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Tempered Stable Ornstein–Uhlenbeck Model for River Discharge Time Series with Its Application to Dissolved Silicon Load Analysis

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Abstract. We identify stochastic process models describing the time series of inflow and outflow discharges of Obara Dam in Hii River, Japan. These models are based on tempered stable Ornstein–Uhlenbeck (TSOU) processes that have not been utilized in hydrological analysis but can capture both large and small fluctuation of the time series data. In addition, the models can be exactly simulated in a statistical sense by utilizing a recent tailored discretization algorithm, serving as efficient stochastic tools. We show that the identified models accurately reproduce statistical moments of the time series data and probability density functions. Based on the mathematical framework of backward stochastic differential equation (BSDE), the identified model is applied to a unique dynamic stochastic analysis of dissolved silicon (DSi) load flowing into the reservoir associated with Obara Dam. We thus contribute to the first application of TSOU processes and a BSDE to hydrological analysis.

1. Introduction

Modelling and analysis of river discharge time series is a long-standing topic in hydrology, hydraulics, and related research areas. River discharge is not only a fundamental quantity in water planning and management like flood risk analysis [1] and water resources availability analysis [2], but also plays critical roles in analysing water quality dynamics [3] and biological dynamics [4].

Stochastic process models have been utilized as efficient tools of river discharge time series from statistical viewpoints. Usually, time series of discharge data at a fixed point in a river is represented by exponentially decreasing spikes triggered by rainfall events. Linear stochastic differential equations (SDEs) driven by compound Poisson processes have been widely employed as the simplest mathematical models to emulate the river discharge time series and associated environmental and ecological processes [5-6]. Nonlinear models driven by the compound Poisson noises have been proposed as well to capture the flood spikes more accurately [7-8]. However, these models have a common limitation that only large spikes corresponding to flood events can be captured. In principle, they cannot capture small fluctuation that would be important especially when considering dynamics of aquatic species such as the filamentous benthic algae whose effective growth rate depends on both large and small flow fluctuation [9]. A stochastic process model that is able to reasonably reproduce both large and small fluctuation of river discharge time series would provide useful information for water planning and management.

We propose a new mathematical model for emulating river discharge time series based on tempered stable Ornstein–Uhlenbeck (TSOU) processes [10]. Tempered stable processes are Lévy processes that are jump white noises having independent jump increments that are possibly very large or infinitesimally small [11]. TSOU processes are linear auto-regressive stochastic processes driven by tempered stable processes. TSOU and related mathematical models have been utilized in financial and economic time series analysis [12-13], but not in hydrology and hydraulics to the best of our knowledge.



We demonstrate that TSOU models can accurately capture the statistical moments (average, standard deviation, skewness, and kurtosis) of the river discharge times series just upstream and downstream of an existing dam. Another advantage of using the TSOU models is that they can be simulated exactly in a statistical sense by Monte-Carlo simulation [14]. Here, the term “exact” means that the sample paths can be generated without numerical errors at each discrete time and the errors simply comes from using pseudo-random numbers in numerical computation. This remarkable property is useful when assessing statistical properties of the generated time series. As an application, we apply the TSOU model to dynamically analysing dissolved silicon (DSi) load flowing into a dam.

2. TSOU model

We explain a TSOU model without going deeply into its mathematics. Mathematical properties of TSOU processes are well-documented in the literature [10, 15].

2.1 SDE formulation

Time is denoted as t and the river discharge at some point at time is denoted as Q_t . Loosely speaking, a TSOU model is an SDE having an exponential decay part (first term of (1)) and a jump noise part of a tempered stable type (second term of (1)):

$$dQ_t = -\lambda(Q_t - \underline{Q})dt + dZ_{\lambda t} \quad (1)$$

for $t > 0$ with an initial condition $Q_0 > 0$. Here, $\underline{Q} \geq 0$ is the constant representing base flow, $\lambda > 0$ is the decay coefficient representing flood decay and Z represents a tempered stable process having the Lévy measure [10]

$$\nu(dz) = \frac{ae^{-bz}}{z^{1+\alpha}} dz, \quad (2)$$

where $z > 0$ represents jump size of the river discharge with parameters $a > 0$, $b > 0$, and $\alpha \in (0,1)$. Using $Z_{\lambda t}$ instead of Z_t is for simplicity of the formulation in the numerical computation [14].

A Lévy measure uniquely determines statistical properties of the corresponding pure jump Lévy process. From a hydrological viewpoint, a is the parameter representing frequency of jumps (larger a gives more frequent jumps), b controls occurrence of large jumps (larger b gives less frequent large jumps), and α represents intermittence of the spikes (smaller α gives more intermittent sample paths). A tempered stable process has infinitely many infinitely small jumps if the integral of the corresponding Lévy measure diverges. In the context of river discharge, this means that both small and large fluctuation can be simulated using this stochastic process. For the TSOU case, we notice that integrating ν of (2) in the interval $(0, +\infty)$ yields $+\infty$ and thus the above-mentioned property is satisfied. Note that the integral of any compound Poisson processes do not diverge.

Finally, the auto-correlation function of the TSOU model is the exponential

$$\text{Cor}(Q_t, Q_s) = e^{-\lambda|t-s|} \text{ for } t, s \geq 0. \quad (3)$$

By (3), λ^{-1} is characterized as an effective memory length of the time series.

2.2 Discretized version

The TSOU model has the four parameters to be determined, which are λ , a , b , and α . They are estimated using Monte-Carlo simulation using the discretized counterpart. We use the discretization following Kawai and Masuda [14]. The time increment for the discretization is denoted as Δ and the discrete times are set as $t_k = k\Delta$ ($k = 0, 1, 2, \dots$). Set $Y_t = Q_t - \underline{Q}$. The exact discrete version of (1) is formulated as

$$Y_{t_{k+1}} = e^{-\lambda\Delta} Y_{t_k} + \eta_0(\Delta) + \sum_{l=1}^{N(\Delta)} \eta_l(\Delta) \quad (k=0,1,2,\dots). \quad (4)$$

Here, $\eta_0(\Delta)$ is a tempered stable variable with the exponent b based on the stable variable [14]

$$\left(\frac{a(1-e^{-\alpha\lambda\Delta})\Gamma(1-\alpha)}{\alpha \cos(V)} \right)^{\frac{1}{\alpha}} \sin(\alpha(V+\pi/2)) \left(\frac{\cos(V-\alpha(V+\pi/2))}{E} \right)^{\frac{1-\alpha}{\alpha}} \quad (5)$$

with Γ the Gamma function, where V follows a uniform distribution in the unit interval $(0,1)$ and E follows an exponential distribution with the intensity 1, $N(\Delta)$ represents a Poisson process having the intensity $-a(1-e^{-\alpha\lambda\Delta})\Gamma(-\alpha)b^\alpha$, and each $\eta_l(\Delta)$ ($l=1,2,3,\dots$) are independent random variables following the common probability density function (PDF)

$$p(\eta) = \frac{1}{(1-e^{-\alpha\lambda\Delta})\Gamma(-\alpha)b^\alpha} \eta^{-(1+\alpha)} (e^{-b\eta} - e^{-be^{\lambda\Delta}\eta}), \quad \eta > 0. \quad (6)$$

The last summation term of (4) is replaced by 0 if $N(\Delta) = 0$. It follows that the stationary density of Y does not depend on λ . This means that we can identify the parameters λ and (a, b, α) separately.

The discretized equation (4) is seemingly complicated, but an important point is that it is exact in the statistical sense that each random variable appearing in the equation can be simulated without severe computational costs. Therefore, we can carry out both fine and coarse sampling of TSOU processes by tuning the increment Δ . Because λ and Δ appear in the discretization algorithm as the product $\lambda\Delta$, one can also tune the parameter λ if his/her focus is on the stationary distributions of TSOU processes. If one is interested in small fluctuation, then he/she should use small Δ , and vice versa. For simulating each independent random variable appearing in this algorithm, we use the Mersenne twister for generating each random variable [16] and an acceptance-rejection method [17].

3. Application to Hii River

3.1 Study sites

The study sites to apply the proposed model are the upstream and downstream reaches of Hii River, Japan (Figure 1). This river is a first-class river flowing in the eastern part of Shimane prefecture with the length of 153 (km) and the catchment area of 2,070 (km²) [18]. Obara Dam as a multi-purpose dam (located at 35°13'29"N 132°57'42"E, crest height 90 (m), crest length 443 (m), reservoir capacity 6.08×10^7 (m³)) was constructed at the midstream of Hii River and has been operated since 2014 [19], and has been playing a central role in flood mitigation and regional water resources supply. After the dam construction, thick growth of filamentous green algae has been an environmental and fishery problem in the downstream river of the dam. In fact, their growth has been considered to hinder growth of benthic diatoms serving as a staple food source of *Plecoglossus altivelis altivelis* (*P. altivelis*, Ayu. This fish is the most important inland fishery resource in the river [9].

The authors have been measuring water depth, water temperature, and water quality indices (turbidity, dissolved oxygen, DSi etc.) at upstream and downstream observation stations of Obara Dam to investigate impacts of the dam on environment and ecology of Hii River. In addition, hourly operation data of Obara Dam is publicly available since April 1, 2016 [20] (Figure 2: inflow and outflow discharges: daily averaged values are presented in the figure for the sake of presentation). Both the inflow and outflow discharges in 2019 are smaller than those in 2016 through 2018 possibly due to the smaller amount of precipitation in this year.



Figure 1. Map around Obara Dam in Hii River.

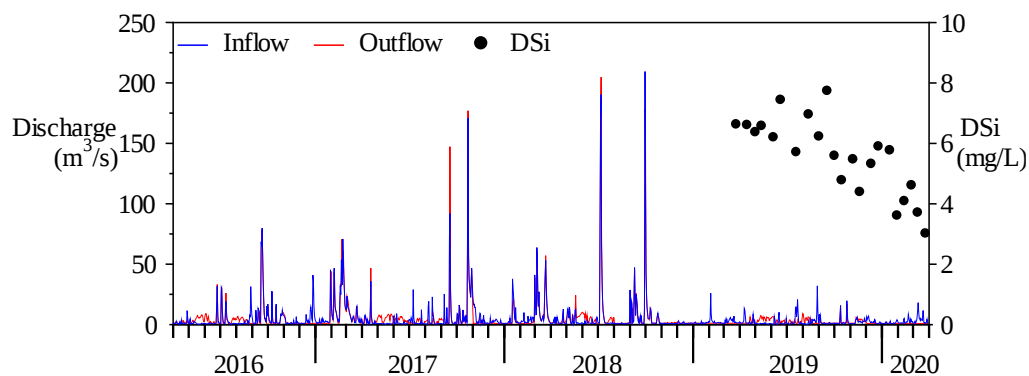


Figure 2. Inflow and outflow time series data. The DSi concentration data is also plotted in the figure.

3.2 Parameter identification

We consider the inflow and outflow discharge times series data as sample paths of TSOU processes. Hence, seasonality of the data is not explicitly considered. Instead, we focus on statistical properties of the data and their application.

Our identification approach is explained below. Firstly, identify λ by comparing the observed and theoretical auto-correlation functions (3). The empirical and computed autocorrelation functions for $0 \leq t, s \leq 300$ (h) are used to identify λ in the least-square sense. Secondly, identify (a, b, α) using a trial and error approach based on the Monte-Carlo simulation (Number of sample points: no less than $365 \times 24 \times 10,000$ with the resolution $\Delta = 1$ (h)) with (4). The triplet (a, b, α) is judged to be the identified one if the sum of the absolute values of the relative errors of the four moments (average, standard deviation, skewness, kurtosis) becomes the smallest. Note that the two steps in this approach is completely decoupled. We use $\underline{Q} = 1$ (m^3/s) for both the inflow and outflow models based on the observed time series data.

Table 1 summarizes the identified parameter values. The empirical and computed autocorrelation functions have the coefficients of determinant 0.99 for both the inflow and outflow. The values of the parameter λ imply that the memories of the time series remain two to three days for both the inflow and outflow. Table 2 shows the empirical statistical moments and the computed ones using the identified parameter values, showing that each moment is computed with the relative error less than 3.8 % for the inflow and less than 7.9 % for the outflow. Note that, for the outflow, the maximum relative error is 7.9 % (skewness), and the other three moments are reproduced with the relative error smaller than 2.1 %. The identified parameter values imply that the inflow and outflow discharges are comparably

intermittent (values of α are close with each other) and less small discharges (smaller kurtosis). This is considered to be due to mitigating flood as well as drought by operating the dam.

Figure 3 shows three sample paths generated by the identified model for the inflow, showing that the intermittent nature of the observed inflow time series is reasonably reproduced. **Figures 4** and **5** show the computed PDFs $P = P(Q)$ of the inflow and outflow discharges, respectively. The empirical inflow PDF is quite accurately reproduced by the model, while the PDFs of the outflow are slightly different around the values of 4 to 8 (m^3/s). This discrepancy is considered due to the dam operation that is not explicitly considered in the proposed model.

In summary, the identified model can be utilized as a simple and exactly computable mathematical tool for generating sample paths of the inflow and outflow discharge time series data, especially the inflow discharge, of Obara Dam.

Table 1. Identified parameter values.

	Inflow	Outflow
λ (1/h)	0.029	0.027
a ($\text{m}^{3\alpha}/\text{s}^\alpha$)	0.195	0.185
b (s/m^3)	0.007	0.009
α (-)	0.50	0.45

Table 2. Comparison of the empirical and computed moments.

	Inflow		Outflow	
Statistical moment	Empirical	Computed	Empirical	Computed
Average (m^3/s)	5.290	5.130	5.013	5.007
Standard deviation (m^3/s)	16.58	17.71	15.39	15.71
Skewness (-)	12.84	12.44	11.98	11.03
Kurtosis (-)	258.1	259.1	198.0	201.2

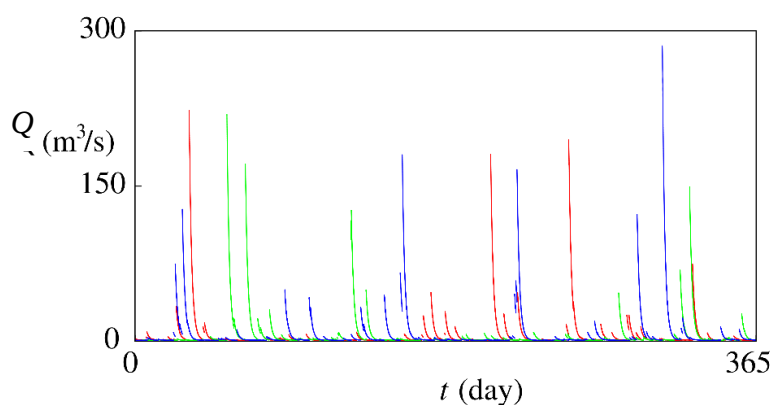


Figure 3. Three sample paths of the inflow discharge generated using $\Delta = 0.1$ (h). Different colours represent different sample paths.

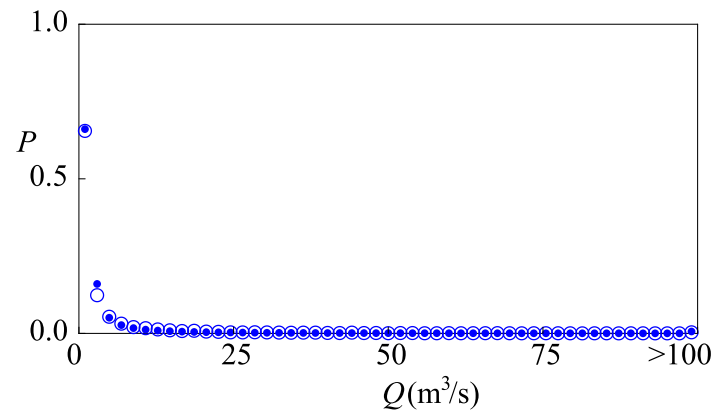


Figure 4. Empirical (○) and computed (●) PDFs for the inflow discharge. The PDFs are plotted with 2 (m^3/s) intervals.

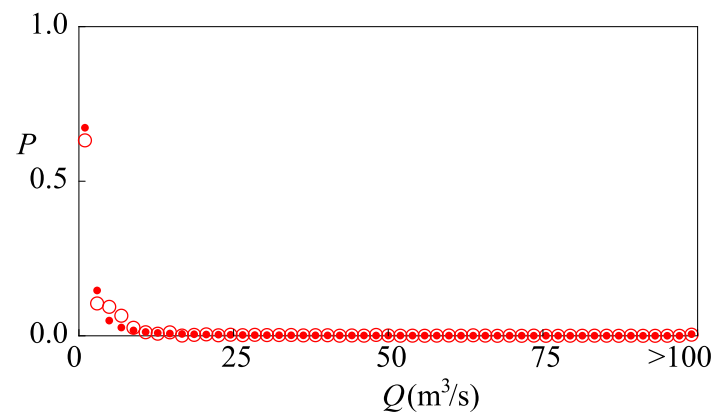


Figure 5. Empirical (○) and computed (●) PDFs for the outflow discharge. The PDFs are plotted with 2 (m^3/s) intervals.

3.3 DSi analysis

Along with collecting the hydrological information, we have been measuring DSi concentration as a key water quality index at several points including the upstream of Obara Dam (Again, see **Figures 1-2**). The analysis at the downstream of the dam is not carried out in this paper, but will be addressed in future analysis by concurrently considering inflow and outflow discharges and the dam operation. Seasonality of DSi concentration is not clearly understood in this river at this stage, but will be clarified in future by continuing sampling river waters. The DSi analysis here thus focuses on the statistical analysis without explicitly considering the seasonality for simplicity.

The DSi load L (g/s) defined as the product of the DSi concentration C (mg/L) and the discharge Q (m^3/s) is an important indicator of eutrophication in rivers and reservoirs [20, 21]. We present a demonstrative application example of the identified TSOU model of the inflow into Obara Dam to estimate the DSi load dynamically. In this context, the term “dynamic” estimation means that the expected cumulative DSi load from the current time t to some future time T is estimated based on the current discharge observation. This problem setting is realistic since the inflow discharge is observed at least the hourly resolution.

As a preparation for the analysis, we investigate correlation between the DSi load and the inflow. The inflow discharge is measured hourly, while the DSi concentration is measured almost twice in each month, meaning that their sampling frequencies are significantly different with each other. Therefore, we examine the measured DSi concentration and the daily-averaged inflow. We identified the load-

inflow relationship using a nonlinear least-squares method assuming $L = uQ^v$ (g/s) with some constants u, v . The least-squares fitting result gives $u = 6.75$ and $v = 0.823 < 1$ ($R^2 = 0.943$), implying a dilution effect of the DSI concentration. This is considered because the precipitation triggering large river discharge does not contain significant DSI. Due to the observation restriction, the data for relatively large inflow ($Q > 20$ (m³/s)) is not available. The fitted curve for the large inflow discharge is therefore an extrapolation.

The expectation of the cumulative DSI load into the reservoir of Obara Dam during the time interval (t, T) ($t \leq T$) is given the observation $Q_t = q$ as the conditional expectation

$$W(t, q) = \mathbb{E} \left[\int_t^T u Q_s^v ds \middle| Q_t = q \right]. \quad (7)$$

Clearly, we have $W(T, q) = 0$. Evaluating the conditional expectation W is carried out by computing a backward SDE (BSDE) as an efficient mathematical tool to evaluate conditional expectations of functional related to SDEs [22, 23]. In addition, the BSDE approach allows us to handle the water quality analysis having multiple water quality indices in future. In fact, one conventional Monte-Carlo simulation only gives one scalar variable $W(t, q)$ for each fixed t and q . On the other hand, with the BSDE we can compute $W(t, q)$ for all t and q backward in time, and is therefore a more informative than the conventional one. Application of the BSDE approach to water quality analysis has not been carried out so far to the best of the authors' knowledge.

Set $\theta_t = W(t, Q_t)$. The BSDE in our case is

$$d\theta_t = u Q_t^v dt + U_t d(Z_{\lambda t} - r_t), \quad (8)$$

where $Z_{\lambda t} - r_t$ with some process r_t is a Martingale term (formally) having the expectation 0 and U_t is an auxiliary process so that θ_t is adapted to the information up to the time t . For more technical details, see the literature [22, 23]. What is important is that the BSDE (8) can be computed using the exact discretization (4) and that the second term in the right-hand side of (8) is not necessary if we use the least-squares Monte-Carlo regression at each discrete time [22]. In addition, using this numerical method, we can obtain all $W(t, q) = 0$ for $x \geq 0$ at each time t .

Set the piece-wise constant approximation of W on each $[q_i, q_{i+1})$ ($i = 0, 1, 2, \dots, I$) of the length dx with $q_i = i \times dq + \underline{Q}$ ($i = 0, 1, 2, \dots, I$) and $q_{I+1} = +\infty$. Set $dq = 10$ (m³/s) and $I = 20$. This range of inflow has been chosen since the inflow larger than 200 (m³/s) is rare according to the identified TSOU model. The computation starts from the terminal time $t = T = 25$ (day) in a time-backward manner. Owing to considering the piece-wise constant approximation, we can analytically perform the least-squares Monte-Carlo procedure without any matrix inversion methods [22].

Based on randomly generated 100,000 sample paths of the inflow Q , **Figure 7** shows the computed cumulative DSI load W at each time using the BSDE. We see that the DSI load is a bit concave with respect to the observe discharge at each time, which is due to the concave $L - Q$ relationship utilized in this paper. Although not presented here, using global polynomial basis instead of the piecewise-constant basis leads to similar computational results.

As demonstrated in this section, one can dynamically evaluate the DSI load from the current time to a prescribed terminal time depending on the current observation. This kind of dynamic evaluation maybe difficult if we do not use the mathematical tools like the SDE and BSDE.

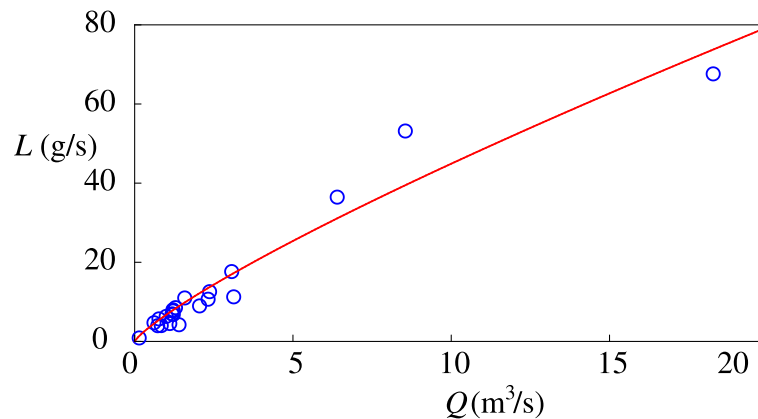


Figure 6. The observed and identified L - Q relationships.

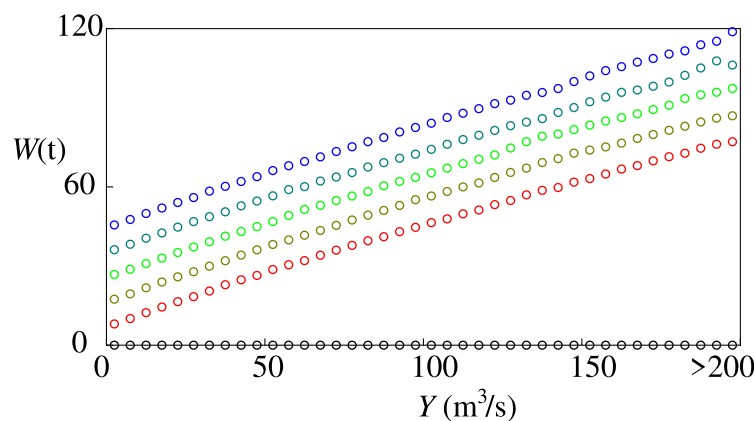


Figure 7. The cumulative DSi load with respect to the observed $Y = Q - \bar{Q}$. The different colours show the results on selected times: $t = 25$ ($= T$), 20, 15, 10, 5, and 0 (day) from the bottom to the top.

4. Conclusions

The proposed TSOU models could capture the statistical properties of the river discharge time series of upstream and downstream rivers of Obara Dam in Hii River, Japan. The identified models can be utilized for considering operation policies of the dam, which were not considered in this paper, and further water quality analysis other than DSi in the river. The BSDE approach as a new way to analyse the stochastic hydrology can then be employed as a versatile mathematical tool.

We will investigate whether the TSOU models can be applied to other rivers. Theoretically, it is possible to incorporate seasonal changes into a TSOU model by considering time-dependent parameters; however, the exact simulation may become impossible in such a case. Development of an exact or very accurate sampling algorithm is a key topic to overcome this difficulty. Currently, we are planning to incorporate a TSOU model to a stochastic control model of a multi-purpose dam-reservoir system. The identification approach of the model can also be improved by considering some likelihood methods, but may become more complex.

It is also important to examine the TSOU model to hydrological analysis of Hii River in Future. A hydropower station located upstream of Obara Dam is idling from Late June in 2020, meaning that the inflow discharge to the dam will change at least quantitatively. We are continuing water sampling and hydraulic measurement in Hii River to tackle this issue.

Acknowledgments

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