# On the stability of trajectories in the task of pursuit

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# On the stability of trajectories in the task of pursuit

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**Abstract.** In two-dimensional and three-dimensional statements, a generalization of the pursuit problem is considered, when the object being pursued moves in a circle (plane case) or in a helical line (three-dimensional case), and various cases of dependence of object velocities in the problem are considered. Conditions are analytically obtained for the stability of the trajectories of motion of objects in the problem. Possible realization of this problem - the pursuit of one drone by another one as flying objects.

## 1. Introduction
Let a simple movement be a movement with a limited speed, and the direction of movement can be changed arbitrarily. Let two material points *A* (catching up) and *B* (running away) move in the same plane with speeds *u* and *v* of constant value. The velocity of point *B* does not change direction, and the velocity vector of point *A* rotates and is always directed to *B*. The task of describing the movement of these points is called the pursuit problem, and the trajectory of the movement of the catcher is called the pursuit curve.

## 2. Description of a problem
The trajectory of the object-chaser in a coordinate system rotating around the origin with angular velocity $\omega = \frac{V_0}{R_0}$ of the target object is the solution of an autonomous system of differential equations:

$$\begin{cases} \dot{x} = \frac{V_0}{R_0} y + V \frac{R_0 - x}{\sqrt{(R_0-x)^2+y^2}} \\ \dot{y} = -\frac{V_0}{R_0} x - V \frac{y}{\sqrt{(R_0-x)^2+y^2}} \end{cases} \tag{1}$$

where $V$ is the velocity of the object-chaser $V = f(x, y) < V_0$ and the target object in this system has coordinates$(R_0; 0)$.

There are two natural problem statements:

The first statement: the velocity of the object-chaser depends on the distance to the origin: $V = f\left(\sqrt{x^2+y^2}\right) = f(r)$

The second statement: the velocity of the object of the pursuer depends on the distance to the object – goal $V = f\left(\sqrt{(R_0-x)^2+y^2}\right)$

For both statements, we have:

The matrix of the linear part of system (1) at a singular point $(x=\frac{R_0}{V_0^2}V^2;\ y=-\frac{R_0}{V_0^2}V\sqrt{V_0^2-V^2})$:

$$\frac{V_0}{R_0}\begin{pmatrix} -\frac{\frac{V}{V_0}((\frac{V}{V_0})^2-\left(1-(\frac{V}{V_0})^2\right)\frac{R_0}{V_0}f'(r))}{\sqrt{1-(\frac{V}{V_0})^2}} & 1+\left(\frac{V}{V_0}\right)^2-(1-\left(\frac{V}{V_0}\right)^2)\frac{R_0}{V_0}f'(r) \\ -1+\left(\frac{V}{V_0}\right)^2+\left(\frac{V}{V_0}\right)^2\frac{R_0}{V_0}f'(r) & -\left(\frac{V}{V_0}\right)^2\sqrt{1-\left(\frac{V}{V_0}\right)^2}(1+\frac{R_0}{V_0}f'(r)) \end{pmatrix}$$

**3. For the first statement in the plane problem**

The eigenvalues of the matrix of the linear part of the system in the equilibrium position:

$$\lambda_{1,2}=\frac{-VV_0\pm\sqrt{(VV_0)^2+4\left(1-\left(\frac{V}{V_0}\right)^2\right)(\frac{R_0}{V_0}f'(r)-1)}}{2R_0\sqrt{V_0^2-V^2}}$$

If $f'(r)-\frac{V_0}{R_0}>0$, then $\lambda_1>0$ and the equilibrium position is unstable. ($V_0^2-V^2>0$)

If a $f'(r)-\frac{V_0}{R_0}<0$, then $\lambda_{1,2}<0$ or $\operatorname{Re}\lambda_{1,2}<0$ and the equilibrium position is stable.

**4. For the second statement in the plane problem**

The matrix of the linear part of system (1) at a singular point:

$$\begin{pmatrix}\dot{x}'_x & \dot{x}'_y \\ \dot{y}'_x & \dot{y}'_y\end{pmatrix}\rightarrow\begin{vmatrix}\dot{x}'_x-\lambda & \dot{x}'_y \\ \dot{y}'_x & \dot{y}'_y-\lambda\end{vmatrix}$$

The trajectory will be stable:

- $\frac{-\dot{x}'_x-\dot{y}'_y}{2}>0$
- $\dot{x}'_x\cdot\dot{y}'_y-\dot{y}'_x\cdot\dot{x}'_y>0$
- $\frac{-\dot{x}'_x-\dot{y}'_y}{2}<0$

The trajectory will be unstable:

- $\begin{cases}\frac{-\dot{x}'_x-\dot{y}'_y}{2}>0\\ \dot{x}'_x\cdot\dot{y}'_y-\dot{y}'_x\cdot\dot{x}'_y<0\end{cases}$
- $\begin{cases}\frac{-\dot{x}'_x-\dot{y}'_y}{2}=0\\ \dot{x}'_x\cdot\dot{y}'_y-\dot{y}'_x\cdot\dot{x}'_y<0\end{cases}$

It is not clear what will happen with sustainability:

- $$\left[\begin{array}{l} \frac{-\dot{x}'_x - \dot{y}'_y}{2} > 0 \\ \dot{x}'_x \cdot \dot{y}'_y - \dot{y}'_x \cdot \dot{x}'_y = 0 \end{array}\right.$$

Due to the difficulties of expression of stability conditions in an explicit form through function of dependence of speed on distance to the target it is necessary to provide explicit examples showing in what cases the trajectory of the movement will be stable and in what will be not (from what parameters and as far as it depends) [1-3].

$$x'(t) = \frac{(-x + \cos(t))\left(\frac{1}{2} + \sqrt{(-x + \cos(t))^2 + (-y + \sin(t))^2}\right)}{\sqrt{(-x + \cos(t))^2 + (-y + \sin(t))^2}}$$

$$y'(t) = \frac{(-y + \sin(\mathrm{t}))\left(\frac{1}{2} + \sqrt{(-x + \cos(t))^2 + (-y + \sin(t))^2}\right)}{\sqrt{(-x + \cos(t))^2 + (-y + \sin(t))^2}}$$

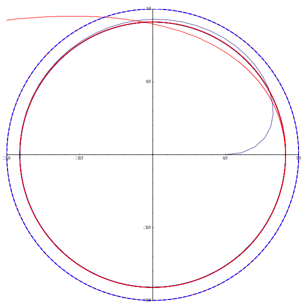

**Figure 1.** Stable trajectory.

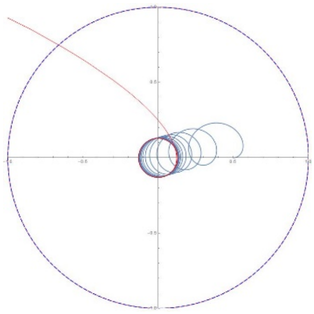

**Figure 2.** Unstable trajectory.

The equation to find the speed limit:

$$\mathrm{NSolve}[x == \mathrm{VMin} + \sqrt{(1 - x^2)\text{^}2 + (1 - x^2)x^2}, x]$$

Solution: $(\mathrm{VMin} + \sqrt{2 - \mathrm{VMin}^2})/2$

With VMin = 0.5 we get speed limit equal to 0.9114378277661477

**5. Study of the trajectory of the pursuer when moving along a helical path in 3D-space**

Statement. If the trajectory of the target does not contain straight segments, then the distance between the points *P(t)* of different paths of the pursuer decreases with increasing time *t*.

The target moves along a helix.

In figure 3 trajectories of the target and pursuer depending on time are shown. From the visual representation, it can be concluded that pursuers coordinates *x (t), y (t)*, z (t) are the same as target ones (the distance between them decreases) with increasing *t.*

The target moves along a closed line with a periodic ascent and descent.

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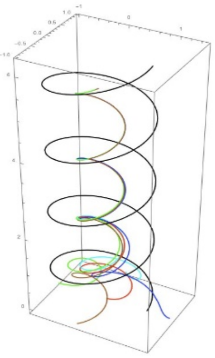

**Figure 3.** Movement of the target (black trajectory) and the pursuer.

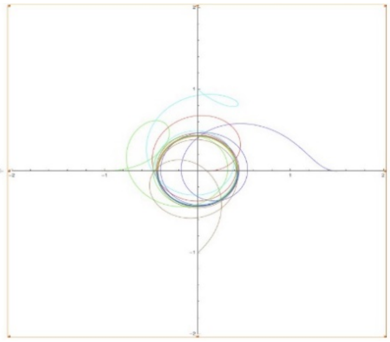

**Figure 4.** Analog to figure 3 top view.

Figures 5 and 6 show trajectories of the target and the pursuer depending on the time. From the visual representation we conclude that the coordinates of the pursuers x *(*t)*, y (*t)*, z (*t) visually coincide with increasing *t*.

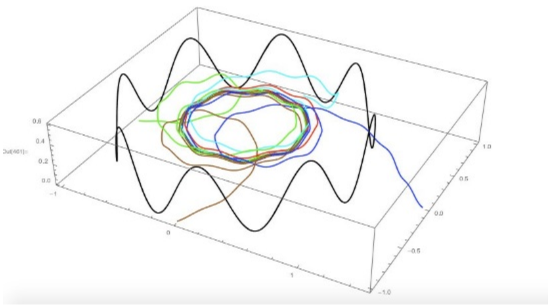

**Figure 5.** Spatial picture of the goal and the pursuer.

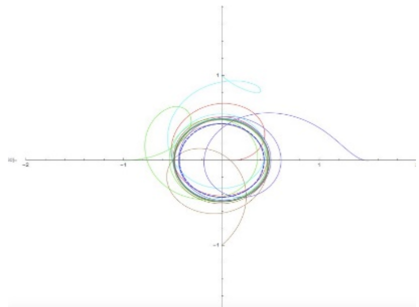

**Figure 6.** Analogue figure 5 top view.

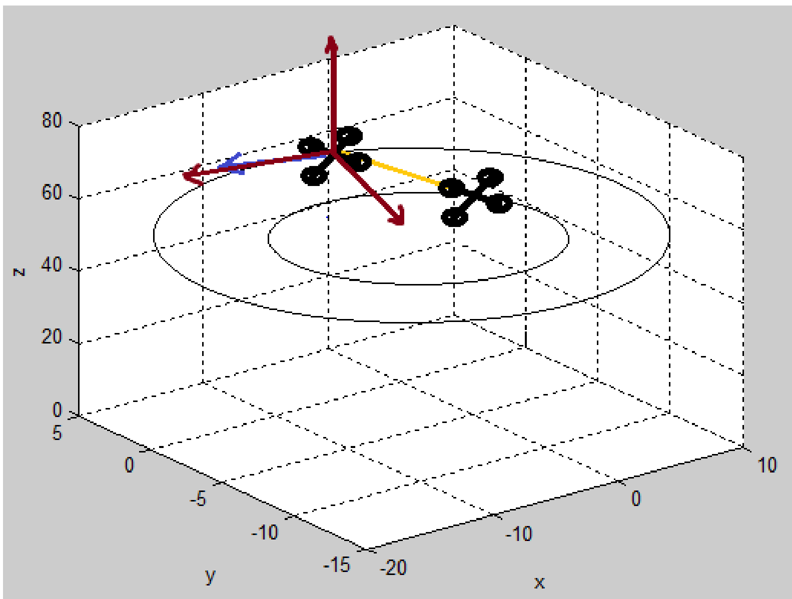


**Figure 7.** The scheme of the pursuit problem for two drones in space.

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## 6. Conclusion

Under certain conditions imposed both on the trajectory of the pursuer object and the target object, we can obtain both a stable trajectory and an unstable one. Using Wolfram Mathematica Package shows that the behavior of numerical solutions of systems of differential equations of motion corresponds to the following property of solutions given earlier: the trajectories of the pursuer with different initial data at $t = 0$ approach each other as $t$ increases. Moreover, from figures 3-6, it can be concluded that the distance between points of different trajectories taken at the same $t$ tends to zero when $t \to \infty$. From the behavior of the trajectories, we can conclude that when the target moves along a helix, the trajectory of the pursuer also approaches the helical line with a reduced radius; when the target moves periodically, the trajectory of the pursuer also approaches the periodic one with reduced amplitude of oscillations.

## References

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