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The Action of some Subgroups of Symmetric Group S_{3n} on An Invariant Markov basis

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Abstract. In this paper, we will find some subgroups H_1 and H_2 of symmetric group S_3n, n∈N and n≥2, such that the Markov basis B is H invariant for (25n^3-66n^2+41n) ×3×n - contingency tables with fixed two dimensional marginal, where B is the Markov basis. We will get another 3×n - contingency tables have same the Markov basis B by using the action of H on these contingency tables.

Keywords linear transformation, contingency table, Markov basis, Symmetric group, action..

1. Introduction

P. Diaconis and B. Sturmfels's published in 1998 found a new path in the rapidly-advancing field of computational algebraic statistics [1] and [2]. In 2008, A. Takemura, and S. Aoki defined an invariant Markov basis for the connected Markov chain over a set of contingency tables, with fixed margins and derived some characterizations of minimal invariant basis, they give the necessary and the sufficient condition for uniqueness of a minimal invariant Markov bases, also present minimal invariant Markov basis for all 2×2×2×2 hierarchical models [3]. In the same year, A. Takemura, and S. Aoki defined a large group of invariance for given Toric ideal and an associated Markov basis. They also give explicit forms of a large group of invariance for several standard statistical problems [4]. In 2018, H. S. Mohammed Hussein, A. H. Majeed found a Markov basis and Toric ideals for $(25n^3 - 66n^2 + 41n) \times 3 \times n$ - contingency tables with fixed two dimensional margins, $n \geq 2$ [5].

If a group G act on $A^{-1}[\boldsymbol{t}]$ on the left, B is a Markov basis, and $G\ (B) = \{gz: z \in B,\ g \in G\}. B$ is called an invariant under G (or G- invariant) if $G(B) = B$. We will denote to the polynomials in the p in determinate (polynomial variables) $p_1,\ p_2,\ \ldots,\ p_p$ over the complex field ₵ by either $₵\{p_1,\ p_2,\ \ldots,\ p_p\}$ or $₵[P]$, $P = (p_1, p_2, \ldots, p_p)$. Let $A: \mathbb{Z}^n \to \mathbb{Z}^d$ be a linear transformation, the Toric ideal I_A is the ideal $< P^u - P^v:\ u\ ,\ v\ \in \mathbb{N}^n,\ A(u) = A(v) > \subseteq ₵[P_1,\ \ldots,\ P_p]$ where $P^u = {P_1}^{u_1}{P_2}^{u_2} \ldots {P_p}^{u_p}$[3].
In [6] H. H. Abbass and H. S. Mohammed Hussein found the largest subgroup H of dihedral Group D_{3^m}, m ∈ ℕ**,** such that **B** is H-invariant.

Contingency tables are used in statistics to store date from sample surveys. One of related problems for a survey of contingency tables is how to generate tables from the set of all non-negative $K_1 \times K_2$ integer tables with given row and column sums. In this work, we use the action of the largest subgroup H of symmetric group $S_{3n}, n \in \mathbb{N}$ and $n \geq 2$ on the set of $(25n^3 - 66n^2 + 41n) \times 3 \times n$ −contingency tables such that $\boldsymbol{B}$ is H-invariant to generate non-negative $3 \times n$ integer tables with given row and column sums.

2 . Preliminaries

In this section, we review some basic definitions and notations of contingency tables, moves, Markov basis, symmetric group, and action of the symmetric group on the set of contingency tables and the set of Markov basis that we need in our work.

Definition (2.1): [3]

Let I be a finite set $n = |I|$ elements, we call an element of I a cell and denoted by $\boldsymbol{i} \in I$. $\boldsymbol{i}$ is often multi-index $\boldsymbol{i} = i_1 \ \dots \ i_l$. A non-negative integer $x_{\boldsymbol{i}} \in \mathbb{N} = \{1, 2, \dots\}$ denotes the frequency of a cell $\boldsymbol{i}$. The set of frequencies is called ***a contingency table*** and denoted as $\boldsymbol{x} = \{x_{\boldsymbol{i}}\}_{\boldsymbol{i} \in I}$, with an appropriate ordering of the cell, we treat a contingency table $\boldsymbol{x} = \{x_{\boldsymbol{i}}\}_{\boldsymbol{i} \in I} \in \mathbb{N}^n$ as a n-dimensional column vector of non-negative integers. Not that a contingency table can also be considered as a function from I to $\mathbb{N}$ defined as $\boldsymbol{i} \longmapsto x_{\boldsymbol{i}}$.

Definition (2.2): [3]

The L_1-norm of $\boldsymbol{x} \in \mathbb{N}^n$ is called the ***sample size*** and denoted as $|\boldsymbol{x}| = \sum_{\boldsymbol{i} \in I} x_{\boldsymbol{i}}$. We will denote $\mathbb{Z}$ be the set of integer numbers, also we denote to the $a_j \in \mathbb{Z}^n$, $j = 1, \dots, v$, as fixed column vectors consisting of integers. A v-dimentional column vector $\boldsymbol{t} = (t_{\mathbf{1}}, \dots, t_v)' \in \mathbb{Z}^v$ as $t_{\boldsymbol{j}} = a'_j \boldsymbol{x}$, $j = 1, \dots, v$. Here $'$ denotes the transpose of a vector or matrix. We also define a $v \times p$ matrix A, with its j-row being a'_j given by $A = \begin{bmatrix} a'_1 \\ \vdots \\ a'_v \end{bmatrix}$, and if $\boldsymbol{t} = A\boldsymbol{x}$ is a v-dimensional column vector, we define the set $T = \{\boldsymbol{t}: \boldsymbol{t} = A\boldsymbol{x},\ \boldsymbol{x} \in \mathbb{N}^n\} = A\mathbb{N}^n \subset \mathbb{Z}^v$, where $\mathbb{N}$ is the set of natural numbers. In typical situations of a statistical theory, $\boldsymbol{t}$ is *sufficient statistic* for the nuisance parameter. The set of $\boldsymbol{x}$'s for a given $\boldsymbol{t}$**,** $A^{-1}[\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^n: A\boldsymbol{x} = \boldsymbol{t}\}$ ($\boldsymbol{t}$-fibers), is considered for performing *similar tests*, for the case of the independence model of two–way contingency tables, for example, $\boldsymbol{t}$ is the row sums and column sums of $\boldsymbol{x}$ **,** and $A^{-1}[\boldsymbol{t}]$ is the set of $\boldsymbol{x}'$s with the same row sums and column sums to $\boldsymbol{t}$. The set of $\boldsymbol{t}$-fibers gives a decomposition of $\mathbb{N}^n$. An important observation is that $\boldsymbol{t}$-fiber depends on given only through its kernel, ker (A). For different A's with the same kernel, the set of $\boldsymbol{t}$-fibers are the same. In fact, if we define $\boldsymbol{x}_1 \sim \boldsymbol{x}_2 \Leftrightarrow \boldsymbol{x}_1 - \boldsymbol{x}_2 \in \ker\ (A)$. This relation is an equivalence relation and $\mathbb{N}^n$ is partitioned into disjoint equivalence classes. The set of $\boldsymbol{t}$-fibers is simply the set of these equivalence classes. Furthermore, $\boldsymbol{t}$ may be considered as labels of these equivalence classes.

Definition (2.3): [3]

A n-dimensional column vector of integer $\boldsymbol{z} = \{z_{\boldsymbol{i}}\}_{\boldsymbol{i} \in I} \in \mathbb{Z}^n$ Is called a move if it is in the kernel of A, i.e. $A\boldsymbol{z} = 0$.

Definition (2.4): [7]

A set of finite moves B is called **Markov basis** if for all $\boldsymbol{t}$, $A^{-1}[\boldsymbol{t}]$ constitutes one B equivalence class.

Definition (2.5): [8]

If $S(X)$ is the set of bijections from any set X to itself, then $(S(X), \circ)$ is a group under composition. This group is called the **symmetric group** or **permutation group** of X.

Remark (2.6): [8]

If $X = \{a, b\}$ is a two-element set, the only bijections from X to itself are the identity and the symmetry $f : X \to X$, defined by $f(a) = b, f(b) = a$, that interchanges the two elements.

Now, we give some concepts about the action of a group on a set that we use later.

Definition (2.7):- [3]

Let G be a group and W be a set. A left action of G in W is a function from $G \times W$ in to W, usually denote by $(g, w) \to w \in W$ such that $g(hw) = (gh)w$ and $ew = w$ for all $g, h \in G$ and $w \in W$ where e is the identity element of G. We also say that G acts on W on the left.

Definition (2.8):-[3]

Let a group G act on a set W, and $U \subseteq W$, $G_{(U)} = \{g: gu = u, \forall\ u \in U\}$ is called the pointwise stabilizer of U.

Definition (2.9): [3]

Let a group G acts on a set W, $U \subseteq W$, and $GU = \{gu: u \in U, g \in G\}$. We call U invariant under G (or G-invariant) if $GU = U$.

Remark (2.10) :- [7]

For a move $\boldsymbol{z}$, the positive part $\boldsymbol{z}^{+} = \{z^{+}{}_{\boldsymbol{i}}\}_{\boldsymbol{i}\in I}$ and the negative part $\boldsymbol{z}^{-} = \{z^{-}{}_{\boldsymbol{i}}\}_{\boldsymbol{i}\in I}$ are defined by $z^{+}{}_{\boldsymbol{i}} = \max(z_i, 0)$, $z^{-}{}_{\boldsymbol{i}} = \max(-z_i, 0)$, respectively, Then $\boldsymbol{z} = \boldsymbol{z}^{+} - \boldsymbol{z}^{-}$ and $\boldsymbol{z}^{+}, \boldsymbol{z}^{-} \in \mathbb{N}^n$moreover, $\boldsymbol{z}^{+}$ and $\boldsymbol{z}^{-}$ are in the same $\boldsymbol{t}$-fiber, i.e., $\boldsymbol{z}^{+}, \boldsymbol{z}^{-} \in A^{-1}[\boldsymbol{t}]$ for $\boldsymbol{t} = A\boldsymbol{z}^{+} = A\boldsymbol{z}^{-}$**.** We define the *degree* of $\boldsymbol{z}$ **as the sample size of** $\boldsymbol{z}^{+}$or $(\boldsymbol{z}^{-})$ and denote it by $\deg(\boldsymbol{z}) = |\boldsymbol{z}^{+}| = |z^{-}|$. In the following we denote the set of moves (for a given A) by $M = M_A = \mathbb{Z}^n \cap \ker(A)$.

Theorem (2.11) :- [7]

A collection of binomials $\{p^{z^{+}} - p^{z^{-}} : z \in B\} \subset I_A$ is generating set of toric ideal I_A if and only if $\pm B$ is a Markov basis for A.

In [2] the Authors gave a Markov basis $\boldsymbol{B}$ for $(25n^3 - 66n^2 + 41n) \times 3 \times n$ - contingency tables with fixed two dimensional marginal as follows:

Remark (2.12):- [5]

Let n be a natural number, and let $\boldsymbol{x}_j \in A^{-1}[t]$, $j = 1, \dots, k$ be the representative elements of the set of $3 \times n$ −contingency tables and $\mathbf{B} = \{\boldsymbol{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{\mathrm{k}}\}$ such that each $\boldsymbol{z}_m$,$m = 1,2, \dots k$, is a matrix of dimension $3 \times n$ either has two non-zero columns and the other columns are zero denoted by $\mathbf{2}\boldsymbol{z}_m$, or it has tree non-zero columns and the other columns are zero denoted by $\mathbf{3}\boldsymbol{z}_m$,like

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ -2 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Also, we write the elements of $\mathbf{B}$ as one dimensional column vector as follows:

$\boldsymbol{z_m} = (z_1, \dots, z_{3n})', m = 1, \dots, k$ and $z_s = 0, 1, -1, 2 \; or -2$, $s = 1,2, \dots, 3n$ such that

If $s = 1,2, \dots, n$, then

$$z_s = \begin{cases} 1 & if \; z_{s+n} + z_{s+2n} = -1 \; and \sum_{\substack{i=1 \\ i\neq s}}^{n} z_i = -1 \\ 2 & if \; z_{s+n} + z_{s+2n} = -2 \; and \sum_{\substack{i=1 \\ i\neq s}}^{n} z_i = -2 \\ 0 & if \; z_{s+n} + z_{s+2n} = 0 \; and \sum_{\substack{i=1 \\ i\neq s}}^{n} z_i = 0 \\ -1 & if \; z_{s+n} + z_{s+2n} = 1 \; and \sum_{\substack{i=1 \\ i\neq s}}^{n} z_i = 1 \\ -2 & if \; z_{s+n} + z_{s+2n} = 2 \; and \sum_{\substack{i=1 \\ i\neq s}}^{n} z_i = 2 \end{cases} \quad (1)$$

If $s = n + 1, n + 2, \dots, 2n$, then

$$z_s = \begin{cases} 1 & if \; z_{s-n} + z_{s+n} = -1 \; and \sum_{\substack{i=n+1 \\ i\neq s}}^{2n} z_i = -1 \\ 2 & if \; z_{s-n} + z_{s+n} = -2 \; and \sum_{\substack{i=n+1 \\ i\neq s}}^{2n} z_i = -2 \\ 0 & if \; z_{s-n} + z_{s+n} = 0 \; and \sum_{\substack{i=n+1 \\ i\neq s}}^{2n} z_i = 0 \\ -1 & if \; z_{s-n} + z_{s+n} = 1 \; and \sum_{\substack{i=n+1 \\ i\neq s}}^{2n} z_i = 1 \\ -2 & if \; z_{s-n} + z_{s+n} = 2 \; and \sum_{\substack{i=n+1 \\ i\neq s}}^{2n} z_i = 2 \end{cases} \quad (2)$$

If $s = 2n + 1, 2n + 2, \dots, 3n$, then

$$z_s = \begin{cases} 1 & if\ z_{s-n} + z_{s-2n} = -1\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = -1 \\ 2 & if\ z_{s-n} + z_{s-2n} = -2\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = -2 \\ 0 & if\ z_{s-n} + z_{s-2n} = 0\ \ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = 0 \\ -1 & if\ z_{s-n} + z_{s-2n} = 1\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = 1 \\ -2 & if\ z_{s-n} + z_{s-2n} = 2\ and \sum_{\substack{i=2n+1 \\ i \neq s}}^{3n} z_i = 2 \end{cases} \qquad (3)$$

Theorem (2.13) :- [5]
The number of elements in **B** equal to $25n^3 - 66n^2 + 41n$.
Remark (2.14):- [5]
Given a contingency table **,** the entry of the matrix A in the column indexed by $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and row $(\sum_{i=1}^{n} x_i, \sum_{i=n+1}^{2n} x_i, \sum_{i=2n+1}^{3n} x_i, x_1 + x_{n+1} + x_{2n+1}, x_2 + x_{n+2} + x_{2n+2}, \dots, x_n + x_{2n} + x_{3n})$ will be equal to one if x_i a pears in the $(\sum_{i=1}^{n} x_i)$ and it will zero otherwise. Then

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(n+3)\times 3n}$$

Theorem (2.15):- [5]
$\mathbf{B} = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_{(25n^3-66n^2+41n)}\}$ is a set of moves.
Remark (2.16):- [5]
We find the elements $\boldsymbol{x}_i \in A^{-1}[t],\ i = 1, \dots, (25n^3 - 66n^2 + 41n)$ by using the elements of the set $= \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_{(25n^3-66n^2+41n)}\}$, Let $\boldsymbol{z}_m$ be an element of **B** such that $\boldsymbol{z_m} = \boldsymbol{x}_m - \boldsymbol{x}_{m-1}$, $m = 1, 2, \dots, (25n^3 - 66n^2 + 41n) - 1$ and $\boldsymbol{z}_{(25n^3-69n^2+44n)} = \boldsymbol{x}_0 - \boldsymbol{x}_{(25n^3-69n^2+44n)-1}$, where $\boldsymbol{x_i} \in A^{-1}[\boldsymbol{t}],\ \boldsymbol{i} = 0, 1, \dots,\ (25n^3 - 66n^2 + 41n) - 1$.
Corollary (2.17):- [5]
The set **B** of moves in theorem (2.15) is a Markov basis**.**

3. The Main Results

Let $n \in \mathbb{N}$ and $n \geq 2$, and let $\boldsymbol{x}_j \in A^{-1}[t],\ j = 0, \dots, (25n^3 - 66n^2 + 41n) - 1$ be representative elements of the set of $3 \times n$ –contingency tables and $\mathrm{B} = \{\mathbf{z_1}, \mathbf{z_2}, \dots, \mathbf{z}_{(25n^3-66n^2+41n)}\}$ is a set of Markov basis , and $H_1 = \{e, r^n, r^{2n}, l_{1,2}, l_{1,3}, l_{2,3}\}$ and $H_2 = < v_{i,j}: i, j = 1,2, \dots, n$ such that $i \neq j >$ are subgroup of symmetric group S_{3n} where $r = (1\ 2\ \dots\ 3n)$, $l_{1,2} = (1\ \ n+1)(2\ \ n+2) \dots (n\ \ 2n)$, $l_{1,3} = (1\ \ 2n+1)(2\ \ 2n+2) \dots (n\ \ 3n)$, $l_{2,3} = (n+1\ \ 2n+1)(n+2\ \ 2n+2) \dots (2n\ \ 3n)$ and $v_{i,j} = (i\ \ j)(n+i\ \ n+j)(2n+i\ \ 2n+j)$ then write each $g \in S_{3n}$ as a $3n \times 3n$ permutation matrix $T_g = \{p_{ij}\} = \{\delta_i, g(i)\}$, where δ is the Kronecker's delta such that $T_{g_1.g_2} = T_{g_1}T_{g_2}$ for $g_1, g_2 \in S_{3n,}$ and$T_{g^{-1}} = T^t{}_g$. Then the identity matrix of order n is denoted by E_n for the unit element $e \in D_n$.
Now, we consider a left action of symmetric group S_{3n} , $3n = |I|$, on $A^{-1}[t]$ the set of $(25n^3 - 66n^2 + 41n) \times 3 \times n$ - contingency tables, and the action of symmetric group S_{3n} on the set of Markov basis **B**.
Definition (3.1):
let H be a group and $A^{-1}[t]$ be a set of $3 \times n$ –contingency tables . A left action of H on $A^{-1}[t]$ is a function $H \times A^{-1}[t] \ni (h, \boldsymbol{x}) \to h\boldsymbol{x} = T_h\boldsymbol{x} \in A^{-1}[t]$.

Remark (3.2):

A left action of $\boldsymbol{H}$ on the set of all $3n$-diminsional column vectors of integers $\mathbb{Z}^{\boldsymbol{3n}}$ is a function $(h, \boldsymbol{v}) \to h\boldsymbol{v} = T_h\boldsymbol{v} \in \mathbb{Z}^{3n}$ of $H \times \mathbb{Z}^{3n}$ into $\mathbb{Z}^{\boldsymbol{3n}}$, where T is a permutation matrix representation of H and T_h is the permutation matrix of h, $\mathrm{A}^{-1}[\boldsymbol{t}]$, $\mathbf{B} \subseteq \mathbb{Z}^{\boldsymbol{n}}$ when the element of $\mathrm{A}^{-1}[\boldsymbol{t}]$ and $\mathbf{B}$ tread as all $3n$-dimensional column vectors**.** If $\boldsymbol{x} \in \mathrm{A}^{-1}[\boldsymbol{t}]$**,** $\boldsymbol{z} \in \mathbf{B}$**,** and $h \in H$**,** then $T_h\boldsymbol{x}$**,** $T_h\boldsymbol{z} \in \mathbb{Z}^{3n}$ but $T_h\boldsymbol{x}$ may not be in $\mathrm{A}^{-1}[\boldsymbol{t}]$ and $T_h\boldsymbol{z}$ may not be in $\mathbf{B}$ as in the following example.

Example(3.3):

Consider 3×3- contingency table

$\boldsymbol{x} =$

3	2	4	9
3	1	2	6
2	1	1	4
8	4	7	19

$\boldsymbol{x}$ can be represented as 9-dimensional column vector of non negative integer $\boldsymbol{x} = (3,2,4,3,1,2,2,1,1)' \in \mathbb{N}^9$. Then $\boldsymbol{x} \in \mathrm{A}^{-1}[\boldsymbol{t}]$, where $\mathrm{A}^{-1}[\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^9 : A\boldsymbol{x} = \boldsymbol{t}\}$,
From Remark (2.14) we get

$$\mathrm{A} = \begin{bmatrix} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{bmatrix}_{6\times 9}$$

, and $\boldsymbol{t} = (9,6,4,8,4,7)'$. If $h = r = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9) \in S_9$, then

$$T_g\boldsymbol{x} = \begin{bmatrix} 0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0 \end{bmatrix} \begin{bmatrix} 3\\2\\4\\3\\1\\2\\2\\1\\1 \end{bmatrix} = \begin{bmatrix} 1\\3\\2\\4\\3\\1\\2\\2\\1 \end{bmatrix} = \begin{bmatrix} x_9\\x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8 \end{bmatrix} \notin \mathrm{A}^{-1}[\boldsymbol{t}]\text{, since}$$

$$\mathrm{A}T_g\boldsymbol{x} = \begin{bmatrix} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{bmatrix} \begin{bmatrix} 1\\3\\2\\4\\3\\1\\2\\2\\1 \end{bmatrix} = \begin{bmatrix} 6\\8\\5\\7\\8\\4 \end{bmatrix} \neq \begin{bmatrix} 9\\6\\4\\8\\4\\7 \end{bmatrix} = \boldsymbol{t},$$

And if $\boldsymbol{z} = (-1, 0, 1, 1, 0, -1, 0, 0, 0)'$,

$$\text{then } T_g\boldsymbol{z} = \begin{bmatrix} 0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0 \end{bmatrix} \begin{bmatrix} -1\\0\\1\\1\\0\\-1\\0\\0\\0 \end{bmatrix} = \begin{bmatrix} 0\\-1\\0\\1\\1\\0\\-1\\0\\0 \end{bmatrix} \notin \mathbf{B}.$$

Now, we will find subgroups of the symmetric group S_{3n}, such that the Markov basis $\mathbf{B}$ invariant under their actions.

Remark (3.4):
Let H be a subgroup of S_{3n}, such that **B** is H-invariant ($H(\mathbf{B}) = \mathbf{B}$). Then a left action of H on **B** is a function $H \times \mathbf{B} \rightarrow h\boldsymbol{z}_j \in \mathbf{B}$.
Theorem(3.5):
The Markov basis **B** is H_1-invariant, where $H_1 = \{e, r^n, r^{2n}, l_{1,2}, l_{1,3}, l_{2,3}\}$.
Proof:
To prove **B** is H_1-invariant
Let $\boldsymbol{z}_j \in \mathbf{B}$, $\boldsymbol{z}_j = (z_1, z_2, \dots, z_n, z_{n+1}, \dots, z_{2n}, z_{2n+1}, \dots, z_{3n-1}, z_{3n})'$.
$\Longrightarrow e\boldsymbol{z}_j = T_e\boldsymbol{z}_j \in \mathbf{B}$.
Since $r = (1\ 2\ \dots\ 3n)$, therefore
$r^n = (1\ \ n+1\ \ 2n+1)(2\ \ n+2\ \ 2n+2)\dots(n\ \ 2n\ \ 3n)$. This implies
$r^n\boldsymbol{z}_j = T_{r^n}\,\boldsymbol{z}_j = (z_{2n+1}, z_{2n+2}, \dots, z_{3n}, z_1, \dots, z_n, z_{n+1}, \dots, z_{2n-1}, z_{2n})' \in \mathbf{B}$.
And $r^{2n} = (1\ \ 2n+1\ \ n+1)(2\ \ 2n+2\ \ n+2)\dots(n\ \ 3n\ \ 2n)$. This implies
$r^{2n}\boldsymbol{z}_j = T_{2n}\,\boldsymbol{z}_j = (z_{n+1}, z_{n+2}, \dots, z_{2n}, z_{2n+1}, \dots, z_{3n}, z_1, \dots, z_{n-1}, z_n)' \in \mathbf{B}$.
Since $l_{1,2} = (1\ \ n+1)(2\ \ n+2)\dots(n\ \ 2n)$, then
$l_{1,2}\boldsymbol{z}_j = T_{l_{1,2}}\,\boldsymbol{z}_j = (z_{n+1}, z_{n+2}, \dots, z_{2n}, z_1, z_2, \dots, z_n, z_{2n+1}, z_{2n+2}, \dots, z_{3n-1}, z_{3n})' \in \mathbf{B}$.
And since $l_{1,3} = (1\ \ 2n+1)(2\ \ 2n+2)\dots(n\ \ 3n)$, this implies
$l_{1,3}\boldsymbol{z}_j = T_{l_{1,3}}\,\boldsymbol{z}_j = (z_{2n+1}, z_{2n+2}, \dots, z_{3n}, z_{n+1}, z_{n+2}, \dots, z_{2n}, z_1, z_2, \dots, z_n)' \in \mathbf{B}$. And
Since $l_{2,3} = (n+1\ \ 2n+1)(n+2\ \ 2n+2)\dots(2n\ \ 3n)$. This implies
$l_{2,3}\boldsymbol{z}_j = T_{l_{2,3}}\,\boldsymbol{z}_j = (z_1, z_2, \dots, z_n, z_{2n+1}, z_{2n+2}, \dots, z_{3n}, z_{n+1}, z_{n+2}, \dots, z_{2n})' \in \mathbf{B}$.
Then $H_1(\mathbf{B}) = \mathbf{B}$, then **B** is H_1-invariant.
Theorem(3.6):
The Markov basis **B** is H_2-invariant, where
$H_2 =< v_{i,j}: i, j = 1,2, \dots, n$ such that $i \leq j\ >$.
Proof:
To prove **B** is H_2-invariant
Let $\boldsymbol{z}_k \in \mathbf{B}$, $\boldsymbol{z}_k = (z_1, z_2, \dots, z_n, z_{n+1}, \dots, z_{2n}, z_{2n+1}, \dots, z_{3n-1}, z_{3n})'$.
$\Longrightarrow v_{i,i}\boldsymbol{z}_k = e\boldsymbol{z}_k = T_e\boldsymbol{z}_k \in \mathbf{B}$.
Since $v_{i,j} = (i\ \ j)(i+n\ \ j+n)(i+2n\ \ j+2n)$. Then
$v_{i,j}\boldsymbol{z}_k = T_{v_{i,j}}\,\boldsymbol{z}_k = (z_1, z_2, \dots, z_{i-1}, z_j, z_{i+1}, \dots, z_{j-1}, z_i, z_{j+1}, \dots z_{i+n-1}, z_{j+n}, z_{i+n+1}, \dots, z_{j+n-1}, z_{i+n},$
$z_{j+n+1}, \dots z_{i+2n-1}, z_{j+2n}, z_{i+2n+1}, \dots, z_{j+2n-1}, z_{i+2n}, z_{j+2n+1}, \dots, z_{3n})' \in \mathbf{B}$
If $v_{i_1,j_1}, v_{i_2,j_2} \in H_2$, this implies
$(v_{i_1,j_1}v_{i_2,j_2})\boldsymbol{z}_k = v_{i_1,j_1}(v_{i_2,j_2}\boldsymbol{z}_k) \in v_{i_1,j_1}\mathbf{B} = \mathbf{B}$
This implies that $H_2(\mathbf{B}) = \mathbf{B}$, then **B** is H_2-invariant.
Corollary (3.7):
The subgroup $H = H_1H_2$ of S_{3n} is the Largest Subgroup of the group S_{3n} such that the Markov basis B is H-invariant.
Proof:
The Markov basis B is H_1-invariant (By Theorem (3.5))
And the Markov basis B is H_2-invariant (By Theorem (3.6))
To show The Markov basis B is H-invariant
Let $h = h_1h_2 \in H = H_1H_2$, where $h_1 \in H_1$ and $h_2 \in H_2$
And let $\boldsymbol{z}_j \in \mathbf{B}$, then $h\boldsymbol{z}_j = h_1(h_2\boldsymbol{z}_j) \in H_1\mathbf{B} = \mathbf{B}$
Therefore $H(\mathbf{B}) = \mathbf{B}$, then **B** is H-invariant
To prove H is the Largest Subgroup of the group S_{3n} suth that $\boldsymbol{B}$ *is* H-invariant
Every elment of $\boldsymbol{B}$ Consists of 3 rows and n columns, so the number of the Largest Subgroup of the group S_{3n} suth that $\boldsymbol{B}$ *is* H-invariant equal to $3! \times n! = 6n! \dots *$.
Now, since $H_1 \cap H_2 = \{e\}$, thus imples that $O(H) = O(H_1H_2) = O(H_1)O(H_2) = 6(n!) \dots **$

From $*$ and $**$ we get H is the Largest Subgroup of the group S_{3n} such that **B** is H-invariant.
Now, use $H(\boldsymbol{B})$ to generate $3 \times n$ Markov basis $\boldsymbol{B}$, and contingincy table with give row and colum sums.

Remark (3.8):

Let $\boldsymbol{t} = (t_1, t_2, t_3, \dots, t_{n+3})'$, $\boldsymbol{x}_i \in \mathrm{A}^{-1}[\boldsymbol{t}]$ and $h \in H$. Then $h\boldsymbol{x}_i \in \mathrm{A}^{-1}[h\boldsymbol{t}]$ where $h\boldsymbol{t} = (ht_1, ht_2, ht_3, \dots, ht_{n+3})'$, $\mathrm{A}^{-1}[h\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^{\boldsymbol{n}} : \boldsymbol{Ax} = h\boldsymbol{t}\}$. So, we have $6n!$ kinds of $h\boldsymbol{t}$-fibers $\mathrm{A}^{-1}[g\boldsymbol{t}]$, $\mathrm{A}^{-1}[\,gr^{n}\boldsymbol{t}]$, $\mathrm{A}^{-1}[\,gr^{2n}\boldsymbol{t}]$, $\mathrm{A}^{-1}\left[gl_{1,2}\boldsymbol{t}\right]$, $\mathrm{A}^{-1}\left[gl_{1,3}\boldsymbol{t}\right]$ and $\mathrm{A}^{-1}\left[gl_{2,3}\boldsymbol{t}\right]$, where $g \in H_2$.

Theorem (3.9):

If $h \in H$. Then $\boldsymbol{B}$ is a Markov basis for $(25n^3 - 66n^2 + 41n)$ contingency tables $h\boldsymbol{x}_0, h\boldsymbol{x}_1, \dots, h\boldsymbol{x}_{(25n^3-66n^2+41n)\,-1}$ in $\mathrm{A}^{-1}[h\boldsymbol{t}]$.

Proof:

By remark (2.16) we have $\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + \boldsymbol{z}_i$, for all $i = 1, 2, \dots, (25n^3 - 66n^2 + 41n) - 1$ and $\boldsymbol{x}_0 = \boldsymbol{x}_{(25n^3-66n^2+41n)\,-1} + \boldsymbol{z}_{(25n^3-66n^2+41n)}$.

New, if $h \in H$

$h\boldsymbol{x}_i = T_h\boldsymbol{x}_i = T_h(\boldsymbol{x}_{i-1} + \boldsymbol{z}_i) = T_h\boldsymbol{x}_{i-1} + T_h\boldsymbol{z}_i$ for all $i = 1, 2, \dots, (25n^3 - 66n^2 + 41n) - 1$ and

$h\boldsymbol{x}_0 = T_h\boldsymbol{x}_0 = T_h\left(\boldsymbol{x}_{(25n^3-66n^2+41n)-1} + \boldsymbol{z}_{(25n^3-66n^2+41n)}\right) = T_h\boldsymbol{x}_{(25n^3-66n^2+41n)-1} + T_h\boldsymbol{z}_{(25n^3-66n^2+41n)}$.

The $T_h\boldsymbol{x}_i = h\boldsymbol{x}_i \in \mathrm{A}^{-1}[h\boldsymbol{t}]$ for all $i = 0, 1, 2, \dots, (25n^3 - 66n^2 + 41n) - 1$, and $\boldsymbol{B}$ generate $(25n^3 - 66n^2 + 41n)$ contingency tables $h\boldsymbol{x}_0, h\boldsymbol{x}_1, \dots, h\boldsymbol{x}_{(25n^3-66n^2+41n)\,-1}$.

To prove $\boldsymbol{B}$ is a Markov basis for the contingency tables $h\boldsymbol{x}_0, h\boldsymbol{x}_1, \dots, h\boldsymbol{x}_{(25n^3-66n^2+41n)\,-1}$

Let $h\boldsymbol{x}_i, h\boldsymbol{x}_j \in A^{-1}[h\boldsymbol{t}]$, such that $j \geq i$.

By Remark (3.7), we get

$h\boldsymbol{x}_j = h\boldsymbol{x}_i + \sum_{k=i+1}^{j} h\boldsymbol{z}_k \Rightarrow h\boldsymbol{x}_j - h\boldsymbol{x}_i = \sum_{k=i+1}^{j} h\boldsymbol{z}_k$, and $h\boldsymbol{z}_k \in Ker(A)$

$\Rightarrow \sum_{k=i+1}^{j} h\boldsymbol{z}_k \in Ker(A) \Rightarrow h\boldsymbol{x}_j - h\boldsymbol{x}_i \in Ker(A) \Rightarrow \boldsymbol{x}_i \sim \boldsymbol{x}_j$

$\Rightarrow A^{-1}[\boldsymbol{t}]$ Constitutes one $\boldsymbol{B}$ equivalence class for all $\boldsymbol{t}$

$\Rightarrow$ The set **B** is a Markov basis.

Example (3.10):- Consider 3×2 $-$contingency table.

$x =$

3	3	6
4	4	8
4	2	6
11	9	20

Then $\boldsymbol{x}$ can be represented as 6-dimensional column vector as $\boldsymbol{x} = (2{,}3{,}5{,}1{,}3{,}1{,}4{,}3{,}3)' \in \mathbb{N}^9$.
Then $\boldsymbol{x} \in \mathrm{A}^{-1}[\boldsymbol{t}]$, where $\mathrm{A}^{-1}[\boldsymbol{t}] = \{\boldsymbol{x} \in \mathbb{N}^9 : A\boldsymbol{x} = \boldsymbol{t}\}$, and $\boldsymbol{t} = (10{,}5{,}10{,}7{,}9{,}9)'$. Then by [2] we can find the elements of $\boldsymbol{B}$

$$\mathbf{z}_1 = \begin{array}{|c|c|}\hline 2 & -2 \\ \hline -2 & 2 \\ \hline 0 & 0 \\ \hline\end{array}, \quad \mathbf{z}_2 = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline 1 & -1 \\ \hline 0 & 0 \\ \hline\end{array}, \quad \mathbf{z}_3 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & -1 \\ \hline -1 & 1 \\ \hline\end{array}$$

$$\mathbf{z}_4 = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline 0 & 0 \\ \hline\end{array}, \quad \mathbf{z}_5 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 2 & -2 \\ \hline -2 & 2 \\ \hline\end{array}, \quad \mathbf{z}_6 = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline 2 & -2 \\ \hline -1 & 1 \\ \hline\end{array}$$

$$\mathbf{z}_7 = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline -1 & 1 \\ \hline 2 & -2 \\ \hline\end{array}, \quad \mathbf{z}_8 = \begin{array}{|c|c|}\hline -2 & 2 \\ \hline 2 & -2 \\ \hline 0 & 0 \\ \hline\end{array}, \quad \mathbf{z}_9 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline -1 & 1 \\ \hline 1 & -1 \\ \hline\end{array}$$

$\mathbf{z}_{10} =$

-1	1
0	0
1	-1

, $\mathbf{z}_{11} =$

1	-1
-2	2
1	-1

, $\mathbf{z}_{12} =$

1	-1
1	-1
-2	2

$\mathbf{z}_{13} =$

2	-2
0	0
-2	2

, $\mathbf{z}_{14} =$

-2	2
1	-1
1	-1

, $\mathbf{z}_{15} =$

1	-1
0	0
-1	1

$\mathbf{z}_{16} =$

2	-2
-1	1
-1	1

, $\mathbf{z}_{17} =$

0	0
-2	2
2	-2

, $\mathbf{z}_{18} =$

-2	2
0	0
2	-2

Then we can find 18 elements in $A^{-1}[\boldsymbol{t}]$ of 3×2 −contingency table as

$\mathbf{x}_0 =$

3	3	6
4	4	8
4	2	6
11	9	20

, $\mathbf{x}_1 =$

5	1	6
2	6	8
4	2	6
11	9	20

, $\mathbf{x}_2 =$

4	2	6
3	5	8
4	2	6
11	9	20

$\mathbf{x}_3 =$

4	2	6
4	4	8
3	3	6
11	9	20

, $\mathbf{x}_4 =$

5	1	6
3	5	8
3	3	6
11	9	20

, $\mathbf{x}_5 =$

5	1	6
5	3	8
1	5	6
11	9	20

$\mathbf{x}_6 =$

4	2	6
7	1	8
0	6	6
11	9	20

, $\mathbf{x}_7 =$

3	3	6
6	2	8
2	4	6
11	9	20

, $\mathbf{x}_8 =$

1	5	6
8	0	8
2	4	6
11	9	20

$\mathbf{x}_9 =$

1	5	6
7	1	8
3	3	6
11	9	20

, $\mathbf{x}_{10} =$

0	6	6
7	1	8
4	2	6
11	9	20

, $\mathbf{x}_{11} =$

1	5	6
5	3	8
5	1	6
11	9	20

$\mathbf{x}_{12} =$

2	4	6
6	2	8
3	3	6
11	9	20

, $\mathbf{x}_{13} =$

4	2	6
6	2	8
1	5	6
11	9	20

, $\mathbf{x}_{14} =$

2	4	6
7	1	8
2	4	6
11	9	20

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$\mathbf{x_{15}} =$

3	3	6
7	1	8
1	5	6
11	9	20

, $\mathbf{x_{16}} =$

5	1	6
6	2	8
0	6	6
11	9	20

, $\mathbf{x_{17}} =$

5	1	6
4	4	8
2	4	6
11	9	20

Therefore we find the set $T_{r^2}\mathbf{B} = \{T_{r^2}\mathbf{z}_1, T_{r^2}\mathbf{z}_2, T_{r^2}\mathbf{z}_3, T_{r^2}\mathbf{z}_4,\ T_{r^2}\mathbf{z}_5, T_{r^2}\mathbf{z}_6, T_{r^2}\mathbf{z}_7,$ $T_{r^2}\mathbf{z}_8 T_{r^2}\mathbf{z}_9, T_{r^2}\mathbf{z}_{10}, T_{r^2}\mathbf{z}_{11}, T_{r^2}\mathbf{z}_{12}, T_{r^2}\mathbf{z}_{13}, T_{r^2}\mathbf{z}_{14}, T_{r^2}\mathbf{z}_{15}, T_{r^2}\mathbf{z}_{16}, T_{r^2}\mathbf{z}_{17}, T_{r^2}\mathbf{z}_{18}\}$
Such that

$T_{r^2}\mathbf{z}_1 =$

0	0
2	-2
-2	2

, $T_{r^2}\mathbf{z}_2 =$

0	0
-1	1
1	-1

, $T_{r^2}\mathbf{z}_3 =$

-1	1
0	0
1	-1

$T_{r^2}\mathbf{z}_4 =$

0	0
1	-1
-1	1

, $T_{r^2}\mathbf{z}_5 =$

-2	2
0	0
2	-2

, $T_{r^2}\mathbf{z}_6 =$

-1	1
-1	1
2	-2

$T_{r^2}\mathbf{z}_7 =$

2	-2
-1	1
-1	1

, $T_{r^2}\mathbf{z}_8 =$

0	0
-2	2
2	-2

, $T_{r^2}\mathbf{z}_9 =$

1	-1
0	0
-1	1

$T_{r^2}\mathbf{z}_{10} =$

1	-1
-1	1
0	0

, $T_{r^2}\mathbf{z}_{11} =$

1	-1
1	-1
-2	2

, $T_{r^2}\mathbf{z}_{12} =$

-2	2
1	-1
1	-1

$T_{r^2}\mathbf{z}_{13} =$

-2	2
2	-2
0	0

, $T_{r^2}\mathbf{z}_{14} =$

1	-1
-2	2
1	-1

, $T_{r^2}\mathbf{z}_{15} =$

-1	1
1	-1
0	0

$T_{r^2}\mathbf{z}_{16} =$

-1	1
2	-2
-1	1

, $T_{r^2}\mathbf{z}_{17} =$

2	-2
0	0
-2	2

, $T_{r^2}\mathbf{z}_{18} =$

2	-2
-2	2
0	0

We find the set$\{T_{r^2}\boldsymbol{x}_0,\ T_{r^2}\boldsymbol{x}_1, T_{r^2}\boldsymbol{x}_2,\ T_{r^2}\boldsymbol{x}_3,\ T_{r^2}\boldsymbol{x}_4, T_{r^2}\boldsymbol{x}_5, T_{r^2}\boldsymbol{x}_6,\ T_{r^2}\boldsymbol{x}_7, T_{r^2}\boldsymbol{x}_8,$ $T_{r^2}\boldsymbol{x}_9,\ T_{r^2}\boldsymbol{x}_{10}, T_{r^2}\boldsymbol{x}_{11}, T_{r^2}\boldsymbol{x}_{12},\ T_{r^2}\boldsymbol{x}_{13}, T_{r^2}\boldsymbol{x}_{14},\ T_{r^2}\boldsymbol{x}_{15},\ T_{r^2}\boldsymbol{x}_{16}, T_{r^2}\boldsymbol{x}_{17}\} \subseteq \mathrm{A}^{-1}[r^2\boldsymbol{t}]$ ($r^3\boldsymbol{t}$-fibers**)** where $r^2\boldsymbol{t} = (6, 6, 8, 11, 9)'$. We have.

$T_{r^2}\mathbf{x_0} =$

4	2	6
3	3	6
4	4	8
11	9	20

, $T_{r^2}\mathbf{x_1} =$

4	2	6
5	1	6
2	6	8
11	9	20

, $T_{r^2}\mathbf{x_2} =$

4	2	6
4	2	6
3	5	8
11	9	20

$$T_{r^2}\mathbf{x_3} = \begin{array}{|c|c|c|}\hline 3 & 3 & 6 \\ \hline 4 & 2 & 6 \\ \hline 4 & 4 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_4} = \begin{array}{|c|c|c|}\hline 3 & 3 & 6 \\ \hline 5 & 1 & 6 \\ \hline 3 & 5 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_5} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 5 & 1 & 6 \\ \hline 5 & 3 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array}$$

$$T_{r^2}\mathbf{x_6} = \begin{array}{|c|c|c|}\hline 0 & 6 & 6 \\ \hline 4 & 2 & 6 \\ \hline 7 & 1 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_7} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 3 & 3 & 6 \\ \hline 6 & 2 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} , T_{r^2}\mathbf{x_8} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 1 & 5 & 6 \\ \hline 8 & 0 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array}$$

$$T_{r^2}\mathbf{x_9} = \begin{array}{|c|c|c|}\hline 3 & 3 & 6 \\ \hline 1 & 5 & 6 \\ \hline 7 & 1 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_{10}} = \begin{array}{|c|c|c|}\hline 4 & 2 & 6 \\ \hline 0 & 6 & 6 \\ \hline 7 & 1 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_{11}} = \begin{array}{|c|c|c|}\hline 5 & 1 & 6 \\ \hline 1 & 5 & 6 \\ \hline 5 & 3 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array}$$

$$T_{r^2}\mathbf{x_{12}} = \begin{array}{|c|c|c|}\hline 3 & 3 & 6 \\ \hline 2 & 4 & 6 \\ \hline 6 & 2 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_{13}} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 4 & 2 & 6 \\ \hline 6 & 2 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_{14}} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 2 & 4 & 6 \\ \hline 7 & 1 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array}$$

$$T_{r^2}\mathbf{x_{15}} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 3 & 3 & 6 \\ \hline 7 & 1 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_{16}} = \begin{array}{|c|c|c|}\hline 0 & 6 & 6 \\ \hline 5 & 1 & 6 \\ \hline 6 & 2 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array} \quad , T_{r^2}\mathbf{x_{17}} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 5 & 1 & 6 \\ \hline 4 & 4 & 8 \\ \hline 11 & 9 & 20 \\ \hline \end{array}$$

And we find the set $T_{r^2}\mathbf{B} = \{T_{v_{1,2}}\mathbf{z}_1, T_{v_{1,2}}\mathbf{z}_2, T_{v_{1,2}}\mathbf{z}_3, T_{v_{1,2}}\mathbf{z}_4,\ T_{v_{1,2}}\mathbf{z}_5, T_{v_{1,2}}\mathbf{z}_6, T_{v_{1,2}}\mathbf{z}_7,$ $T_{v_{1,2}}\mathbf{z}_8, T_{v_{1,2}}\mathbf{z}_9, T_{v_{1,2}}\mathbf{z}_{10}, T_{v_{1,2}}\mathbf{z}_{11}, T_{v_{1,2}}\mathbf{z}_{12}, T_{v_{1,2}}\mathbf{z}_{13}, T_{v_{1,2}}\mathbf{z}_{14}, T_{v_{1,2}}\mathbf{z}_{15}, T_{v_{1,2}}\mathbf{z}_{16}, T_{v_{1,2}}\mathbf{z}_{17},$ $T_{v_{1,2}}\mathbf{z}_{18}\}$, Such that

$$T_{v_{1,2}}\mathbf{z}_1 = \begin{array}{|c|c|}\hline -2 & 2 \\ \hline 2 & -2 \\ \hline 0 & 0 \\ \hline \end{array} \quad , T_{v_{1,2}}\mathbf{z}_2 = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline 0 & 0 \\ \hline \end{array} \quad , T_{v_{1,2}}\mathbf{z}_3 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline -1 & 1 \\ \hline 1 & -1 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{z}_4 = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline 1 & -1 \\ \hline 0 & 0 \\ \hline \end{array} \quad , T_{v_{1,2}}\mathbf{z}_5 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline -2 & 2 \\ \hline 2 & -2 \\ \hline \end{array} \quad , T_{v_{1,2}}\mathbf{z}_6 = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -2 & 2 \\ \hline 1 & -1 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{z}_7 = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline 1 & -1 \\ \hline -2 & 2 \\ \hline \end{array} \quad , T_{v_{1,2}}\mathbf{z}_8 = \begin{array}{|c|c|}\hline 2 & -2 \\ \hline -2 & 2 \\ \hline 0 & 0 \\ \hline \end{array} \quad , T_{v_{1,2}}\mathbf{z}_9 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{z}_{10} = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline 0 & 0 \\ \hline -1 & 1 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{z}_{11} = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline 2 & -2 \\ \hline -1 & 1 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{z}_{12} = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline -1 & 1 \\ \hline 2 & -2 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{z}_{13} = \begin{array}{|c|c|}\hline -2 & 2 \\ \hline 0 & 0 \\ \hline 2 & -2 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{z}_{14} = \begin{array}{|c|c|}\hline 2 & -2 \\ \hline -1 & 1 \\ \hline -1 & 1 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{z}_{15} = \begin{array}{|c|c|}\hline -1 & 1 \\ \hline 0 & 0 \\ \hline 1 & -1 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{z}_{16} = \begin{array}{|c|c|}\hline -2 & 2 \\ \hline 1 & -1 \\ \hline 1 & -1 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{z}_{17} = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 2 & -2 \\ \hline -2 & 2 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{z}_{18} = \begin{array}{|c|c|}\hline 2 & -2 \\ \hline 0 & 0 \\ \hline -2 & 2 \\ \hline \end{array}$$

Then We find the set$\{T_{v_{1,2}}\boldsymbol{x}_0, T_{v_{1,2}}\boldsymbol{x}_1, T_{v_{1,2}}\boldsymbol{x}_2, T_{v_{1,2}}\boldsymbol{x}_3, T_{v_{1,2}}\boldsymbol{x}_4, T_{v_{1,2}}\boldsymbol{x}_5, T_{v_{1,2}}\boldsymbol{x}_6, T_{v_{1,2}}\boldsymbol{x}_7, T_{v_{1,2}}\boldsymbol{x}_8, T_{v_{1,2}}\boldsymbol{x}_9, T_{v_{1,2}}\boldsymbol{x}_{10}, T_{v_{1,2}}\boldsymbol{x}_{11}, T_{v_{1,2}}\boldsymbol{x}_{12}, T_{v_{1,2}}\boldsymbol{x}_{13}, T_{v_{1,2}}\boldsymbol{x}_{14}, T_{v_{1,2}}\boldsymbol{x}_{15}, T_{v_{1,2}}\boldsymbol{x}_{16}, T_{v_{1,2}}\boldsymbol{x}_{17}\} \subseteq \mathrm{A}^{-1}[r^2\boldsymbol{t}]$ ($r^3\boldsymbol{t}$-fibers) where $r^2\boldsymbol{t} = (6, 6, 8, 11, 9)'$. We have.

$$T_{v_{1,2}}\mathbf{x}_{0} = \begin{array}{|c|c|c|}\hline 3 & 3 & 6 \\ \hline 4 & 4 & 8 \\ \hline 2 & 4 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{1} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 6 & 2 & 8 \\ \hline 2 & 4 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{2} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 5 & 3 & 8 \\ \hline 2 & 4 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{x}_{3} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 4 & 4 & 8 \\ \hline 3 & 3 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{4} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 5 & 3 & 8 \\ \hline 3 & 3 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{5} = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 3 & 5 & 8 \\ \hline 5 & 1 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{x}_{6} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 1 & 7 & 8 \\ \hline 6 & 0 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{7} = \begin{array}{|c|c|c|}\hline 3 & 3 & 6 \\ \hline 2 & 6 & 8 \\ \hline 4 & 2 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{8} = \begin{array}{|c|c|c|}\hline 5 & 1 & 6 \\ \hline 0 & 8 & 8 \\ \hline 4 & 2 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{x}_{9} = \begin{array}{|c|c|c|}\hline 5 & 1 & 6 \\ \hline 1 & 7 & 8 \\ \hline 3 & 3 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{10} = \begin{array}{|c|c|c|}\hline 6 & 0 & 6 \\ \hline 1 & 7 & 8 \\ \hline 2 & 4 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{11} = \begin{array}{|c|c|c|}\hline 5 & 1 & 6 \\ \hline 3 & 5 & 8 \\ \hline 1 & 5 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}$$

$$T_{v_{1,2}}\mathbf{x}_{12} = \begin{array}{|c|c|c|}\hline 4 & 2 & 6 \\ \hline 2 & 6 & 8 \\ \hline 3 & 3 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{13} = \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 2 & 6 & 8 \\ \hline 5 & 1 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}, T_{v_{1,2}}\mathbf{x}_{14} = \begin{array}{|c|c|c|}\hline 4 & 2 & 6 \\ \hline 1 & 7 & 8 \\ \hline 4 & 2 & 6 \\ \hline 9 & 11 & 20 \\ \hline \end{array}$$

$T_{v_{1,2}}\mathbf{x_{15}} =$

3	3	6
1	7	8
5	1	6
9	11	20

$, T_{v_{1,2}}\mathbf{x_{16}} =$

1	5	6
2	6	8
6	0	6
9	11	20

$, T_{v_{1,2}}\mathbf{x_{17}} =$

1	5	6
4	4	8
4	2	6
9	11	20

And in the same method we can find 18 elements in $A^{-1}[r^4\boldsymbol{t}]$ of 3×2 −contingency table ($r^4\boldsymbol{t}$-fibers) where, $r^4\boldsymbol{t} = (8\,, 6, 6, 11, 9)'$, and we can find 18 elements in $A^{-1}\left[l_{1,2}\boldsymbol{t}\right]$ of 3×2 −contingency table ($l_{1,2}\boldsymbol{t}$-fibers) where, $l_{1,2}\boldsymbol{t} = (8\,, 6, 6, 11, 9)'$, and we can find 18 elements in $A^{-1}\left[l_{1,3}\boldsymbol{t}\right]$ of 3×2 −contingency table ($l_{1,3}\boldsymbol{t}$-fibers) where, $l_{1,3}\boldsymbol{t} = (6, 8\,, 6, 11, 9)'$, and we can find 18 elements in $A^{-1}\left[l_{2,3}\boldsymbol{t}\right]$ of 3×2 −contingency table ($l_{2,3}\boldsymbol{t}$-fibers) where, $l_{2,3}\boldsymbol{t} = (6, 6, 8\,, 11, 9)'$, and we can find 18 elements in $A^{-1}\left[r^2 v_{1,2}\boldsymbol{t}\right]$ of 3×2 −contingency table ($r^2 v_{1,2}\boldsymbol{t}$-fibers) where, $r^2 v_{1,2}\boldsymbol{t} = (6, 6, 8, 9, 11)'$, and we can find 18 elements in $A^{-1}\left[r^4 v_{1,2}\boldsymbol{t}\right]$ of 3×2 −contingency table ($r^4 v_{1,2}\boldsymbol{t}$-fibers) where, $r^4 v_{1,2}\boldsymbol{t} = (8, 6, 6, 9, 11)'$, and we can find 18 elements in $A^{-1}\left[l_{1,2} v_{1,2}\boldsymbol{t}\right]$ of 3×2 −contingency table ($l_{1,2} v_{1,2}\boldsymbol{t}$-fibers) where, $l_{1,2} v_{1,2}\boldsymbol{t} = (8, 6, 6, 9, 11)'$, and we can find 18 elements in $A^{-1}\left[l_{1,3} v_{1,2}\boldsymbol{t}\right]$ of 3×2 −contingency table ($l_{1,3} v_{1,2}\boldsymbol{t}$-fibers) where, $l_{1,3} v_{1,2}\boldsymbol{t} = (6, 8, 6, 9, 11)'$, and we can find 18 elements in $A^{-1}\left[l_{2,3} v_{1,2}\boldsymbol{t}\right]$ of 3×2 −contingency table ($l_{2,3} v_{1,2}\boldsymbol{t}$-fibers) where, $l_{2,3} v_{1,2}\boldsymbol{t} = (6, 6, 8, 9, 11)'$.

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