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Idempotent and Pure Gamma Subacts of Multiplication Gamma Acts

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Abstract.
The purpose of this article is to investigate pure gamma subacts of multiplication gamma acts and some related concepts. Some characterizations of pure gamma subacts are given . For this reason, we introduce the concept of idempotent gamma subacts and study some of its properties .Also, we discussed the relation among the pure , multiplication and idempotent gamma acts. Moreover , some result of pure gamma subacts of multiplication gamma acts are considered. Finally, we prove that the product of two pure gamma subacts is also pure.

Introduction

Let R be a commutative ring and M an left R-module. Ribenboim [1] called the submodule N a pure submodule of M if $rM \cap N = rN$ for each $r \in R$. Anderson and Fuller [2] defined N to be pure in M if $IN = N \cap IM$ for every ideal I of R. After that ,in 1988 Z. A. El-Bast and P. F. Smith [3], introduced the concept of multiplication modules(Let M be an R-module. Then M is a multiplication provided for each submodule N of M, there exists an ideal I of R such that $N = IM$) and they studied various properties about it. The concept of gamma semigroup was introduced by M.K. Sen [4] ,as a generalization of semigroup and they studied several notions of semigroups have been extended to gamma semigroups. Recently, M.S. Abbas and Abdulqader Faris [5], introduced the notion of gamma act over Γ-semigroup and they studied some of their basic properties.

Our work is to introduce pure gamma subacts of multiplication gamma acts. In Section 2, we review some basic notions and properties of gamma semigroup , and gamma acts . Also , we give some results and properties of such gamma subacts which are need in our work . In Section 3 , the concepts of idempotent and pure gamma subacts was introduced . Also, we give some characterizations and properties of such gamma subacts. For example we prove that if the gamma subact and gamma ideal are idempotent then so is their product . The relation between gamma subact and its residual is studied . Several results about this concepts are studied . Throughout this paper, our definition of gamma purity will be as a generalization to Anderson and Fuller [2] ,and S will be denote a commutative gamma semigroup with identity .

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Basic Conceptes

In this section we review some basic definitions and notions of gamma semigroup and gamma act which are need in our work.

Definition 2.1. [4] Let S and Γ be nonempty sets, S is called a gamma semigroup (denoted by Γ-semigroup)if there is a mapping: $S \times \Gamma \times S \rightarrow S$ written by $(s_1, \alpha, s_2) \longmapsto s_1\alpha s_2$ which is satisfying the condition $(s_1\alpha s_2)\beta s_3 = s_1\alpha(s_2\beta s_3)$ for all $s_1, s_2, s_3 \in S$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup S, is called commutative if $s\alpha t=t\alpha s$ for all $s,t \in S$ and $\alpha \in \Gamma$.

Definition 2.2.[4] Let S be a Γ-semigroup. An element a in S is said to be left (right)identity of if $a\alpha s = s$ ($s\alpha a = s$) for all $s \in S$ and $\alpha \in \Gamma$. An element a of a Γ-semigroup S is said to be a identity if it is both a left and right identity of S. A Γ-semigroup S with identity is called a Γ-monoid. The identity of a Γ-semigroup (if exists) is denoted by 1.

Definition 2.3.[4] Let S be a Γ-semigroup. A nonempty subset A of S is called Left (right) Γ-ideal if $S\Gamma A \subseteq N$ ($A\Gamma S \subseteq N$) such that $S\Gamma A \coloneqq \{ s\alpha a \mid a \in A , \alpha \in \Gamma \text{ and } s \in S\}$, where Γ-ideal means Left and right Γ-ideal.

Definition 2.4.[4] An Γ-ideal A of a Γ-semigroup S is said to be a maximal Γ-ideal if A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

Definition 2.5.[4] Let S be a Γ-semigroup .An element s in S is called α-idempotent if $s\alpha s = s$ for some $\alpha \in \Gamma$. If all elements of S are α –idempotent, then S is said to be an idempotent**.** An element **s** in **S** is called an idempotent if $s\alpha s = s$ for all $\alpha \in \Gamma$. If all elements of S are idempotent, then S is said to be a strongly idempotent**.** A Γ-ideal A of a Γ-semigroup S is said to be globally idempotent (gl-idempotent for short) if $A\Gamma A = A$.

Proposition 2.6.[6]Let S be a Γ-semigroup . If A and B are globally idempotent Γ-ideals of S , then AΓB is gl-idempotent Γ-ideal .

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We introduce the following definition

Definition 2.7.A Γ-semigroup S, is said to be completely gl-idempotent if every Γ-ideal of S, is gl-idempotent .

The concepts of acts over semigroups is generalized to the following in [2].

Definition 2.8.[5] Let S be Γ-semigroup. A nonempty set M is called left gamma act over S (denoted by S_Γ-act) if there is a mapping S ×Γ× M $\rightarrow$ M written (s, α, m) by sαm, such that the following condition is satisfied$(s_1\alpha s_2)\beta m = s_1\alpha(s_2\beta m)$ for all $s_1, s_2 \in S$, α, β ∈ Γ and m ∈ M. Similarity one can define a right gamma acts. From now on " S_Γ-act" means "left S_Γ-act". An S_Γ-act M is called unitary if S is a Γ-monoid and 1αm =m for all m ∈ M and α∈ Γ.

Definition 2.9.[5] Let M be an S_Γ-act. An element θ ∈ M is called a zero of M if sα θ = θ for all s ∈ S and α ∈ Γ. If S is a Γ-semigroup with zero then 0αm= θ.

Definition 2.10.[5] Let M be an S_Γ-act, A nonempty subset N of M is called S_Γ-subact if SΓN ⊆ N, Where SΓN={ sαn | s ∈ S , α ∈ Γ and n ∈ N }. For S_Γ-subact N of M, [N : M] is defined as : [N:M] = {s ∈ S | sαm ∈ N for all α ∈ Γ and m ∈ M}.

Let { N_i |i ∈ I} be an arbitrary collection of S_Γ-subacts of M .Then **,** $\bigcup_{i\in I} N_i$ is S_Γ-subact of M ,and if $\bigcap_{i\in I} N_i$ is nonempty, then $\bigcap_{i\in I} N_i$ is S_Γ-subact of M **.**Also, if N and L are S_Γ-subacts of M and A , B are nonempty subsets of M . Then**,**

1. if A ⊆ B implies that [N : B]⊆ [N :A] .

2. [N⋂L : A] = [N: A] ⋂[N : A] . [5]

For S_Γ-subact N of S_Γ-act M, it's clear to show that [N:M] is a Γ-ideal of a Γ-semigroup S.

Definition 2.11.[5] Let M be an S_Γ-act. Then M is a simple S_Γ-act, if it contain no gamma subact other than M . A Γ-semigroup S is said to be simple if S is S_Γ-act .

Definition 2.12.[5] Let M and N be two S_Γ-acts. A mapping f : M → N is called **S_Γ-**homomorphism if f(sαm) = sαf(m). for all s ∈ S, α ∈ Γ and m ∈ M.

Definition 2.13.[7] An S_Γ-act M is called a multiplication if for every S_Γ-subact N of M , there exists a Γ-ideal A of S ,such that N = AΓM. Γ-ideal A of S, is multiplication if A is S_Γ-subact of S_Γ-act S.

Clearly, M is multiplication S_Γ-act if and only if N=[N:M]ΓM for every S_Γ -subact N of M.

Examples and Remarks 2.14.

i. A Γ-monoid S, is called multiplication if all its Γ-ideals are multiplication.

ii. If S is a completely gl-idempotent Γ-semigroup, then is a multiplication .

iii. Cyclic S_Γ -acts are multiplication . [7]

iv. The nonempty intersection of two multiplication S_Γ-acts is multiplication. [7]

v. A union of two multiplication S_Γ –acts not necessary that being multiplication S_Γ –act.[7]

In [7], if M is a S_Γ-act and P a maximal Γ-ideal of S, then we define :
$T_p(M) = \{m \in M : m = p\alpha m$ for some $p \in P$ and $\alpha \in \Gamma\}$. Clearly $T_p(M)$ is an S_Γ-subact of M..
We say that M is P-cyclic provided there exist $q \in S\backslash P$ such that $q\Gamma M \subseteq S\Gamma m$, for all $m \in M$. As generalization of $T_p(M)$, we can define ,$\overline{T}_p(M) = \{m \in M : m \in p\Gamma m$,for some $p \in P$ }, it's clear that $T_p(M) \subseteq \overline{T}_p(M)$. We proved that, if S is a Γ-monoid ,then M is a multiplication S_Γ-act if and only if for every maximal Γ- ideal P of S, either $M = T_p(M)$ or M is P-cyclic. Thus, we have the following corollary.

Corollary 2.15.[7] Let S be Γ-monoid. Then, M is a multiplication S_Γ-act if and only if AΓM is a multiplication S_Γ-act for all multiplication Γ-ideal A of S.

Recall that an S_Γ-act M is faithful(globally faithful (gl-faithful for short)) ,if the equality $s\alpha m = t\alpha m$, ($s\Gamma m = t\Gamma m$) implies that s = t for all m ∈ M and α ∈ Γ . It's clear that every gl- faithful S_Γ-act M is faithful.[7]

Theorem 2.16.[7] Let S be a Γ-monoid and M a faithful S_Γ-act. Then M is a multiplication if and only if :

i. $\bigcap_{i\in I} (A_i \Gamma M) = (\bigcap_{i\in I} A_i)\Gamma M$ for any nonempty collection of Γ-ideals A_i ,($i \in I$) of S, with $(\bigcap_{i\in I} A_i) \neq \emptyset$,and

ii. For any S_Γ-subact N of M and Γ-ideal A of S such that $N \subset A\Gamma M$ there exists an Γ-ideal B with $B \subset A$ and $N \subseteq B\Gamma M$.

Now, we give some Lemmas which are need in our work.

Lemma 2.17. Let S be a Γ-monoid and K , N S_Γ-subacts of S_Γ-act M. If [K⋂N:M]=S ,then ([K:M]⋂[N:M])Γ([K:M] ∩ [N:M])=([K:M]Γ[K: M])⋂ ([N:M]Γ[N:M]).

Proof. Let aαb ∈ ([K:M] ∩ [N:M])Γ([K:M] ∩ [N:M]) where a,b ∈ [K:M] ∩ [N:M] and $\alpha \in \Gamma$ .Thus, aαb ∈ [K:M]Γ[K: M] and aαb ∈ [N:M]Γ[N:M] and hence , aαb∈ [K:M]Γ[K: M]⋂ [N:M]Γ[N: M]. For the other inclusion,

$$\begin{aligned}([K:M]\Gamma[K:M])\cap([N:M]\Gamma[N:M]) &\subseteq [K:M]\cap[N:M]\\ &= ([K:M]\cap[N:M])\Gamma S\\ &= ([K:M]\cap[N:M])\Gamma([K\cap N:M])\\ &= [K:M]\cap[N:M])\Gamma([K:M]\cap[N:M] .\end{aligned}$$

Therefore, ([K:M]⋂[N:M])Γ([K:M]⋂[N:M])=([K:M]Γ[K: M])⋂([N:M]Γ[N:M]).

Lemma 2.18. Let S be a Γ-semigroup and A, B are Γ-ideals of S . If AΓM =BΓM **,** then $A^k\Gamma M$ =$B^k\Gamma M$ for positive integer k. Where $A^k = A \cdot \Gamma \cdot A \cdot \Gamma \cdot \ldots \cdot \Gamma \cdot A$ **,** (k times)**.**

Proof: We will prove by induction on k. If, k=1,then it's cleat . Suppose it's true when $k = n$, and we show that for $n = k + 1$ **.** Thus, $A^{n+1}\Gamma M = A^n\Gamma(A\Gamma M) = A^n\Gamma(B\Gamma M) = B\Gamma(A^n\Gamma M) = B\Gamma(B^n\Gamma M) = B^n\Gamma(B\Gamma M) = B^{n+1}\Gamma M$**.**

Lemma 2.19. Let A be a Γ-ideal of Γ–monoid S ,and M an S_Γ-act .If M is faithful multiplication ,then A=[AΓM:M] .

Proof: Let a ∈ A .Then , aαm ∈ AΓM for all $\alpha \in \Gamma$and $m \in$ M .So, aΓM ⊆ AΓM. Hence, a ∈ [AΓM:M]. Conversely, suppose s ∈ [AΓM:M] then sΓM ⊆ AΓM. By faithfulness of M, $s \in$ A. Therefore, A=[AΓM:M].

The proof of the following two Lemmas are immediate

Lemma 2.20. Let { $N_i, i \in I$} be a nonempty collection of S_Γ–subacts of an S_Γ -act M ,and A be a Γ-ideal of S. Then $A\Gamma(\cup_{i\in I}N_i) = \cup_{i\in I}(A\Gamma N_i)$.

Lemma 2.21. Let S be a Γ-semigroup ,and A be Γ-ideal of S. Then for any collection of Γ-ideals { B_i: $i \in$ I} of S, we have;

1. $[A : \bigcup_{i\in I} B_i] = \bigcap_{i\in I}[A : B_i]$
2. $[\bigcap_{i\in I} B_i : A] = \bigcap_{i\in I}[B_i : A]$.

Lemma 2.22. Let S be a Γ-monoid and M an S_Γ-act. Then K is a multiplication S_Γ-subact of M if and only if N⋂K =[N : K]ΓK for every S_Γ-subact N of M.

Proof. (⟹) Let $x \in$ N ⋂ K, then $x \in$ K, since K is a multiplication S_Γ-subact of M, there is an Γ-ideal A in S such that, SΓx= AΓK ⊆ N. It follows that A ⊆ [N:K] and hence AΓK ⊆ [N:K]ΓK . Therefore , $x \in$ [N : K]ΓK. Now , we show the other inclusion. Let $y \in$ [N:K]ΓK .Then $y= s\alpha k$ for some s ∈[N:K] , $\alpha \in \Gamma$ and $k \in$ K. So, sΓK ⊆ N thus $y = s\alpha k \in$ N. Also, since [N:K]ΓK ⊆ SΓK ⊆ K then $y \in$ K. We conclude that y ∈ N⋂K. Hence , N⋂K= [N:K]ΓK.

(⟸) Let L be a S_Γ-subact of K . Then by hypothesis , L=L⋂K= [L:K]ΓK .

Idempotent and Pure Gamma Subacts of Multiplication Gamma Acts.

In this section we introduce the definitions of idempotent and pure gamma subacts and we give several characterizations and properties of such gamma subacts.

Definition 3.1. An S_Γ-subact N of an S_Γ-act M is said to be idempotent if N = [N:M]ΓN.

It's clear that the trivial S_Γ-subacts are idempotent .For M=S=Γ=ℤ and N=2ℤ. Then, [N:M]= 2ℤ. So, [N : M]ΓN =2ℤ=N.

Now , we study some properties of idempotent gamma subacts.

Proposition 3.2. Let A be an Γ-ideal of a Γ-semigroup S and K ,N S_Γ-subacts of S_Γ-act M.

(1) If K and N are idempotent then, so is K ⋃ N.

(2) Let M be a faithful multiplication and K , N idempotent in M. If S= [K⋂N:M] then K ⋂ N is idempotent in M.

(3) If K is idempotent in N and N is idempotent in M then K is idempotent in M.

(4) Let M be a faithful multiplication S_Γ-act. Then [N:M] is gl-idempotent Γ-ideal of S if and only if N is idempotent.

Proof.**(1)**By Lemma 2.20**,** we have $[K \cup N: M]\Gamma(K \cup N) = ([K \cup N: M]\Gamma K) \cup ([K \cup N: M]\Gamma N) \supseteq ([K: M]\Gamma K) \cup ([N: M]\Gamma N) = K \cup N$. Also, $[K \cup N: M]\Gamma(K \cup N) \subseteq K \cup N$. Hence, $K \cup N$ is idempotent in M.

(2) By Theorem 2.16 ,and Lemma2.17 we have,

$$
\begin{aligned}
[K \cap N: M]\Gamma(K \cap N) &= ([K: M] \cap [N:M])\Gamma(K \cap N)\\
&= ([K:M] \cap [N:M])\Gamma([K:M]\Gamma M \cap [N:M]\Gamma M)\\
&=([K:M] \cap [N:M])\Gamma([K:M] \cap [N:M])\Gamma M\\
&= [([K:M] \cap [N:M])\Gamma([K:M] \cap [N:M])]\Gamma M\\
&= [([K:M]\Gamma[K: M]) \cap ([N:M]\Gamma[N:M])]\Gamma M\\
&= [K:M]\Gamma([K: M]\Gamma M) \cap [N:M]\Gamma([N:M]\Gamma M)\\
&= [K:M]\Gamma K \cap [N:M]\Gamma N = K \cap N.
\end{aligned}
$$

(3) By hypothesis, $K =[K:N]\Gamma K$ and $N =[N:M]\Gamma N$. It follows that $[K:N]\Gamma N= [K:N]\Gamma([N:M]\Gamma N)\subseteq [K:M]\Gamma N \subseteq [K:N]\Gamma N$. Thus, $[K:N]\Gamma N= [K:M]\Gamma N$. Also, $K =[K:N]\Gamma K \subseteq [K:N]\Gamma N = [K:M]\Gamma N \subseteq [K:M]\Gamma M \subseteq K$. So , $K= [K:M]\Gamma M$. Hence, $[K:N]\Gamma N = [K:M]\Gamma N \subseteq [K:M]\Gamma M= K$. Therefore , $K= [K:N]\Gamma N= [K:M]\Gamma N$. Finally, $K =[K:N]\Gamma K=[K:N]\Gamma([K:M]\Gamma N) = ([K:M] \Gamma[K:N])\Gamma N=[K:M]\Gamma([K:N] \Gamma N) =[K:M]\Gamma K$.

(4) ($\Longrightarrow$)If $[N:M]$ is an gl-idempotent Γ-ideal of S, then $N =[N : M]\Gamma M =([N : M]\Gamma[N: M])\Gamma M = [N : M]\Gamma([N:M]\Gamma M)= [N:M]\Gamma N$, and hence N is idempotent in M.

($\Longleftarrow$)By hypothesis ,$[N : M]\Gamma M =N=[N : M]\Gamma N$. Thus, $[N :M]\Gamma M=[N : M]\Gamma N= [N : M]\Gamma([N:M] \Gamma M) = ([N:M] \Gamma[N:M])\Gamma M$. By faithfulness of M, we have $[N : M] =[N : M]\Gamma[N:M]$. Therefore, $[N:M]$ is a gl-idempotent.

Proposition 3.3. Let M be a multiplication S_Γ-act .If A is gl-idempotent Γ-ideal of S and N is idempotent S_Γ-subact of M ,then $A\Gamma N$ is an idempotent S_Γ-subact of M.

Proof: Since M is a multiplication. Then $A\Gamma N=[A\Gamma N:M]\Gamma M$ .Also, $A\Gamma N= A\Gamma([N:M]\Gamma M) = (A\Gamma [N:M])\Gamma M$ .By assumptions, $A\Gamma N=(A\Gamma A)\Gamma([N : M]\Gamma N) = (A\Gamma[N : M])\Gamma A\Gamma N = (A\Gamma[N : M]\Gamma[A\Gamma N:M])\Gamma M =[A\Gamma N:M]\Gamma(A\Gamma N)$.

Corollary 3.4. Let M be a multiplication S_Γ-act. If A is gl-idempotent Γ-ideal of S, then AΓM is an idempotent S_Γ-subact of M. The converse is true if M is faithful S_Γ-act .

Proof: It's clear by Proposition 3.3 .Conversely, since AΓM is an idempotent then by Lemma 2.19 , AΓM= [AΓM:M]Γ(AΓM) = AΓ[M:M]Γ(AΓM) =(AΓS)Γ(AΓM)= (AΓA)Γ(SΓM) = (AΓA)ΓM. By faithfulness of M**,** A=AΓA.

Proposition 3.5. Let M be a multiplication S_Γ act and $\{N_i, i\in I\}$ a nonempty collection of S_Γ-subacts of M such that M= $\dot{\bigcup}_{i\in I} N_i$ .Then N_i is idempotent for all $i \in I$.

Proof : Let $i \in I$,by Lemma 2.20 we get N_i= $[N_i:M]\Gamma M$= $[N_i:M]\Gamma(\bigcup_{i\in I} N_i)$ = $[N_i:M]\Gamma(N_i \cup (\bigcup_{i\neq j} N_j)$=$([N_i:M]\Gamma N_i)\cup([N_i:M]\Gamma(\bigcup_{i\neq j} N_j)$ .Since M is multiplication then $[N_i:M]\Gamma(\bigcup_{i\neq j} N_j)$=$[N_i:M]\Gamma([\bigcup_{i\neq j} N_j:M]\Gamma M)$=$[\bigcup_{i\neq j} N_j: M]\Gamma([N_i:M]\Gamma M)$= $[\bigcup_{i\neq j} N_j:M]\Gamma N_i \subseteq N_i$. It follows that $[N_i:M]\Gamma(\bigcup_{i\neq j} N_j) \subseteq N_i \cap (\bigcup_{i\neq j} N_j)$= θ. We have N_i=$[N_i:M]\Gamma N_i$ and hence N_i is idempotent.

Now, we introduce the definition of pure gamma subact and we discussed some of basic properties .

Definition 3.6. An S_Γ-subact N of S_Γ-act M is called pure ,if AΓN = N ∩ AΓM for each Γ-ideal A of S. A Γ-ideal A of S, is pure if A is S_Γ-subact of S_Γ-act S.

Example 3.7. Let S = $\{x , y , z\}$,Γ=$\{\alpha, \beta, \gamma\}$ and M=S .Define a binary operation in M as shown in the following table:

.	x	y	z
x	x	x	x
y	x	x	x
z	x	y	z

Then M is an S_Γ-act under the mapping S ×Γ× M → M which defined by $x\alpha y = xy$ for all $x, y \in$ M. Here, N_1=$\{x , y\}$ and N_2=$\{x , z\}$ are S_Γ-subacts of M . Clearly, N_1 is a pure S_Γ-subact ,but N_2 is not pure S_Γ-subact of M.

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Proposition 3.8. Pure S_Γ-subacts of multiplication S_Γ-act are multiplication and idempotent.
Proof. Let N be a pure S_Γ-subact of multiplication S_Γ-act M, and K an S_Γ-subact of N. Thus K = AΓM for some Γ-ideal A of S. But N is pure, so we have K = K ∩ N = AΓM ∩ N = AΓN. Hence, N is a multiplication .Now, we show that N is idempotent. Since N=[N:M]ΓM, and N is pure then [N:M]ΓN= N∩[N:M]ΓM=N. Therefore, N=[N:M]ΓN, hence N is idempotent in M.

Proposition 3.9. If each of K and N are pure S_Γ-subact of M then so is K ∪ N.
Proof: Let A be an Γ-ideal of S. Since K and N are pure in M, then AΓK = K∩ AΓM and AΓN = N ∩ AΓM. So , by Lemma **2.20**, (K ∪ N) ∩ AΓM = (K ∩ AΓM) ∪ (N ∩ AΓM) = AΓK ∪ AΓN= AΓ(K ∪ N), and hence K ∪ N is pure S_Γ-subact of M.

Proposition 3.10. Let S be a Γ-semigroup ,and K , N S_Γ-subacts of S_Γ-act M . If K is pure S_Γ-subact of N and N is pure of M ,then K is pure of M.
Proof: Let A be a Γ-ideal of S. By hypothesis AΓK= K ∩ AΓN = K ∩ (N ∩ AΓM) = K ∩ AΓM. Therefore, K is pure.

An S_Γ-act M is said to be pure-multiplication if for each pure S_Γ-subact N of M there exist, a Γ-ideal A of S such that N=AΓM. An S_Γ-act M is said pure-simple, if it contain no pure S_Γ–subact other than M . An S_Γ-act M is said to regular if all its S_Γ-subacts are pure. A Γ-semigroups is called regular if a S is regular S_Γ-act.

It's clear that every pure-simple S_Γ-act is pure multiplication. But the convers is not true in general as the following example:

Example 3.11. Let S = { i, 0, -i} and S = Γ = M. Then M is S_Γ-act under the multiplication over complex numbers. Here $N_1=\{0\}$ and N_2=M ,are the only S_Γ–subacts of M. It's clear that N_i, i=1,2 are pure S_Γ-subacts of M, and there is Γ-ideal A of S, such that N_i =AΓM for i=1,2 . Therefore , M is pure multiplication S_Γ-act but not simple. Also, M is regular.

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Proposition 3.12. Let M be an S_Γ –act and $\{N_i, i \in I\}$ a nonempty collection of S_Γ–subacts of M. If $M = \dot{\bigcup}_{i\in I} N_i$,then N_i is a pure S_Γ-subacts of M for all $i \in I$.

Proof: Let A be a Γ-ideal of S. By Lemma 2.20, $A\Gamma M = A\Gamma(\bigcup_{i\in I} N_i) = \bigcup_{i\in I}(A\Gamma N_i)$. Now, let $j \in I$ then, $A\Gamma M \cap N_j = \bigcup_{i\in I}(A\Gamma N_i) \cap N_j = (A\Gamma N_j \cap N_j) \cup (\bigcup_{i\neq j\in I} A\Gamma N_i \cap N_j) = A\Gamma N_j \cup \theta = A\Gamma N_j$. Hence, N_i is a pure S_Γ-subacts of M for all $i \in I$.

In the following Theorem we give a relation between pure gamma subacts, multiplication gamma acts and idempotent gamma subacts.

Theorem 3.13.Let S be a Γ-monoid and N a S_Γ-subact of faithful multiplication S_Γ-act M. The following statement are equivalent

- **i.** N is a pure S_Γ-subact of M.
- **ii.** N is multiplication and idempotent in M.
- **iii.** AΓ[N:M] =A ∩ [N:M] for every Γ-ideal A of S.

Proof. (i)⟹(ii) It's clear by Proposition 3.8 .
(ii)⟹(iii) Let K be a S_Γ-subact of M .Then [K:N]ΓN=[K:N]Γ([N:M]ΓN) =([K:N]Γ[N:M])ΓN ⊆ [K: M]ΓN .Since [K: M]ΓN ⊆ [K:N]ΓN. So, [K:N]ΓN = [K:M]ΓN .Since N is multiplication , then by Lemmas 2.22 and 2.19,for every Γ-ideal A of S, AΓM∩N= [AΓM:N]ΓN = [AΓM:M]ΓN =AΓN. Thus, AΓN=AΓM∩N.So, (AΓ[N:M])ΓM =AΓN=AΓM∩N =AΓM∩[N:M]ΓM . Therefore, AΓM∩[N:M]ΓM = (AΓ[N:M])ΓM and by Theorem 2.16, (A ∩ [N:M])ΓM = (AΓ[N:M])ΓM .By faithfulness of M, AΓ[N:M] =A∩[N:M].
(iii) ⟹ (i) Let A be an Γ-ideal of S. By Theorem 2.16, and hypothesis N ∩ AΓM =[N:M]ΓM ∩ AΓM = ([N:M] ∩ A)ΓM=([N:M]ΓA)ΓM =(AΓ[N:M]) ΓM =AΓ[N:M]ΓM)= AΓN .Therefore, N is a pure S_Γ-subact of M.

As a special case of Theorem 3.13,the following corollary gives a characterization of pure Γ-ideal in Γ-monoids .

Corollary 3.14.If S is a faithful and A Γ-ideal of S. Then the following statement are equivalent:

i. A is a pure Γ-ideal of S.

ii. A is multiplication and gl-idempotent in S.

iii. BΓ[A:S] =B ∩ [A:S] for every Γ-ideal B of S.

Corollary 3.15. Let S be a Γ-monoid and M a gl-faithful multiplication S_Γ-act .

i. If N (resp. A) is a pure S_Γ-subact (resp. Γ-ideal) of M(resp. S) then AΓN is a pure S_Γ-subact of M.

ii. The nonempty intersection of two pure S_Γ-subacts of M is pure S_Γ-subact of M.

Proof. **(i)** Let N (A) be a pure S_Γ-subact (Γ-ideal) of M (S). Then by Theorem 3.13, N and (A) is a multiplication and idempotent (gl-idempotent). By Corollary 2.15**,** and Proposition 3.3, AΓN is multiplication an idempotent. By Theorem 3.13, AΓN is pure S_Γ-subact of M.

(ii) Let N_1 and N_2 be a pure S_Γ-subacts of M, with $\mathrm{N}_1 \cap \mathrm{N}_2 \neq \emptyset$. We will show that $\mathrm{N}_1 \cap \mathrm{N}_2$ is an idempotent. Let $x \in \mathrm{N}_1 \cap \mathrm{N}_2$,then $x \in \mathrm{N}_1$and $x \in \mathrm{N}_2$ .By Theorem 3.13 **,** $\mathrm{N}_1$and N_2 are idempotent S_Γ-subacts, then $\mathrm{N}_1 = [\mathrm{N}_1\colon \mathrm{M}]\Gamma\mathrm{N}_1$ and $\mathrm{N}_2 = [\mathrm{N}_2\colon \mathrm{M}]\Gamma\mathrm{N}_2$ .Thus, there is $y_1 \in [\mathrm{N}_1\colon \mathrm{M}]$ and $y_2 \in [\mathrm{N}_2\colon \mathrm{M}]$ such that x=$y_1\alpha x$,and x=$y_2\beta x$ for some $\alpha , \beta \in \Gamma$. Also, since $y_1 \in [\mathrm{N}_1\colon \mathrm{M}]$ and $y_2 \in [\mathrm{N}_2\colon \mathrm{M}]$ then $y_1\Gamma y_2 \subseteq [\mathrm{N}_1\colon \mathrm{M}]\Gamma[\mathrm{N}_2\colon \mathrm{M}] \subseteq [\mathrm{N}_1\colon \mathrm{M}]\cap[\mathrm{N}_2\colon \mathrm{M}]$= $[(\mathrm{N}_1 \cap \mathrm{N}_2)\colon \mathrm{M}]$. Now, x=$y_1\alpha(y_2\beta x)$ = $(y_1\alpha y_2)\beta x \in [(\mathrm{N}_1 \cap \mathrm{N}_2)\colon \mathrm{M}]\Gamma(\mathrm{N}_1 \cap \mathrm{N}_2)$ .Thus, $\mathrm{N}_1 \cap \mathrm{N}_2$ is idempotent and by Remark 1.19 (1), $\mathrm{N}_1 \cap \mathrm{N}_2$ is a multiplication. So, by Theorem 3.13, $\mathrm{N}_1 \cap \mathrm{N}_2$ is a pure S_Γ-subact of M .

Proposition 3.16.Let S be a Γ-monoid and N a pure S_Γ-subact of gl-faithful multiplication S_Γ-act M. Then every S_Γ-subact of N is pure.
Proof: By Theorem 3.13, N is idempotent, and multiplication. Let K be a S_Γ-subact of N .Then K = [K :N]ΓN. By Corollary3.15 , K is a pure S_Γ-subact of M.

Theorem 3.17. Let S be a Γ-monoid and M a gl-faithful multiplication S_Γ-act. Then :

i. An S_Γ-subact N of M is pure if and only if [N : M] is a pure Γ-ideal of S .

ii. An Γ-ideal A of S is pure if and only if AΓM is a pure S_Γ-subact of M.

Proof: (i) (⟹) Let A be a Γ-ideal of S , and N be a pure S_Γ-subact of M .Then ,by Theorem3.13**,** $A\Gamma[N:M] = A \cap [N:M]$. It follows that, [N:M] is pure Γ-ideal of S.

(⟸) Let N be an S_Γ-subact of M such that [N : M] is a pure Γ-ideal of S. Then, corollary 3.14 implies that [N : M] is multiplication and gl-idempotent. Since, $N = [N : M]\Gamma M$,then N is multiplication and by Proposition 3.2(4), N is idempotent. Hence, N is pure.
(ii) (⟹) Let B be a Γ-ideal of S .Then, by Theorem 2.16**,** $A\Gamma M \cap B\Gamma M=(A \cap B)\Gamma M= (A\Gamma B)\Gamma M=A\Gamma(B\Gamma M)= B\Gamma(A\Gamma M)$. Hence , $A\Gamma M$ is a pure S_Γ-subact of M.
(⟸) Let A be an Γ-ideal of S ,such that $A\Gamma M$ is a pure S_Γ-subact of M. By Lemma 2.19, $A = [A\Gamma M:M]$, by (1) A is a pure Γ-ideal of S .

In following Proposition**,** we give a characterization of completely gl-idempotent Γ-monoids in terms of pure Γ-ideals.

Corollary 3.18 Let S be a Γ-monoid. Then S is completely gl-idempotent if and only if S is regular.
Proof: (⟹)Let A be a Γ-ideal of S. Then A is gl-idempotent and A is multiplication. Hence , by Corollary 3.14, A is pure .
(⟸) Let B be a pure Γ-ideal of S. By Theorem 3.13, B is gl-idempotent. Hence , S is completely gl-idempotent.

An S_Γ-subact N of S_Γ-act M is said to be meet principal if $A\Gamma N\cap K= (A\cap[K:N])\Gamma N$,for all Γ-ideal A of S and S_Γ-subact K of M.

Theorem 3.19.[6] Let S be a Γ-monoid. If M is meet principal S_Γ-act then M is a multiplication. The converse is true, if M is faithful.

It's easy to show that if A and B are Γ-ideals of S , then so is $A\Gamma B$.

Corollary 3.20.[6] Let S be a Γ-monoid . If A and B are faithful multiplication Γ-ideals of S ,then $A\Gamma B$ is a multiplication Γ-ideal of S.

Proposition 3.21. If S is a faithful and A and B are pure Γ-ideals of S ,then AΓB is a pure Γ-ideal of S.

Proof: By Corollary 3.20, and Proposition 2.6, AΓB is multiplication and gl-idempotent Γ-ideal of S. So, by Corollary 3.14, AΓB is pure.

Recall that [6], If N_1 , N_2 are S_Γ-subacts of a multiplication S_Γ-act M such that N_1= AΓM and N_2 =BΓM for some Γ-ideals A and B of S, then the product of N_1 and N_2 is denoted by $N_1 * N_2$ is defined by $N_1 * N_2$=(AΓB)ΓM. Where, $N_1 * N_2$ is an S_Γ-subact of M .

Proposition 3.22. Let S be a Γ-monoid and M a multiplication S_Γ-act. If N_1 and N_2 are pure S_Γ-subacts of M ,then $N_1 * N_2$ is a pure S_Γ-subact of M.

Proof. Without loss the generality every Γ-ideals of S is a faithful. Let $N_1$and N_2 be a pure S_Γ-subacts of M. Since M is multiplication then there is Γ-ideals A and B of S, such that N_1=AΓM and N_2=BΓM. By Theorem3.17, A and B are pure Γ-ideals and by Proposition 3.21**,** AΓB is pure Γ-ideal. Thus, (AΓB)ΓM is pure S_Γ-subact of M (By Theorem3.17). Hence, $N_1 * N_2$ is a pure.

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