Modelling FGM materials. An accurate function approximation algorithms

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Abstract. The study is focused on development of an accurate and cost effective function approximation techniques for modelling functionally graded materials. Different grading functions (exponential, power law) are expanded into Haar wavelet series based on higher order Haar wavelet approach. The proposed techniques can be utilized also for modelling load cases, complex boundary conditions, grading functions etc.

1. Introduction

Recently, the Haar wavelet method (HWM) and higher order Haar wavelet method (HOHWM) have been utilized with success for solving wide class of engineering problems [1-7]. The differential governing equations of the structures, solved using HWM or HOHWM, involve often set of functions describing loading, boundary conditions, varying stiffness, geometry, density and material properties. In order to provide comprehensive approach, it is reasonable to treat these function in terms of Haar wavelets i.e. to expand these functions into Haar wavelet series [8-10]. Most commonly the functions accompanying the governing equations are expanded into Haar wavelets. However, here is reason to be careful, because it is shown in [11] that in the case where function is expanded into Haar wavelets the order of convergence with respect to mesh/resolution is equal to one. Since the order of convergence of the HWM is equal to two [12] and HOHWM higher, such an approach may lead to loss of accuracy of the solution, especially in the case of use of HOHWM. Thus, more accurate Haar wavelet based approach is needed to overcome possible loss of accuracy of the solution.
Current study is focused on approximation of grading functions of the functionally graded (FG) materials using higher order Haar wavelet method based function approximation. The HOHWM was introduced by authors for solving differential equations as improvement of the HWM [13]. Based on HOHWM, the higher order approach for functions approximation was introduced in [14]. Herein the latter higher order function approximation technique is extended for modelling grading functions of the FG materials. As result the generalized algorithm can be used for solving governing equations covering different grading functions (exponential, power law, four parameter functions, etc.). In the current study the second order derivatives of the grading functions are expanded into Haar wavelets providing fourth order convergence with respect to mesh.

2. Haar wavelets

The Haar wavelets are introduced as [1]

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\xi_1(i), \xi_2(i)) \\ -1 & \text{for} \quad x \in [\xi_2(i), \xi_3(i)), \\ 0 & \text{elsewere} \end{cases} \tag{1}$$

where $i = m + k + 1$, $m = 2^j$ is a maximum number of square waves deployed in interval $[A, B]$ and the parameter k indicates the location of the particular square wave,

$$\xi_1(i) = A + 2k\mu\Delta x, \quad \xi_2(i) = A + (2k+1)\mu\Delta x, \quad \xi_3(i) = A + 2(k+1)\mu\Delta x,$$
$$\mu = M/m, \ \Delta x = (B-A)/(2M), M = 2^J. \tag{2}$$

The parameters j and J stand for the resolution and maximum resolution, respectively. Any square integrable and finite function in the interval $[A, B]$ can expanded into Haar wavelets.

3. Approximation of grading functions of the FG materials

Based on idea of higher order Haar wavelet method introduced by authors in [13] an accurate function approximation is derived for four simple and widely used grading functions described in the following subsections.

3.1. Exponential grading function

Due to its simplicity, the exponential grading function is widely used. In the case of axially graded materials the exponential functions are utilized commonly for describing the elasticity modulus $E(x)$ and the density $\rho(x)$ as

$$E(x) = E(0) * e^{2\beta x/L}, \ \rho(x) = \rho(0) * e^{2\beta x/L}. \tag{3}$$

In (3) L is a length of the beam and β is a grading parameter, $E(0)$ and $\rho(0)$ stand for the reference values of the elasticity modulus and density at $x = 0$. However, in order to provide unique approach for handing different grading functions, herein the volume fraction of the first constituent V_1 is expanded into Haar wavelet. The modulus of elasticity and the density can be expressed in terms of volume fraction V_1 as

$$E = (E_1 - E_2)V_1 + E_2, \quad \rho = (\rho_1 - \rho_2)V_1 + \rho_2, \tag{4}$$

where the indexes 1 and 2 refers to the constituents (materials) 1 and 2, respectively. In the case of exponential grading function (3), the volume fractions of the constituents (materials) V_1 and V_2 can be derived as

$$V_1 = \frac{e^{2\beta} - e^{2\beta x/L}}{e^{2\beta} - 1}, \ V_2 = \frac{e^{2\beta x/L} - 1}{e^{2\beta} - 1}. \tag{5}$$

The higher order Haar wavelet expansion can be introduced as

$$f^{(n)}(x) = \sum_{i=1}^{2M} b_i h_i(x), \quad n = 0,1,2,..., \tag{6}$$

where b_i and $h_i(x)$ stand for the wavelet coefficients and Haar functions, respectively. In the case of $n = 0,1,2\ldots$ the functions itself, its first derivative, second derivative, etc. is expanded into series of the Haar functions. In the following the value $n = 2$ is utilized in order to provide fourth order convergence

$$\frac{e^{2\beta} - e^{2\beta x/L}}{e^{2\beta} - 1} = \sum_{i=1}^{2M} b_i p_{2,i} + d_1 x + d_2. \tag{7}$$

In (7) $p_{2,i}$ stand for second order integrals of the Haar functions (1). Two integration constants d_1 and d_2 can be determined by satisfying equation (5) in boundary points $x = 0$ and $x = L$, respectively. The Haar wavelet expansion coefficients b_i can be determined by satisfying the equation (7) in grid points. In the case of uniform mesh used the grid points x_l are given as

$$x_l = \frac{2l-1}{4M}, \quad l = 1, \dots, 2M. \tag{8}$$

The volume fraction $V_1(x)$ be calculated for each coordinate x value by substituting the coefficients b_i, integration constants d_1 and d_2 in (7). The volume fraction $V_2(x)$ can be evaluated as $1 - V_1(x)$. The detailed expressions of the coefficients b_i, integration constants d_1 and d_2 are omitted for conciseness sake.

3.2. Power law grading function

The power law relation for describing FG materials is given as [15]

$$E = (E_1 - E_2)\left(1 - \frac{x}{L}\right)^k + E_2\,, \quad \rho = (\rho_1 - \rho_2)(1 - \frac{x}{L})^k + \rho_2. \tag{9}$$

In (9) k is a grading parameter. Despite to presence of one parameter in both, exponential and power law grading function, the power law function is more flexible/general. In exponential grading function (3) the parameter β value is determined by mechanical characteristic value at the point $x = L$. However, in the case of power law grading function (9) the values of the mechanical characteristics in boundary points are provided and the parameter k remains as design parameter. Varying the value of the parameter k allow to obtain different distributions of the mechanical characteristics. The higher order Haar wavelet expansion can be employed for the volume fraction of the constituent (material) 1 as

$$\left(1 - \frac{x}{L}\right)^k = \sum_{i=1}^{2M} b_i p_{2,i} + d_1 x + d_2. \tag{10}$$

Similarly, to above, the second order derivative of the volume fraction function is expanded into Haar wavelet and grading function is expressed in terms of second integrals of Haar functions. The elasticity modulus and density can be evaluated by substituting (10) in (9).

3.3. Four parameter power law grading function

In order to provide higher flexibility for describing the distribution of the FGM, the following four parameter model is considered for approximation of the volume fraction V_1 [15]

$$V_1 = C * \left[1 - \frac{x}{L} + \alpha\left(\frac{x}{L}\right)^\beta\right]^\gamma, \tag{11}$$

where parameters C, α , β and γ control the volume fraction V_1variation through the length of the axially graded structure. The higher order Haar wavelet expansion is given as

$$C * \left[1 - \frac{x}{L} + \alpha\left(\frac{x}{L}\right)^\beta\right]^\gamma = \sum_{i=1}^{2M} b_i p_{2,i} + d_1 x + d_2. \tag{12}$$

Note, that four parameter grading function can be utilized for design optimization of FGM structures.

3.4. Four parameter trigonometric grading function

Four parameter trigonometric grading function can be considered as an alternative to four parameter power law grading function. The corresponding volume fraction V_1 can be expressed as [15]

$$V_1 = C * \left[\frac{1}{2} - \frac{\alpha}{2} sin\left(\frac{n\pi x}{L} + \phi\right)\right]^{\gamma}. \quad (13)$$

Expanding the second order derivative of the volume fraction (13) into Haar wavelet series and integrating twice one obtains approximation of the function V_1 as

$$C * \left[\frac{1}{2} - \frac{\alpha}{2} sin\left(\frac{n\pi x}{L} + \phi\right)\right]^{\gamma} = \sum_{i=1}^{2M} b_i p_{2,i} + d_1 x + d_2. \quad (14)$$

The approximations used for above four grading functions are second order polynomials including global and local terms.

4. Numerical results

The values of the grading functions, its absolute errors and convergence rates corresponding to the exponential, power law and four parameter grading functions are given in tables 1 and 2, respectively. In Table 1 the approximation results are given for exponential and power law grading functions.

Table 1. Approximation for exponential and power law grading functions ($\beta = -0.549306, k = 1.5$).

	$V_1(x) = \frac{e^{2\beta} - e^{2\beta x/L}}{e^{2\beta} - 1}$			$V_1(x) = \left(1 - \frac{x}{L}\right)^k$		
2M	Function value at point x=L/2	Converg. rate	Absolute error	Function value at point x=L/2	Absolute error	Converg. rate
4	0.365989778		3.57E-05	0.353652500		9.91E-05
8	0.366023040	3.8946	2.40E-06	0.353549500	4.6608	3.92E-06
16	0.366025287	3.9954	1.50E-07	0.353553000	3.3869	3.74E-07
32	0.366025428	3.9996	9.40E-09	0.353553400	3.9854	2.36E-08
64	0.366025437	3.9999	5.87E-10	0.353553400	3.9969	1.48E-09
128	0.366025437	3.9994	3.67E-11	0.353553400	3.9990	9.26E-11

Table 2. Approximation for four parameter grading functions ($C = 1, \alpha = 1, \beta = \theta = 2, \phi = \frac{\pi}{2}, \gamma = 1, \eta = 1.2$).

	$V_1(x) = C * \left[1 - \frac{x}{L} + \alpha\left(\frac{x}{L}\right)^{\beta}\right]^{\theta}$			$V_1(x) = C * \left[\frac{1}{2} - \frac{\alpha}{2} sin\left(\frac{n\pi x}{L} + \phi\right)\right]^{\gamma}$		
2M	Function value at point x=L/2	Converg. rate	Absolute error	Function value at point x=L/2	Absolute error	Converg. rate
4	0.561810700		6.89E-04	0.653466700		1.04E-03
8	0.562454300	3.9150	4.57E-05	0.654447800	4.1021	6.07E-05
16	0.562497100	3.9975	2.86E-06	0.654504800	4.0206	3.74E-06
32	0.562499800	4.0000	1.79E-07	0.654508300	4.0050	2.33E-07
64	0.562500000	4.0000	1.12E-08	0.654508500	4.0013	1.45E-08
128	0.562500000	3.9999	6.99E-10	0.654508500	4.0003	9.08E-10

It can be observed from Table 1 that the order of convergence tends to four as expected and the absolute error reduces up to 10^{-11}. The results in Tables 2 are similar, high accuracy and fourth order rate of convergence is achieved. Thus, the obtained results can be used for HWM and also HOHWM (s=1) where convergence is not higher than four (convergence rate of the HOHWM is equal to 2+2s, where s is method parameter).
In future study the functions approximations developed, are planned to apply for structural analysis and design optimization of engineering structures [16-21] and production processes [22-25]. An another challenging research area is fractional calculus [26]. From one side, fractional calculus allows to describe a number of real world problems more naturally/objectively. For example, modelling the behavior of the viscoelastic material. From other side, the mainstream numerical methods cannot by applied directly for fractional calculus, but need adaption, refinement. The proposed function approximation technique can be applied for expansion of the fractional derivatives included in differential equations of integer order derivatives used for approximation of fractional derivatives. Solution of fractional differential and integro-differential equations is foreseen.

5. Conclusions

An accurate function approximation technique is proposed for modelling grading functions of the FG materials, based on idea of higher order Haar wavelet method. According to proposed approach the second order derivative of the grading function is expanded into Haar wavelet series. The two complementary integration constants are determined by using function values on boundary. As result fourth order convergence with respect to mesh was achieved. The obtained results are accurate enough utilizing with HWM and HOHWM (s=1). The proposed approach covers obviously further increase of accuracy by increasing the order of derivative expended into Haar wavelet series in formula (6).

6. References

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