IOP Conference Series: Materials Science and Engineering

Finite Element Method to Solve Poisson's Equation Using Curved Quadratic Triangular Elements

To cite this article: G Shylaja *et al* 2019 *IOP Conf. Ser.: Mater. Sci. Eng.* **577** 012165

IOP Conf. Series: Materials Science and Engineering **577** (2019) 012165 doi:10.1088/1757-899X/577/1/012165

Finite Element Method to Solve Poisson's Equation Using Curved Quadratic Triangular Elements

Shylaja G, Venkatesh B and Kesavulu Naidu V
Department of Mathematics, Amrita School of Engineering, Bengaluru, Amrita Vishwa Vidyapeetham, India

Email: b_venkatesh@blr.amrita.edu

Abstract. The paper discusses the finite element method to solve Poisson's equation using quadratic order curved triangular elements. We use quadratic order point transformation to solve the partial differential equation. We observe that with quadratic order as the discretization of the domain element is increased, the error of the solution decreases.

1. Introduction

In the finite element method, we can find the approximate solution for a complicated problem. By discretization of the continuum states of matter such as solid, liquid or gas into the same kind of element called triangular elements. These elements are connected to an adjacent element at a point called nodes. The nodes lie within the boundary of the domain where the adjacent elements are connected. Since the field variable like stress, displacement, force, temperature, pressure or velocity are not known inside the continuum, this field variable can be approximated using interpolation functions. These interpolants function are defined in terms of field variable at each nodes. Writing the finite element equation for the entire domain, the new unknown will be nodal value of field variable. By solving the finite element equation which is in the form of matrix, the nodal value of field variable will be known. Knowing these nodal value the interpolants function defines the field variable for the entire assembly of element. The domain may have curved boundaries which sometimes becomes difficult to find the accurate solution. The curved boundaries was first studied by Ergatoudis et.al.[1]. Bassi and Rebey[2] presented that curved boundaries gives higher order accurate solution. The technique of considering finite element method with exact boundary condition was presented by Zlamal[3,4]. Similarly Scott[5] used triangular elements with one curved side representing the exact boundary. The point transformation for curved boundaries are discussed in detail by Rathod et.al.[6]. In this paper, a numerical example is demonstrated to show that for quadratic order as the number of elements is increased, the accuracy of the solution increases. Mathematica 7.0 codes are written for solving these problems.

2. Finite element method for irregular domain

We consider a problem described by two dimension Poisson equation,

$$\nabla^2 u = -f(x,y) \quad in \quad \Omega \tag{1}$$

subject to the boundary condition

$$u = 0 \quad on \quad \partial\Omega \tag{2}$$

where u and $f(x,y)$ are real-valued functions on the Euclidean space, Ω is the interior domain and $\partial\Omega$ is the boundary of the domain see Fig.1.

As in Rathod et.al.[6], the Lagrange interpolants for the field variable u is given by,

$$u = \sum_{i=1}^{n} N_i(\xi,\eta) u_i \tag{3}$$

where $u_i's$ are nodal values of u and N_i's are prescribed functions of position called shape functions or Lagrange interpolants.

IOP Conf. Series: Materials Science and Engineering **577** (2019) 012165 doi:10.1088/1757-899X/577/1/012165

The natural co-ordinate system t is given by,

$$t = \sum_{i=1}^{6} N_i^{(2)}(\xi,\eta)t_i \qquad t = (x,y) \tag{4}$$

where t_i are nodal values of the triangular element.
The point transformation $t(\xi,\eta)$ for curved boundary is given by Rathod et.al.[6],

$$t(\xi,\eta) = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + (4t_4 - 2t_1 - 2t_2)\xi\eta \tag{5}$$

The Galerkin weighted residual technique is used in finite element method to obtain approximate solutions of linear boundary value problem (1) and (2) it can also be seen from Bathe[7], Zienkiewicz et.al.[8], Bhatti[9], Reddy[10] and Murali et.al.[11]. The domain Ω is discretized into number of triangular elements Ω_e and hence the boundary $\partial\Omega$ is also discretized as shown in Figs. 2-4. The Finite element equation of each element is obtained by,

$$\{K\}_{n\times n} \times \{U\}_{n\times 1} = \{F\}_{n\times 1}$$

where, K is the stiffness matrix, U is the nodal displacement and F is the nodal force.
The stiffness matrix and Nodal force matrix are derived as follows,

$$K_{i,j}^e = \iint_{\Omega_e} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) dxdy \tag{6}$$

$$F_i^e = \iint_{\Omega_e} f(x,y)\, N_i dxdy \tag{7}$$

$$K_{i,j}^e = K_{x,x}^{i,j} + K_{y,y}^{i,j} \tag{8}$$

$$K_{x,x}^{i,j} = \int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} \left(\frac{1}{J}\right)\left(\frac{\partial N_i}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial N_i}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial N_j}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial N_j}{\partial \eta}\frac{\partial y}{\partial \xi}\right) d\eta d\xi \tag{9}$$

$$K_{y,y}^{i,j} = \int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} \left(\frac{1}{J}\right)\left(\frac{\partial N_i}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial N_i}{\partial \xi}\frac{\partial x}{\partial \eta}\right)\left(\frac{\partial N_j}{\partial \eta}\frac{\partial x}{\partial \xi} - \frac{\partial N_j}{\partial \xi}\frac{\partial x}{\partial \eta}\right) d\eta d\xi \tag{10}$$

$$F_i^e = \int_{\xi=0}^{1}\int_{\eta=0}^{1-\xi} J\left(f\big(x(\xi,\eta), y(\xi,\eta)\big)\right) N_i(\xi,\eta) d\eta d\xi \tag{11}$$

Where e is the element number and Ω_e is the region of each element and J is the determinant of Jacobian matrix of the transformation of equation (5) which is expressed as

$$J(\xi,\eta) = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \tag{12}$$

The above integrals (6-11) are obtained by gauss quadrature rules over triangle from Rathod et.al.[12,13]. This gives the matrix of each element then the matrix are assembled and boundary condition are applied. We get a system of n equations in n unknowns. Solving this we can find the solution at each nodal points.

3. Numerical example

Consider a two dimensional Poisson's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2 \quad within \quad \Omega \tag{13}$$

subject to the boundary condition (see Fig.1.)

$$u = 0 \ on \ curve \ \partial\Omega \ and \ \frac{\partial u}{\partial n} = 0 \ on \ x - axis \ and \ y - axis. \tag{14}$$

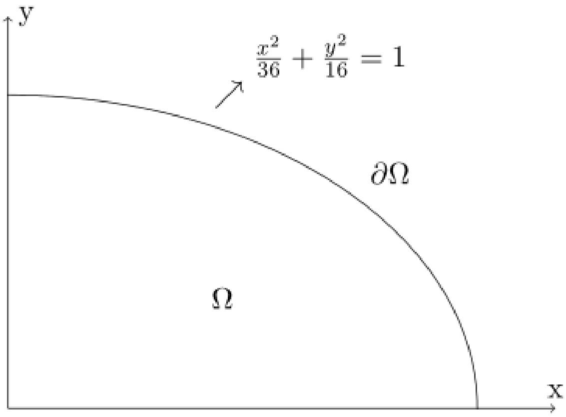

Figure 1: Physical Domain

The exact solution of the partial differential equation is,

$$u(x, y) = 11.07\left(1 - \frac{x^2}{36} - \frac{y^2}{16}\right) \tag{15}$$

The domain is discretized into 4, 8 and 12 elements taking only quarter elliptical domain as it is a axisymmetric problem.The discretization of domain for quadratic order is shown in Figs.2-4. The finite element method(FEM) solution and exact solution at each co-ordinate are tabulated in the Table.1-3. The maximum error for 4, 8 and 12 elements are tabulated in the Table.4.

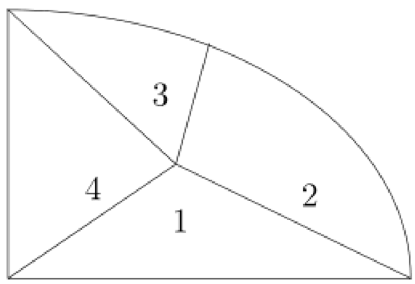

Figure 2: Discretization of domain into 4 elements.

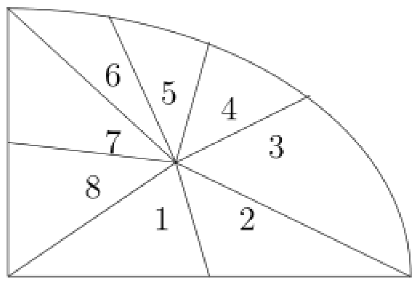

Figure 3: Discretization of domain into 8 elements.

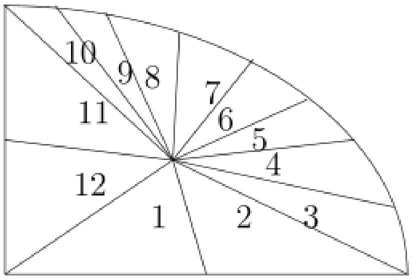

Figure 4: Discretization of domain into 12 elements.

The solution at each nodal point is tabulated as follows,

Table 1: Solution at each co-ordinate in the 4 elements discretization for quadratic order.

Nodes	x	y	FEM Solution	Exact Solution	Absolute Error
1	0.000000	0.000000	11.099289	11.070000	0.029289
2	3.000000	0.000000	7.985890	8.302500	0.316610
3	6.000000	0.000000	0.000000	0.000000	0.000000
4	4.500000	2.645751	0.000000	0.000000	0.000000
5	3.000000	3.464102	0.000000	0.000000	0.000000
6	1.500000	3.872983	0.000000	0.000000	0.000000
7	0.000000	4.000000	0.000000	0.000000	0.000000
8	0.000000	2.000000	7.852564	8.302500	0.449936
9	1.273277	0.848851	9.942578	10.072941	0.130363
10	4.273277	0.848851	4.594640	4.956244	0.361604
11	2.773277	2.580902	4.186197	4.096379	0.089818
12	1.273277	2.848851	4.970337	4.956244	0.014093
13	2.546554	1.697703	6.899963	7.081763	0.181800

Table 2: Solution at each co-ordinate in the 8 elements discretization for quadratic order.

Nodes	x	y	FEM Solution	Exact Solution	Absolute Error
1	0.000000	0.000000	11.034550	11.070000	0.035450
2	1.500000	0.000000	10.330493	10.378125	0.047632
3	3.000000	0.000000	8.197468	8.302500	0.105032
4	4.500000	0.000000	4.716014	4.843125	0.127111
5	6.000000	0.000000	0.000000	0.000000	0.000000
6	5.250000	1.936492	0.000000	0.000000	0.000000
7	4.500000	2.645751	0.000000	0.000000	0.000000
8	3.750000	3.122499	0.000000	0.000000	0.000000
9	3.000000	3.464102	0.000000	0.000000	0.000000
10	2.250000	3.708099	0.000000	0.000000	0.000000
11	1.500000	3.872983	0.000000	0.000000	0.000000
12	0.750000	3.968627	0.000000	0.000000	0.000000
13	0.000000	4.000000	0.000000	0.000000	0.000000
14	0.000000	3.000000	4.832451	4.843125	0.010674
15	0.000000	2.000000	8.283889	8.302500	0.018611

16	0.000000	1.000000	10.346764	10.378125	0.031361
17	1.273277	0.848851	10.034501	10.072941	0.038440
18	2.773277	0.848851	8.129006	8.206468	0.077462
19	4.273277	0.848851	4.801482	4.956244	0.154763
20	3.523277	2.171727	3.965595	3.989696	0.024101
21	2.773277	2.580902	4.092955	4.096379	0.003424
22	2.023277	2.785343	4.439279	4.443542	0.004263
23	1.273277	2.848851	4.947316	4.956244	0.008929
24	1.273277	1.848851	8.182791	8.206468	0.023677
25	2.546554	1.697703	7.048205	7.081763	0.033558

Table 3: Solution at each co-ordinate in the 12 elements discretization for quadratic order.

Nodes	x	y	FEM Solution	Exact Solution	Absolute Error
1	0.000000	0.000000	11.059181	11.070000	0.010819
2	1.500000	0.000000	10.362271	10.378125	0.015854
3	3.000000	0.000000	8.256291	8.302500	0.046209
4	4.500000	0.000000	4.783677	4.843125	0.059448
5	6.000000	0.000000	0.000000	0.000000	0.000000
6	5.625000	1.391941	0.000000	0.000000	0.000000
7	5.250000	1.936492	0.000000	0.000000	0.000000
8	4.875000	2.331845	0.000000	0.000000	0.000000
9	4.500000	2.645751	0.000000	0.000000	0.000000
10	4.125000	2.904738	0.000000	0.000000	0.000000
11	3.750000	3.122499	0.000000	0.000000	0.000000
12	3.375000	3.307189	0.000000	0.000000	0.000000
13	3.000000	3.464102	0.000000	0.000000	0.000000
14	2.625000	3.596874	0.000000	0.000000	0.000000
15	2.250000	3.708099	0.000000	0.000000	0.000000
16	1.875000	3.799671	0.000000	0.000000	0.000000
17	1.500000	3.872983	0.000000	0.000000	0.000000
18	1.125000	3.929058	0.000000	0.000000	0.000000
19	0.750000	3.968627	0.000000	0.000000	0.000000
20	0.375000	3.992180	0.000000	0.000000	0.000000
21	0.000000	4.000000	0.000000	0.000000	0.000000
22	0.000000	3.000000	4.838680	4.843125	0.004445
23	0.000000	2.000000	8.298324	8.302500	0.004176
24	0.000000	1.000000	10.368692	10.378125	0.009433
25	1.273277	0.848851	10.061364	10.072941	0.011577
26	2.773277	0.848851	8.175797	8.206468	0.030671
27	4.273277	0.848851	4.882837	4.956244	0.073408
28	3.898277	1.817097	4.078282	4.112594	0.034312
29	3.523277	2.171727	3.971176	3.989696	0.018520
30	3.148277	2.410101	3.994006	4.003352	0.009346
31	2.773277	2.580902	4.091816	4.096379	0.004563
32	2.398277	2.702901	4.244083	4.246729	0.002646
33	2.023277	2.785343	4.441391	4.443542	0.002151
34	1.648277	2.833165	4.678749	4.681020	0.002271
35	1.273277	2.848851	4.953562	4.956244	0.002683
36	1.273277	1.848851	8.200764	8.206468	0.005704
37	2.546554	1.697703	7.075128	7.081763	0.006635

IOP Conf. Series: Materials Science and Engineering **577** (2019) 012165 doi:10.1088/1757-899X/577/1/012165

Table 4: Maximum Error.

Number of elements	4	8	12
Maximum Absolute Error	0.352348	0.154763	0.073408

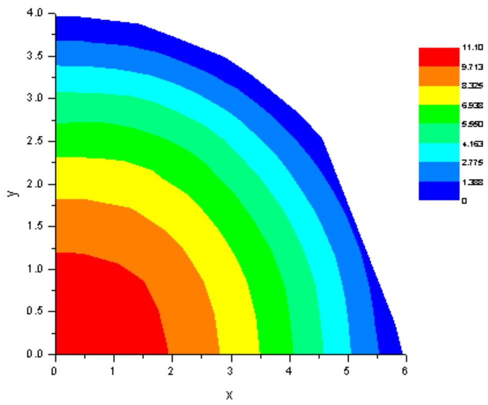

Figure 5: Contour plot for quadratic order of 4 elements FEM solution.

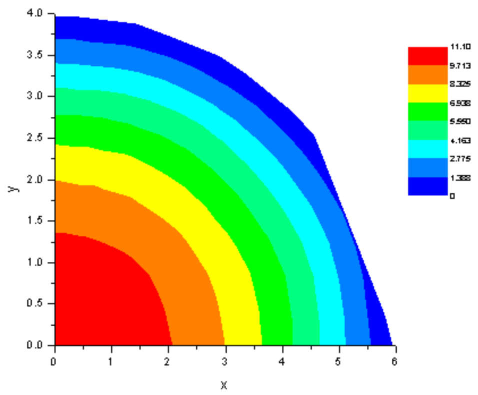

Figure 6: Contour plot for quadratic order of 4 elements exact solution.

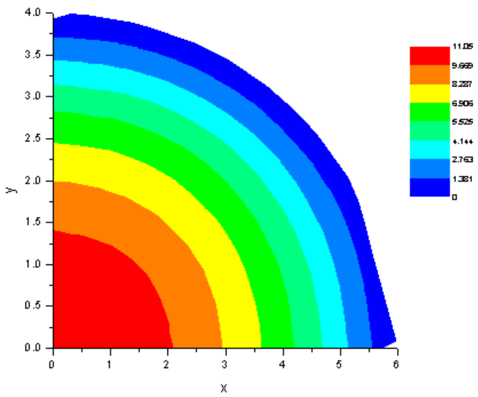

Figure 7: Contour plot for quadratic order of 8 elements FEM solution.

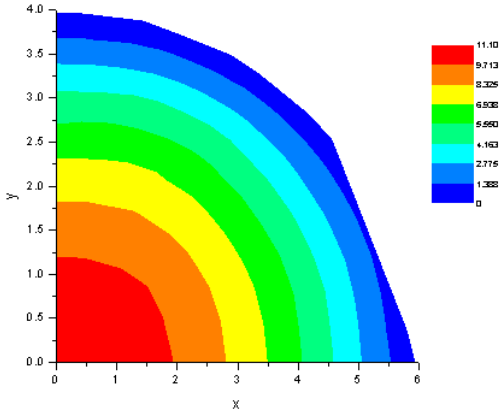

Figure 8: Contour plot for quadratic order of 8 elements exact solution.

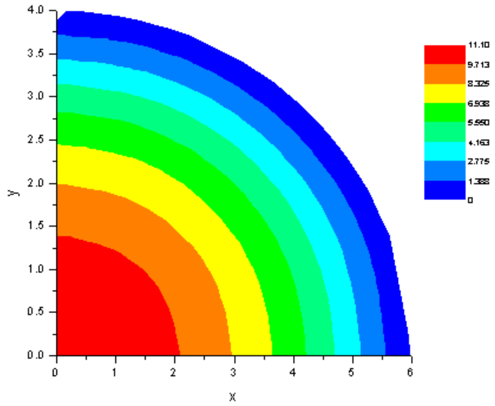

Figure 9: Contour plot for quadratic order of 12 elements FEM solution.

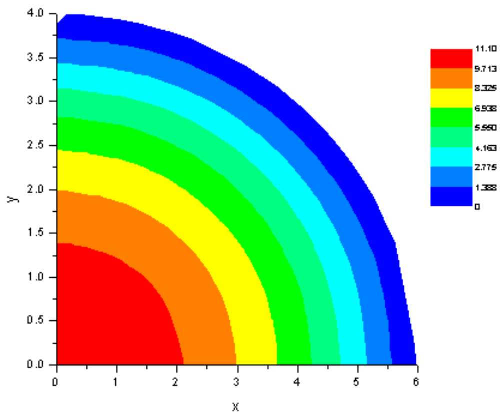

Figure 10: Contour plot for quadratic order of 12 elements exact solution.

4. Conclusion

Initially, the curved domain is discretized into 4 elements, then 8 elements and then into 12 elements for the curved boundaries of quadratic order. The solution at each co-ordinate for 4, 8 and 12 elements are tabulated in the Tables 1-3. and we observe from Table 4. that the maximum error is decreased as the number of elements increased. The contour plot from Figs.5-10 shows that the obtained FEM solution matches very well with the exact solution. Poisson's equation physically describes how a function diffuses in space and broadly used in mechanical engineering and theoretical physics. These method shows the efficiency and effectiveness which can be implemented in various physical phenomenon.

Acknowledgment
The work as been funded by National Board of Higher Mathematics(NBHM), Department of Atomic Energy, Mumbai, India. Ref.No.2/48(10)/2016/NBHM(R.P.)/R&D II/14092.

References

[1] Ergatoudis I, Irons B M and Zienkiewicz O C Curved isoparametric quadrilateral finite element analysis 1968 *Int. J. Solids Struct.,* **4** 31-42
[2] Bassi F and Rebay S High-order accurate discontinuous finite element solution of the 2D Euler equations 1997 *J. Comput. Phys.* **138** 837-861
[3] Zlamal M Curved elements in the finite element method 1973 *SIAM J. Numer. Anal.* **10** 229-240
[4] Zlamal M The finite element method in domains with curved boundaries 1973 *Int. J. Numer. Methods Eng.* **5** 367-373
[5] Scott L Finite element techniques for curved boundaries 1973 Ph.D. thesis Massachusetts Institute of Technology
[6] Rathod H T, Nagaraja K V, Kesavulu Naidu V and Venkatesudu B The use of parabolic arcs in matching curved boundaries by point transformations for some higher order triangular elements 2008 *Fin. Elem. Anal. Des.* **44** 920-932
[7] Bathe K J 1996 *Finite Element Procedures* Prentice Hall Pearson Education Inc.,
[8] Zienkiewicz O C, Taylor R L and Zhu J Z 2005 *The Finite Element Method: Its Basis and Fundamentals* 6th ed., Oxford Elsevier
[9] Bhatti M A 2005 *Fundamental Finite Element Analysis and Applications* John Wiley & Sons New York
[10] Reddy J N 2005 *An Introduction to the Finite Element Method* 3rd Ed., Tata McGraw-Hill New-Delhi
[11] Murali K, Kesavulu Naidu V and Venkatesh B Optimal subparametric finite element approach for a Darcy-Brinkman fluid flow problem through a rectangular channel with one curved side using curved finite elements 2017 *IOP Conf. Series: Materials Science and Engineering* art. no. 012145 310
[12] Rathod H T, Nagaraja K V and Venkatesudu B Symmetric Gauss Legendre quadrature formulas for composite numerical integration over a triangular surface 2007a *Appl. Math. Comput.* **188** 865-876
[13] Rathod H T, Nagaraja K V, Venkatesudu B and Ramesh N L Gauss Legendre quadrature over a Triangle 2004 *J. Indian Inst. Sci.* **84** 183-188