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Numerical methods to find the slope at a point on map using level curves. Application in road designing

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Abstract. The problem of determining the slope at a point on a map using level curves has practical applicability in designing the roads. This problem is reduced to the problem of finding the shortest length segment through a point that connects two coplanar lines. Two methods to solve this problem are presented: a fast approximate method and a slower but exact algorithmic method. The results are compared at the end.

1. Introduction

When designing a new road some points are fixed on the map (where the new road will pass). It is very important for designer to know as exact as possible the slope at each such point. Instead of sending workers on the field to measure the slope at different points these values can be calculated using the methods from this paper fast and with very good accuracy. The slope at a point can be determined using the closest two level curves to that point (figure 1). The slope is given by the shortest length segment that connects the two level curves. Since a level curve consists in consecutive coplanar segments of straight lines, our problem can be reduced to finding the shortest length segment that connects two coplanar line segments located on two consecutive level curves.

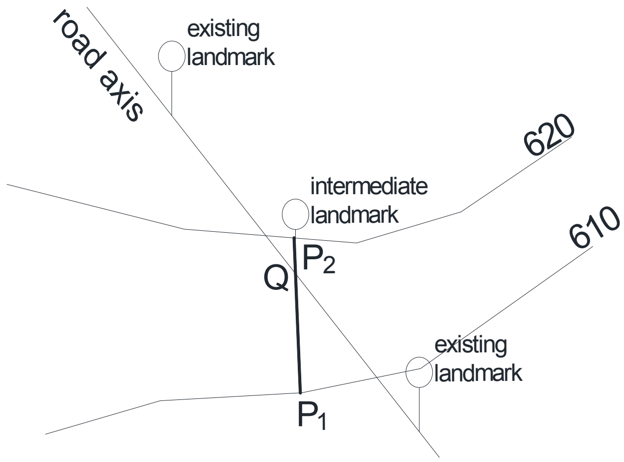

Figure 1. Finding the slope at a point using level curves.

This problem has also applications in railways, roads and bridges when circular curves are needed to draw as can be seen in figure 2 [1].

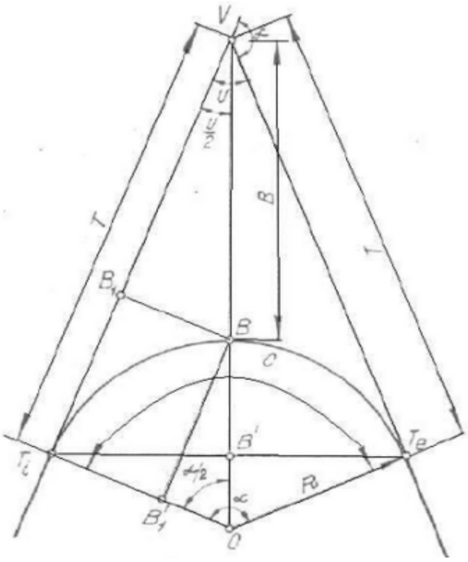

Figure 2. Drawing a circular curve.

We will show next how to calculate the shortest segment that passes through a given point Q and connects two coplanar lines d_1 and d_2 (figure 3). We shall denote this problem as SSTPC2L (Shortest Segment Through a Point that Connects 2 Lines problem). In figure 3 the points A_{11} and A_{12} determine the line d_1 and the points A_{21} and A_{22} determine the line d_2.

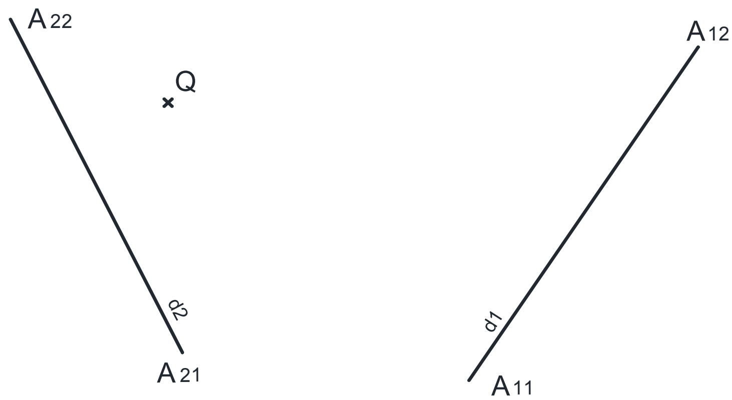

Figure 3. Connecting the point Q to the lines d_1 and d_2.

We present two methods of solving SSTPC2L: an approximate method presented in section 2 and an exact mathematical method presented in section 3. In section 4, the experimental one, six different positions of the Q points are presented for which the length of the connecting segments was determined by both methods, thus observing the variation of the error of the first method.

2. Approximate method

The method consists in drawing from the point Q the perpendiculars on the two given lines d_1 and, respectively, d_2. We construct a line passing through Q and it is parallel to the line determined by the feet of the two perpendiculars. The obtained line is denoted by d_3, and its intersection with the lines d_1 and d_2 is the searched segment $[P_1P_2]$ (approximate solution of SSTPC2L), like in figure 4.

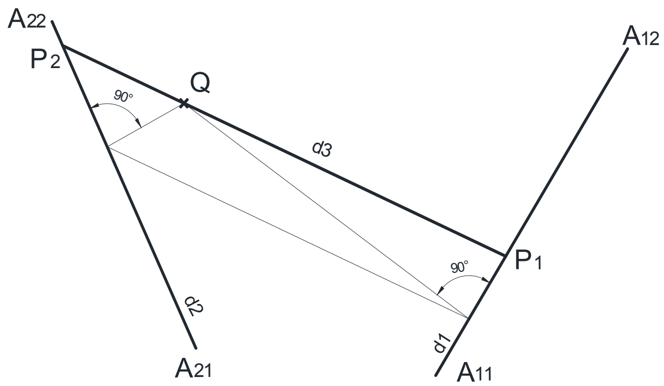

Figure 4. Approximate method of drawing the line d_3.

3. The mathematical method

In order to have simpler calculus we apply some initial transformations (rotations and translations) to the points A_{11}, A_{12}, A_{21}, A_{22} and Q so that the bisector of the angle between the lines d_1 and d_2 becomes parallel to Oy axis and the point Q is in the origin O, i.e., $Q = O(0,0)$.

The following 2 systems of equations have to be solved in order to find the points $P_i(x_i,y_i)$ (i = 1,2) where the lines d_i (i = 1,2) intersect with the line d:

$$\begin{cases} y = a_i x + b_i \\ y = ax \end{cases} \tag{1}$$

Since A_{i1}, A_{i2} define the line d_i (i = 1,2) in (1) we have:

$$a_i = \frac{(y_{i2} - y_{i1})}{(x_{i2} - x_{i1})} \tag{2}$$

$$b_i = y_{i1} - a_{i1} x_{i1}. \tag{3}$$

Solving the 2 systems of equations, the points $P_i(x_i,y_i)$ (i = 1,2) are found, where:

$$x_i = \frac{b_i}{a - a_i} \tag{4}$$

$$y_i = \frac{\mathrm{a} b_i}{a - a_i} \tag{5}$$

The distance between P_1 and P_2 must be minimized, i.e.:

$$\min_a f(a) \tag{6}$$

where:

$$f(a) = (y_2 - y_1)^2 + (x_2 - x_1)^2 \tag{7}$$

By replacing (4) and (5) in (7) the following formula for $f(a)$ is obtained:

$$f(a) = \left(\frac{a_2 b_2}{a - a_2} - \frac{a_1 b_1}{a - a_1} + b_2 - b_1\right)^2 + \left(\frac{b_2}{a - a_2} - \frac{b_1}{a - a_1}\right)^2 \tag{8}$$

In order to find the value a_{min} where $f(a)$ is minimum, the equation:

$$f'(a) = 0 \tag{9}$$

must be solved, where:

$$f'(a) = 2\left(\frac{a_2b_2}{a-a_2} - \frac{a_1b_1}{a-a_1} + b_2 - b_1\right)\left(\frac{-a_2b_2}{(a-a_2)^2} + \frac{a_1b_1}{(a-a_1)^2}\right) + 2\left(\frac{b_2}{a-a_2} - \frac{b_1}{a-a_1}\right)\left(\frac{-b_2}{(a-a_2)^2} + \frac{b_1}{(a-a_1)^2}\right) \quad (10)$$

The value a is the slope of the line d_3 and, consequently, a_{min} must be searched in the interval given by the slopes of the perpendiculars QQ_1 and QQ_2 from Q on the lines d_1 and, respectively, d_2 (figure 5), i.e.:

$$a \in (a_L, a_R) \quad (11)$$

where:

$$a_L = \min\left\{\frac{-1}{a_1}, \frac{-1}{a_2}\right\} \quad (12)$$

$$a_R = \max\left\{\frac{-1}{a_1}, \frac{-1}{a_2}\right\} \quad (13)$$

Remark 1: Since the bisector of the angle between d_1 and d_2 is parallel to Oy axis, it can be seen that:

$$a_L = -a_R \quad (14)$$

In order to obtain the solution of a_{min} of the equation (9) a numerical method [2] can be used such as bisection method (interval halving method) or tangent method. We preferred bisection method [3] and we performed it on the interval defined in (11).
After a_{min} is calculated, the points P_i (i = 1,2) are obtained using (4) and (5) and the distance between these two points is:

$$\sqrt{f(a_{\min})} \quad (15)$$

In order to go back to the initial system of coordinates, the inverse initial transformations must be applied in reverse order to the points A_{11}, A_{12}, A_{21}, A_{22}, Q, P_1 and P_2. So, the following algorithm is obtained to solve the problem SSTPC2L:
Algorithm for SSTPC2L

- Transform the points A_{11}, A_{12}, A_{21}, A_{22} and Q so that the bisector of the angle between the lines d_1 and d_2 is parallel to Oy axis and the point Q is in the origin O;
- Find the solution a_{min} of the equation $f'(a) = 0$ (10) using bisection method on the interval (a_L, a_R) (11);
- Calculate the coordinates of the points P_i (i = 1,2) using (4) and (5), where $a = a_{min}$;
- Apply the inverse transformations in reverse order to the points A_{11}, A_{12}, A_{21}, A_{22}, Q, P_1 and P_2;
- Calculate the distance between P_1 and P_2 using (15).

We implemented the algorithm above in Visual C++ 2017 and in the bisection method we set the error err = 0.0001 for finding the value a_{min}. In figure 5 a graphically output of our program is presented.

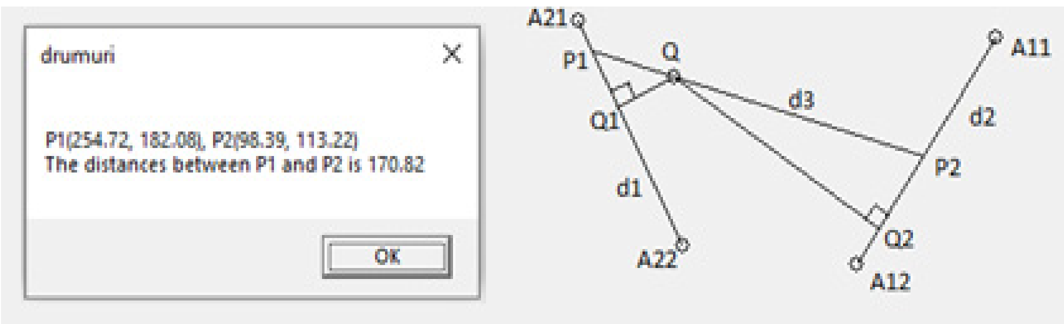

Figure 5. Output of the Visual C++ program.

4. Experiments

In order to compare the results of the two methods, two random lines d_1 and d_2 were chosen and several Q points were considered: near the bisector of the angle formed by d_1 and d_2, near the two straight lines d_1 and d_2 and other intermediate points.

In experiments the two lines d_1 and d_2 are defined by the points $A_{11}(200, 100)$ and $A_{12}(280, 220)$, and, respectively, $A_{21}(100, 110)$ and $A_{22}(40, 230)$. Six different Q points were chosen with the coordinates: $Q_1(70, 175)$, $Q_2(95, 200)$, $Q_3(156, 185)$, $Q_4(170, 210)$, $Q_5(210, 205)$ and $Q_6(250, 180)$ (figure 6).

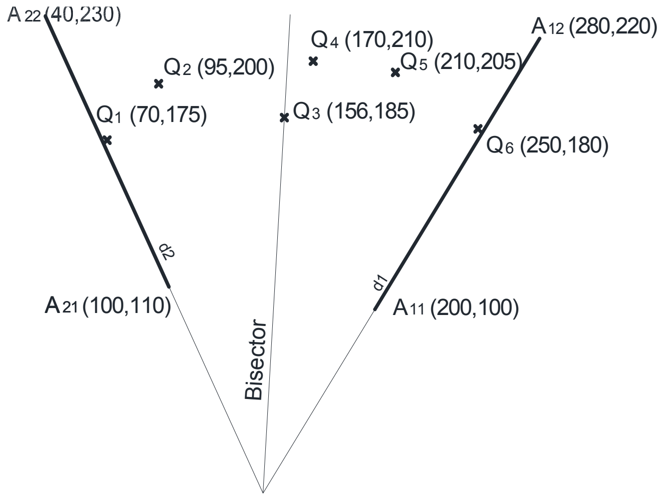

Figure 6. Different positions for Q point.

In figure 6 the 6 positions of the point Q with the obtained results are presented. P_1 and P_2 are the intersection points calculated with the first method. P'_1 and P'_2 are the intersection points calculated with the second method. With L was denoted the length of the segment obtained by the approximate method and with l was denoted the length of the segment obtained by the exact method.

As one can see in table 1 the first approximate method provides good enough solutions. Moreover, as expected, if the point Q is located near the bisector given by the lines d_1 and d_2 or near the segments d_1 and d_2 the solutions given by the two methods are very alike.

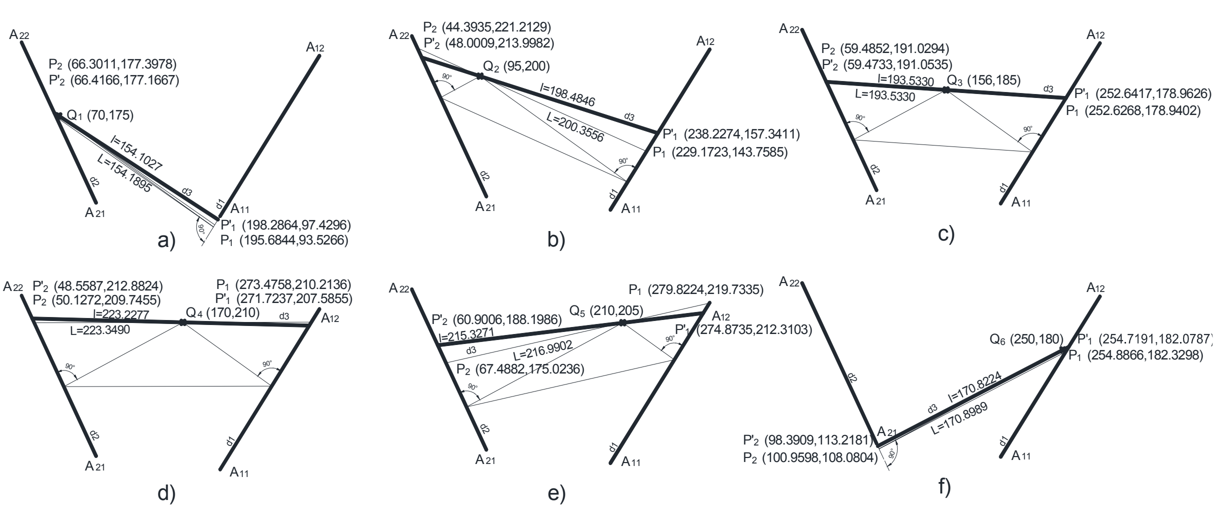

Figure 7. Various positions of point Q and obtained results.

Table 1: Comparison between the two methods.

Coordinate		P1	P2	P'1	P'2	L	l	L-l	Error (%)
Q1	x=	195.6844	66.3011	198.2864	66.4166	154.1895	154.1027	0.0868	0.0563
	y=	93.5266	177.3978	97.4296	177.1667				
Q2	x=	229.1723	44.3935	238.2274	48.0008	200.3556	198.4846	1.8710	0.9426
	y=	143.7585	221.2129	157.3411	213.9982				
Q3	x=	252.6268	59.4733	252.6417	59.4852	193.5330	193.5330	0.0000	0.0000
	y=	178.9402	191.0535	178.9626	191.0294				
Q4	x=	273.4758	50.1272	271.7237	48.5587	223.3490	223.2277	0.1213	0.0543
	y=	210.2136	209.7455	207.5855	212.8824				
Q5	x=	279.8224	67.4882	274.8735	60.9006	216.9902	215.3271	1.6631	0.7724
	y=	219.7335	175.0236	212.3102	188.1986				
Q6	x=	254.8866	100.9598	254.7191	98.3909	170.8989	170.8224	0.0765	0.0448
	y=	182.3298	108.0804	182.0787	113.2181				

5. Conclusions

In conclusion when designing a new road it is very important for designer to know as exact as possible the slope at some points of the future road. Instead of sending workers on the field to measure the slope at different points these values can be calculated with very good accuracy using the methods from this paper. We presented two methods to solve SSTPC2L, a fast and approximate one and an exact and slower one. The error of the first method is low (table 1). It has the advantage of being very fast and accurate enough. So, if we have many points for which SSTPC2L is applied the first method can be preferred. Although, if we look for an exact solution, the second method has to be used.

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