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Singular solutions for the rigid plastic double slip and rotation model under plane strain

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Abstract. In the mechanics of granular and other materials the system of equations comprising the rigid plastic double slip and rotation model together with the stress equilibrium equations under plane strain conditions forms a hyperbolic system. Boundary value problems for this system of equations can involve a frictional interface. An envelope of characteristics may coincide with this interface. In this case, the solution is singular. In particular, some components of the strain rate tensor approach infinity in the vicinity of the frictional interface. Such behavior of solutions is in qualitative agreement with experimental data that show that a narrow layer of localized plastic deformation is often generated near frictional interfaces. The present paper deals with asymptotic analysis of the aforementioned system of equations in the vicinity of an envelope of characteristics. It is shown that the shear strain rate and the spin component in a local coordinate system connected to the envelope follow an inverse square root rule in its vicinity.

1. Introduction
Singular solutions have been studied for several rigid plastic models. In particular, the rigid perfectly plastic model has been considered [1]. In this work, three dimensional flow of material obeying an arbitrary smooth yield criterion and its associated flow rule has been analysed. It has been shown that the singular solutions appear near surfaces on which the shear stress is equal to the shear yield stress and that the quadratic invariant of the strain rate tensor follows an inverse square root rule near the surface. This result has been extended to a class of viscoplastic models in [2] for planar flow and in [3] for axisymmetric flow. The constitutive equations of this class of models include a saturation stress. In particular, the yield stress in tension approaches the saturation stress as the quadratic invariant of the strain rate tensor approaches infinity. The qualitative asymptotic behaviour of solutions is controlled by the exact asymptotic behaviour of the dependence of the yield stress on the quadratic invariant of the strain rate tensor at infinity. For a certain class of functions the quadratic invariant of the strain rate tensor follows an inverse square root rule near the surface where the shear stress is equal to the shear saturation stress. The model of anisotropic plasticity proposed in [4] has been investigated in [5] under plane strain conditions. The corresponding system of equations is hyperbolic. The singularity in solutions develops near envelopes of characteristics. The exact asymptotic singular behaviour of solutions depends on the shape of the yield surface near the point corresponding to the state of stress at the envelope. For a certain class of functions the quadratic invariant of the strain rate tensor follows an

inverse square root rule near the envelope. The present paper deals with plane strain deformation of materials that obeys the rigid plastic double slip and rotation model proposed in [6]. The corresponding system of equations is hyperbolic. It is shown that the shear strain rate and the spin component in a local coordinate system connected to an envelope of characteristics follow an inverse square root rule in its vicinity.

   The technological value of the present research is that a narrow layer of localized plastic deformation is generated near frictional interfaces in machining and deformation processes ([7]-[16] among many others). These layers affect the performance of machine parts [17]. The main theoretical result reported in [1]-[3], [5] as well as in the present paper for the rigid plastic double slip and rotation model demonstrates that the models considered in these papers are capable of predicting the generation of the layer of localized plastic deformation near frictional interfaces.

2. The rigid plastic double slip and rotation model under plain strain

The double slip and rotation model has been proposed in [6]. Under plane strain conditions of incompressible material, the rigid plastic constitutive equations of the model are the Mohr - Coulomb yield criterion (other yield criteria may be adopted as well) and the flow rule. In an arbitrary Cartesian coordinate system (x, y) these equations can be written as

$$\left(\sigma_{xx} + \sigma_{yy}\right)\sin\phi + \sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\sigma_{xy}^2} = 2k\cos\phi \tag{1}$$

and

$$\varepsilon_{xx} + \varepsilon_{yy} = 0, \quad \sin 2\psi\left(\varepsilon_{xx} - \varepsilon_{yy}\right) - 2\cos 2\psi\varepsilon_{xy} - 2\sin\phi\left(\omega_{xy} + \Omega\right) = 0. \tag{2}$$

Here σ_{xx}, σ_{yy} and σ_{xy} are the components of the stress tensor referred to the (x, y) coordinate system, ε_{xx}, ε_{yy} and ε_{xy} are the components of the strain rate tensor referred to the (x, y) coordinate system, ω_{xy} is the only non-zero spin (vorticity) component referred to the (x, y) coordinate system, ψ is the angle between the $x-$ direction and the greatest principal stress σ_1 measured from the $x-$ direction anti-clockwise, Ω is the intrinsic spin due to grain rotation, k is the cohesion, and ϕ is the angle of internal friction. The quantity Ω is an unknown variable which is governed by the equation of rotational motion. It is often reasonable to suppose that in the interior of the yielding region the grain rotation is either zero or averages to zero [18]. In the present paper, it is assumed that Ω is constant.

3. Characteristics, characteristic relations and envelopes

The system of equations comprising (1), (2) and the equations of equilibrium is hyperbolic. Let u_x and u_y be the velocity components referred to the Cartesian coordinate system. Then, equation (2) can be rewritten as

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0,$$

$$\sin 2\psi\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) - \left(\cos 2\psi + \sin\phi\right)\frac{\partial u_x}{\partial y} - \left(\cos 2\psi - \sin\phi\right)\frac{\partial u_y}{\partial x} - 2\Omega\sin\phi = 0. \tag{3}$$

The components of the stress tensor can be represented as [19]

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{yy} = -p - q\cos 2\psi, \quad \sigma_{xy} = q\sin 2\psi \tag{4}$$

where p and q are the stress invariant defined as

$$p = -\frac{\left(\sigma_{xx} + \sigma_{yy}\right)}{2}, \quad q = \sqrt{\frac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2}. \tag{5}$$

Then, the yield criterion (1) becomes

$$-p\sin\phi + q = k\cos\phi. \tag{6}$$

The equilibrium equations are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0. \tag{7}$$

Substituting (4) into (7) yields

$$-\frac{\partial p}{\partial x} + \cos 2\psi \frac{\partial q}{\partial x} - 2q\sin 2\psi \frac{\partial \psi}{\partial x} + \sin 2\psi \frac{\partial q}{\partial y} + 2q\cos 2\psi \frac{\partial \psi}{\partial y} = 0,$$

$$\sin 2\psi \frac{\partial q}{\partial x} + 2q\cos 2\psi \frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial y} - \cos 2\psi \frac{\partial q}{\partial y} + 2q\sin 2\psi \frac{\partial \psi}{\partial y} = 0. \tag{8}$$

In these equations, p can be eliminated by means on (6). The resulting system contains two unknowns, q and ψ. This system is hyperbolic and the angle between the direction of the greatest principal stress σ_1 and the characteristic directions is $\pm\left(\pi/4 + \phi/2\right)$ [19]. The Cartesian $\left(x, y\right)$ and characteristic $\left(\xi, \eta\right)$ coordinate systems are illustrated in Fig. 1. It is always possible to rotate the Cartesian coordinate system such that its $x-$axis is tangent to an $\xi-$ characteristic curve at a given point. Then, $\psi = \pi/4 + \phi/2$ at this point and equation (3) becomes

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad \cos\phi \frac{\partial u_x}{\partial x} + \sin\phi \frac{\partial u_y}{\partial x} - \Omega\sin\phi = 0. \tag{9}$$
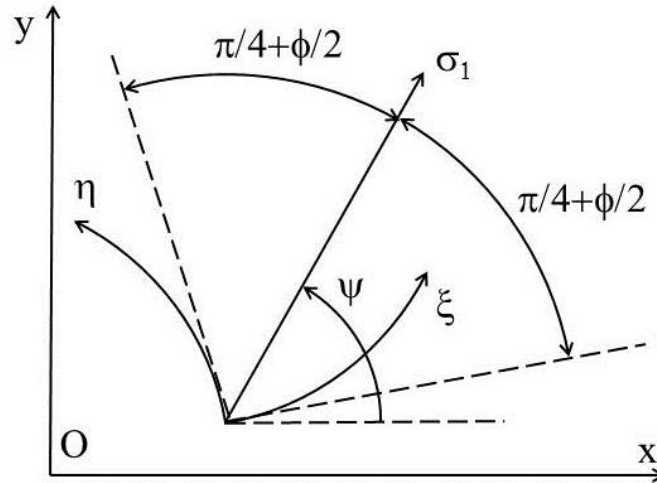

Figure 1. Cartesian and characteristic coordinates.

In the second equation of this system the derivative $\partial u_y / \partial y$ has been eliminated by means of the first equation. The second equation in (9) is one of the characteristic relations. This equation is valid if $\left|\partial u_x / \partial y\right| < \infty$. In order to study the behaviour of solutions in the vicinity of an envelope of $\xi-$ characteristic curves, it is necessary to assume that

$$\left| \frac{\partial u_x}{\partial y} \right| \to \infty \tag{10}$$

as $y \to 0$. In this case the second equation in (9) is not valid because

$$\left(\cos 2\psi + \sin \phi \right) \frac{\partial u_x}{\partial y} \neq 0 \tag{11}$$

at $y = 0$. Eliminating the derivative $\partial u_y / \partial y$ in the second equation in (3) by means of the first equation yields

$$2 \sin 2\psi \frac{\partial u_x}{\partial x} - \left(\cos 2\psi + \sin \phi \right) \frac{\partial u_x}{\partial y} - \left(\cos 2\psi - \sin \phi \right) \frac{\partial u_y}{\partial x} - 2\Omega \sin \phi = 0. \tag{12}$$

In what follows, it is assumed that the in-surface derivatives are bounded. Then, it follows from (12) that

$$\left| \left(\cos 2\psi + \sin \phi \right) \frac{\partial u_x}{\partial y} \right| < \infty \tag{13}$$

at $y = 0$.

4. Asymptotic analysis

Consider the solution behaviour of equations (3) and (8) in the vicinity of the maximum friction surface assuming that equation (11) is satisfied. It is convenient to adopt the Cartesian coordinate system whose x- axis is tangent to the maximum friction surface at a given point (i.e. $\psi = \pi/4 + \phi/2$ at this point) and whose origin is situated at this point (i.e. $y = 0$ at this point). In what follows, it is assumed that the solution is represented by power series of y near $y = 0$. Then, the velocity component u_x is represented as

$$u_x = u_0 + u_1 y^\alpha + u_2 y + o(y) \tag{14}$$

as $y \to 0$. Here u_0 and u_1 are independent of y. Since the velocity component u_x must be bounded, it is obvious from (12) that $\alpha > 0$. On the other hand, equation (11) demands $\alpha < 1$. Therefore,

$$0 < \alpha < 1. \tag{15}$$

The angle ψ is represented as

$$\psi = \frac{\pi}{4} + \frac{\phi}{2} + \psi_0 y^\beta + \psi_1 y + o(y) \tag{16}$$

as $y \to 0$. Here ψ_0 and ψ_1 are independent of y and $\beta > 0$. Substituting (14) and (16) into (11) and taking into account (13) gives $\alpha + \beta - 1 = 0$ or

$$\beta = 1 - \alpha. \tag{17}$$

It is seen from (15) and (17) that $0 < \beta < 1$. Then, it follows from (16) and (17) that

$$\cos 2\psi = -\sin\phi - 2\cos\phi\left(\psi_0 y^{1-\alpha} + \psi_1 y\right) + o(y),$$
$$\sin 2\psi = \cos\phi - 2\sin\phi\left(\psi_0 y^{1-\alpha} + \psi_1 y\right) + o(y),$$
$$\cos 2\psi + \sin\phi = -2\cos\phi\left(\psi_0 y^{1-\alpha} + \psi_1 y\right) + o(y),$$
$$\cos 2\psi - \sin\phi = -2\sin\phi - 2\cos\phi\left(\psi_0 y^{1-\alpha} + \psi_1 y\right) + o(y) \tag{18}$$

as $y \to 0$. Using (14) the first equation in (3) can be represented as

$$\frac{\partial u_y}{\partial y} = -\frac{du_0}{dx} + \frac{du_1}{dx} y^\alpha + \frac{du_1}{dx} y + o(y) \tag{19}$$

as $y \to 0$. Integrating this equation with respect to y leads to

$$u_y = -\frac{du_0}{dx} y + o(y) \tag{20}$$

as $y \to 0$. It has been taken into account here that it is always possible to assume with no loss of generality that the rigid tool is motionless and, therefore, $u_y = 0$ at $y = 0$. Using (14), (18) and (20) the terms involved in (12) are represented as

$$2\sin 2\psi \frac{\partial u_x}{\partial x} = 2\cos\phi \frac{du_0}{dx} + 2\cos\phi \frac{du_1}{dx} y^\alpha - 4\psi_0 \sin\phi \frac{du_0}{dx} y^{1-\alpha} + O(y),$$
$$\left(\cos 2\psi + \sin\phi\right)\frac{\partial u_x}{\partial y} = -2\psi_0 \alpha u_1 \cos\phi - 2\psi_0 u_2 \cos\phi y^{1-\alpha} - 2\psi_1 \alpha u_1 \cos\phi y^\alpha + O(y), \tag{21}$$
$$\left(\cos 2\psi - \sin\phi\right)\frac{\partial u_y}{\partial x} = O(y)$$

as $y \to 0$. Substituting (21) into (12) gives

$$2\left(\cos\phi \frac{du_0}{dx} + \psi_0 \alpha u_1 \cos\phi - \Omega\sin\phi\right) + 2\cos\phi\left(\frac{du_1}{dx} + \psi_1 \alpha u_1\right) y^\alpha +$$
$$2\psi_0\left(u_2 \cos\phi - 2\sin\phi\right) y^{1-\alpha} + O(y) \tag{22}$$

as $y \to 0$. It is seen from this equation that $1 - \alpha = \alpha$ or

$$\alpha = \frac{1}{2}. \tag{23}$$

It remains to demonstrate that this result does not contradict the stress equations. Eliminating p in (8) by means of (6) yields

$$\left(\cos 2\psi - \frac{1}{\sin\phi}\right)\frac{\partial q}{\partial x} - 2q\sin 2\psi \frac{\partial \psi}{\partial x} + \sin 2\psi \frac{\partial q}{\partial y} + 2q\cos 2\psi \frac{\partial \psi}{\partial y} = 0,$$
$$\sin 2\psi \frac{\partial q}{\partial x} + 2q\cos 2\psi \frac{\partial \psi}{\partial x} - \left(\frac{1}{\sin\phi} + \cos 2\psi\right)\frac{\partial q}{\partial y} + 2q\sin 2\psi \frac{\partial \psi}{\partial y} = 0. \tag{24}$$

Using (17) and (23) it is possible to rewrite equations (16) and (18) in the form

$$\psi = \frac{\pi}{4} + \frac{\phi}{2} + \psi_0\sqrt{y} + \psi_1 y + o(y),$$
$$\cos 2\psi = -\sin\phi - 2\cos\phi\left(\psi_0\sqrt{y} + \psi_1 y\right) + o(y), \tag{25}$$
$$\sin 2\psi = \cos\phi - 2\sin\phi\left(\psi_0\sqrt{y} + \psi_1 y\right) + o(y)$$

as $y \to 0$. Then,

$$\cos 2\psi - \frac{1}{\sin\phi} = -\frac{1+\sin^2\phi}{\sin\phi} - 2\cos\phi\left(\psi_0\sqrt{y} + \psi_1 y\right) + o(y),$$
$$\cos 2\psi + \frac{1}{\sin\phi} = \frac{\cos^2\phi}{\sin\phi} - 2\cos\phi\left(\psi_0\sqrt{y} + \psi_1 y\right) + o(y) \tag{26}$$

as $y \to 0$. It is seen from (25) that

$$q\cos 2\psi \frac{\partial\psi}{\partial y} = O\left(\frac{1}{\sqrt{y}}\right), \quad q\sin 2\psi \frac{\partial\psi}{\partial y} = O\left(\frac{1}{\sqrt{y}}\right) \tag{27}$$

as $y \to 0$. Therefore, it follows from (24), (25) and (26) that $\partial q/\partial y = O\left(1/\sqrt{y}\right)$ as $y \to 0$. Then, the function q can be represented as

$$q = q_0 + q_1\sqrt{y} + q_2 y + o(y) \tag{28}$$

as $y \to 0$. Here q_0, q_1 and q_2 are independent of y. It follows from (25), (26) and (28) that

$$\left(\cos 2\psi - \frac{1}{\sin\phi}\right)\frac{\partial q}{\partial x} = -\frac{\left(1+\sin^2\phi\right)}{\sin\phi}\frac{dq_0}{dx} - \left[\frac{\left(1+\sin^2\phi\right)}{\sin\phi}\frac{dq_1}{dx} + 2\psi_0\cos\phi\frac{dq_0}{dx}\right]\sqrt{y} + O(y),$$

$$-2q\sin 2\psi \frac{\partial\psi}{\partial x} = q\frac{\partial\cos 2\psi}{\partial x} = -2q_0\cos\phi\frac{d\psi_0}{dx}\sqrt{y} + O(y),$$

$$\sin 2\psi \frac{\partial q}{\partial y} = \frac{q_1\cos\phi}{2\sqrt{y}} + \left(q_2\cos\phi - q_1\psi_0\sin\phi\right) + O\left(\sqrt{y}\right),$$

$$2q\cos 2\psi \frac{\partial\psi}{\partial y} = q\frac{\partial\sin 2\psi}{\partial y} = -\frac{q_0\psi_0\sin\phi}{\sqrt{y}} + \left(2\psi_1 q_0 + \psi_0 q_1\right)\sin\phi + O\left(\sqrt{y}\right),$$

$$\sin 2\psi \frac{\partial q}{\partial x} = \cos\phi\frac{dq_0}{dx} + \left(\cos\phi\frac{dq_1}{dx} - 2\psi_0\sin\phi\frac{dq_0}{dx}\right)\sqrt{y} + O(y),$$

$$2q\cos 2\psi \frac{\partial\psi}{\partial x} = q\frac{\partial\sin 2\psi}{\partial x} = -2q_0\sin\phi\frac{d\psi_0}{dx}\sqrt{y} + O(y),$$

$$-\left(\frac{1}{\sin\phi} + \cos 2\psi\right)\frac{\partial q}{\partial y} = -\frac{q_1\cos^2\phi}{2\sin\phi\sqrt{y}} + \left(q_1\psi_0\cos\phi - \frac{q_2\cos^2\phi}{\sin\phi}\right) + O\left(\sqrt{y}\right),$$

$$2q\sin 2\psi \frac{\partial\psi}{\partial y} = -q\frac{\partial\cos 2\psi}{\partial y} = \frac{q_0\psi_0\cos\phi}{\sqrt{y}} + \left(2q_0\psi_1 + q_1\psi_0\right)\cos\phi + O\left(\sqrt{y}\right) \tag{29}$$

as $y \to 0$. It is seen from (24) and (28) that each of the equations in (24) reduces to the equation of the form $A + B\sqrt{y} + o\left(\sqrt{y}\right)$ as $y \to 0$ where A and B are independent of y if

$$q_1\cot\phi - 2q_0\psi_0 = 0. \tag{30}$$

In this case, equation (24) is compatible with (23) and the asymptotic representation of the velocity component u_x in the vicinity of the envelope (or the maximum friction surface) follows from (14) and (23) in the form

$$u_x = u_0 + u_1\sqrt{y} + o\left(\sqrt{y}\right) \tag{31}$$

as $y \to 0$. It is seen from this equation that $\partial u_x/\partial y = O\left(1/\sqrt{y}\right)$ as $y \to 0$ and, therefore, both $\varepsilon_{xy} = O\left(1/\sqrt{y}\right)$ and $\omega_{xy} = O\left(1/\sqrt{y}\right)$ as $y \to 0$.

5. Conclusions

It has been shown that plane strain rigid plastic solutions for the double slip and rotation model for incompressible material are singular (in the sense that some space derivatives of velocities and stresses approach infinity) in the vicinity of maximum friction surfaces. The maximum friction surface is defined by the condition that an envelope of characteristics coincides with the surface. The exact asymptotic expansion of velocities, strain rates and stresses in the vicinity of singular surfaces has been found. It has been hypothesized that this singular behaviour of the velocity field can be adopted for describing the evolution of material properties in a narrow layer near frictional interfaces cause by highly localized plastic deformation.

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