 \in U$. Since x_0 is pre-stable, then there exists a (p-o) set V ; $x_0 \in V \subseteq U$ such that, $O(x) \subseteq U, \forall x \in V$. Since U is (p-o) set, then $U \subseteq \overline{U}^{\circ}$, and so $O(x) \subseteq \overline{U}^{\circ}$, $\forall x \in V$.

Hence, x_0 pre-ic-stable. □

3.22 Theorem

Let (X, τ) be a space, $\{f^n\}_{n\in N}$ be a DDS with a fixed point x_0. If x_0 pre-ic-stable, then it is pre-c-stable.

Proof : Let U be a (p-o) set; $x_0 \in U$. Since x_0 is pre-ic-stable, then there exists a (p-o) set V ; $x_0 \in V \subseteq U$ such that, $O(x) \subseteq \overline{U}^{\circ}, \forall x \in V$. Since $\overline{U}^{\circ} \subseteq \overline{U}$, then $O(x) \subseteq \overline{U}, \forall x \in V$.

Then, x_0 pre-c-stable. □

3.23 Theorem

If the phase space X of a DDS has a basis of pairwise disjoint basic open sets, then every fixed point is pre-ic-stable.

Proof : it is clear [Theorem 2.8] and [Theorem 2.21] □

3.24 Theorem

If $\{f^n\}_{n\in N}$ is a DDS with $\tau = \{X, \emptyset, A\}$, $A \subseteq X$, then any fixed point in A is pre-ic-stable, and so it is pre-c-stable.

Proof: Let x_0 be a fixed point. Let $x_0 \in A$, and U be any (p-o) set containing x_0 . Then $V = \{x_0\}$ is a (p-o) set containing x_0 and $V \subseteq U$ with $O(x_0) \subseteq U$ with $O(x_0) \subseteq \overline{U}^{\circ}$.

Hence, x_0 is pre-ic-stable .

Now, if $x_0 \in \mathrm{A}^c$, then any (p-o) set containing x_0 is of the form $\mathrm{U} = \mathrm{A}^c \cup \mathrm{B}$, where $\emptyset \neq \mathrm{B} \subseteq \mathrm{A}$.

Choose $\mathrm{V} = \mathrm{A}^c \cup \{x_0\}$. Then V is a (p-o) set containing x_0, $\mathrm{V} \subseteq \mathrm{U}$.

$\mathrm{O}(x) \subseteq \overline{\mathrm{U}}^{\circ} = \mathrm{X}, \forall \mathrm{x} \in \mathrm{V}$. Hence, x_0 is pre-ic-stable. □

4. Conclusion

Certain types of stability which depend on the pre- open sets had been discussed. Since every open set is (p-o) set, so these types of stability had been discussed the stability in phase spaces in which the collection of (p-o) sets is at most finer than the collection of open sets. This means that we gave a stability in larger phase spaces..

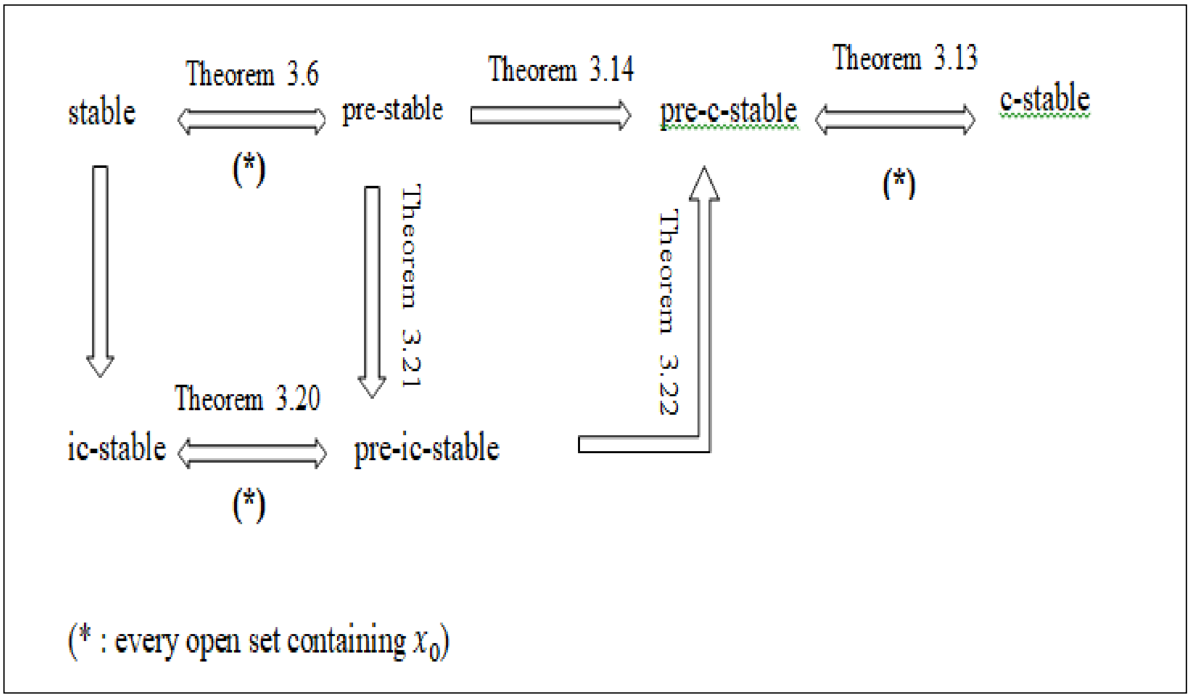

Figure (1): Relationships among certain types of pre-stability of a fixed point x_0 .

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