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# The nonlinear dynamic analysis of sediment particles near river bed

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**Abstract.** According to the equation of discrete solid particles' motion in arbitrary flow field, the dynamic characteristics and trajectories of sediment particles are discussed. The small disturbance method is used to analyze the stability of sediment particles at the equilibrium point, it is shown that the change of turbulence frequency will lead to the sudden change of sediment particles, under the condition of a certain periodic excitation amplitude; and under the condition of certain periodic excitation amplitude, the change of sediment particle size will also lead to the sudden change of sediment particle movement, All of the above prove that the movement of sediment particles near the river bed surface has a catastrophe.

## 1. The general form of motion equation of sparse sediment particles in arbitrary flow field

The sediment movement near the bed surface is very complex and has strong nonlinearity<sup>[1,2]</sup>. In order to track the motion track of sediment particles near the bed, the nonlinear dynamic characteristics of sediment particles on the bed surface are discussed according to the general form of motion equation of sparse sediment particles in arbitrary flow field.

$$\begin{aligned} \frac{du_p}{dt} = & \left[ 1 - \frac{\rho_f}{\rho_p} \right] f + \frac{\rho_f}{\rho_p} \frac{du_f}{u_t} + \frac{18\mu_{eff}}{\rho_p d_p^2} (u_f - u_p) \\ & + \frac{9}{\rho_p d_p} \left( \frac{\rho_f \mu_{eff}}{\pi} \right)^{1/2} \int_0^t \frac{d(u_f - u_p)/d\tau}{\sqrt{t - \tau}} d\tau + d\tau \frac{\rho_f}{2\rho_p} \frac{d(u_f - u_p)}{dt} \\ & + C_{LM} \frac{3}{4\rho_p d_p} |u_f - u_p| (u_f - u_p) + C_{LS} \frac{6K_s \mu}{\rho_p \pi d_p} \left( \frac{\xi}{\nu} \right)^{1/2} (u_f - u_p) \end{aligned} \quad (1)$$

In the formula<sup>[3]</sup>,  $u_p$  is the velocity of sediment particles, m/s;  $\rho_p$  is the density of sediment particles, kg/m<sup>3</sup>;  $\rho_f$  is the density of fluid, kg/m<sup>3</sup>;  $d_p$  is the diameter of sediment particles, m;  $u_p$  is the velocity of sediment particles, m/s;  $P$  is the flow field pressure, Pa;  $\nu$  is the fluid viscosity, m<sup>2</sup>/s;  $C_{LM}$  is Magnus lift,  $C_{LM} = 3/4\rho_p d_p |u_f - u_p| (u_f - u_p)$ ;  $C_{LS}$  is Saffman lift,  $C_{LS} = 6K_s \mu / \rho_p \pi d_p (\xi/\nu)^{0.5} (u_f - u_p)$ , the velocity gradient in the boundary layer is very large, which can



not be ignored; the parabolic velocity distribution near the bottom layer has enough accuracy,  $u(y) = 7.25u_*\sqrt{y/\bar{D}}$ ,  $\bar{D}$  is the average particle size of the particle <sup>[4]</sup>.

## 2. The second order nonlinear differential equation of vertical motion of river sediment particles

The vertical (Y-direction) forces on sediment particles include Stokes resistance, added mass force, differential pressure force and gravity, Saffman lift force, Magnus lift force, etc. When the angular velocity of particles is large, Magnus lift can be ignored in the boundary layer; in the region of high tangential stress in the boundary layer, Saffman<sup>[5,6]</sup> lift must be considered; under the condition of two-dimensional stable horizontal flow, if  $du/dt = 0$ , then formula(1) is simplified as follows:

$$y''(t) + (2\alpha - \pi\chi^2 - \beta y^{1/2})y'(t) + \alpha^2 y - 2\alpha\beta y^{1/2} = Y''(t) \quad (2)$$

In the formula,  $\alpha = 36\nu/[(2\bar{\rho} + 1)d_p^2]$ ,  $\beta = 0.1841gu_*/(\sqrt{\bar{D}}\bar{\rho}\pi)$ ,  $\chi = 18/(2\bar{\rho} + 1)\sqrt{\nu/(\pi d_p^2)}$ ,

$$Y''(t) = -\beta y^{1/2}u_p(0)\sqrt{t} + \chi \int_0^t F/\sqrt{t-\tau}d\tau - \alpha \int_0^t Fd\tau - (\alpha - 1)u_p(0) - \alpha^2 y(0) + (\chi - \alpha)\beta/2y^{1/2} \quad (0)$$

$Y''(t)$  are simplified treatment, in which,  $\omega$  is the turbulence frequency of flow,  $H = E\omega_0^2$ ,  $E$  is the excitation amplitude of flow,  $\omega_0$  is the natural frequency of sediment particles, and  $\mathcal{G}$  is the initial phase angle. Then the formula (2) is as follows:

$$y''(t) + (2\alpha - \pi\chi^2 - \beta y^{1/2})y'(t) + \alpha^2 y - 2\alpha\beta y^{3/2} = H \cos(\omega t + \mathcal{G}) + h \quad (3)$$

## 3. The stability analysis of motion equilibrium point

When  $F(t) = 0$ , from the equilibrium condition  $y'_o = 0$ ,  $u'_{po} = 0$ , it can be found that there are two static equilibrium points  $O(0,0)$   $P(4\beta^2/\alpha^2, 0)$  in the system equation.

The small disturbance method can be used to analyze the stability of sediment particles near the equilibrium point. Here, we first replace the variable:  $y(t)^{1/2} = X(t)$ ; and then translate the variable  $X(t)$  as follows:  $X(t) = x(t) + (2\beta/\alpha)$ . Then (3) is changed into:

$$x''(t) + \beta \left( \frac{2\alpha}{\beta} - \frac{\pi\chi^2}{\beta} - \frac{1}{x + (2\beta/\alpha)} \right) x'(t) + \frac{x'^2(t)}{x + (2\beta/\alpha)} + \frac{\alpha^2}{2} x = \frac{H \cos(\omega t + \mathcal{G}) + h}{x + (2\beta/\alpha)} \quad (4)$$

Equation (4) is the differential equation of single particle sediment movement with the equilibrium position  $P'(2\beta/\alpha, 0)$  as the origin after the variable  $X(t)$  translation. In the formula,  $\omega_0 = \alpha/\sqrt{2}$ . In this paper, we introduce a small parameter  $0 < \varepsilon < 1$  into equation (4) and finally make it 1. Let  $\beta$ ,  $H$ ,  $h$  are small quantities of order  $\varepsilon$ , that is  $\beta = \varepsilon\bar{\beta}$ ,  $\chi = \varepsilon\bar{\chi}$ ,  $H = \varepsilon\bar{H}$ ,  $h = \varepsilon\bar{h}$ , take  $\mathcal{G} = 0$ , then formula (4) becomes:

$$\frac{d^2x}{dt^2} + \varepsilon \left( 2\alpha - \pi\chi^2 - \frac{\beta}{x + (2\beta/\alpha)} + \frac{x'}{x + (2\beta/\alpha)} \right) x' + \frac{\alpha^2}{2} x = \frac{\varepsilon\bar{H} \cos(\omega t + \mathcal{G}) + \varepsilon\bar{h}}{x + (2\beta/\alpha)} \quad (5)$$

In the following calculation, remove the "-" above the letter and use the Multiple scales method to calculate the operation of formula (5). Let the approximate solution of formula (5) be as follows:

$$x = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) + \dots \quad (6)$$

Here  $T_0$  is the fast changing time scale,  $T_0 = t$ ,  $T_1$  and is the slow changing time scale,  $T_1 = \varepsilon t$ . There are differential operators:

$$d/dt = dT_0/dt \cdot \partial/\partial T_0 + dT_1/dt \cdot \partial/\partial T_1 + \dots = D_0 + \varepsilon D_1 + \dots \quad (7)$$

$$d^2/dt^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 D_1^2 + \dots \quad (8)$$

Equation (6) is substituted into formula (5), and make the coefficients of the same power equal of  $\varepsilon$ :

$$\varepsilon^0: D_0^2 x_0 + \alpha^2 x_0 / 2 = 0 \quad (9)$$

$$\varepsilon^1: D_0^2 x_1 + \alpha^2 / 2 x_1 = -2D_1 D_0 x_0 + \pi \chi^2 D_0 x_0 + (\beta D_0 x_0 - (D_0 x_0)^2 + \bar{H} \cos(\omega t + \vartheta) + \bar{h}) \\ (\alpha/(2\beta) - (\alpha/(2\beta))^2 x_0 + (\alpha/(2\beta))^3 x_0^2 - (\alpha/(2\beta))^4 x_0^3 + \dots) \quad (10)$$

$$\text{suppose the solution of formula (10) is as follows: } x_0 = A e^{(i\alpha T_0/\sqrt{2})} + (\bar{A}) e^{(-i\alpha T_0/\sqrt{2})} \quad (11)$$

Considering the principal resonance:  $\omega = \omega_0 + \varepsilon \sigma$ , in the formula,  $\varepsilon$  is a small quantity;  $\sigma$  is a coordination parameter, which reflects the approximate degree of the frequency  $\omega_0$  of sediment particle movement and the frequency  $\omega$  of flow turbulence. Formula (11) is substituted into equation (10), according to the conditions for eliminating the perpetual term, there are

$$-\sqrt{2}i\alpha D_1 A + i\pi\alpha\chi^2 A/\sqrt{2} + \sqrt{2}i\alpha A/4 + i\beta(\alpha/(2\beta))^3 |A|^2 A \\ -(\alpha/(2\beta))^2 |A|^2 A - 3(\alpha/(2\beta))^4 |A|^4 A - H e^{i\omega T_1}/\sqrt{2} = 0 \quad (12)$$

$$\text{In polar coordinates: } A = r e^{i\theta}/2 \quad (13)$$

In the formula,  $r$  and  $\theta$  are real functions of  $T_1$ . By substituting equation (13) into equation (12), the following results are obtained:

$$-i\frac{\alpha}{\sqrt{2}} D_1 r e^{i\theta} + i\pi\frac{\alpha}{2\sqrt{2}} \chi^2 r e^{i\theta} + i\frac{\alpha}{4\sqrt{2}} r e^{i\theta} + i\frac{1}{8}\beta(\frac{\alpha}{2\beta})^3 r^3 e^{i\theta} \\ -(\frac{\alpha}{2\beta})^2 \frac{1}{8} r^3 e^{i\theta} - \frac{3}{32}(\frac{\alpha}{2\beta})^4 r^5 e^{i\theta} - \frac{\sqrt{2}}{2} H e^{(i\omega T_1)} + \dots = 0 \quad (14)$$

By separating the real part from the imaginary part and adding "-" to each coefficient, the average equation in polar form is obtained in the first approximation case:

$$\begin{cases} dr/dt = [\bar{H} \sin(\psi)/\sqrt{2} - \bar{\alpha}\pi\bar{\chi}^2/\sqrt{2} - \sqrt{2}\bar{\alpha}^2 r/4 + \bar{\beta}(\bar{\alpha}r/(2\bar{\beta}))^3/4]/\sqrt{2}\bar{\alpha} \\ r d\theta/dt = -\sigma r - (\bar{\alpha}/(2\bar{\beta}))^2 r^3/64 - 3(\bar{\alpha}/(2\bar{\beta}))^4 r^5/1024 - \sqrt{2}\bar{H} \cos(\psi)/2 \end{cases} \quad (15)$$

At this time, the periodic solution of nonlinear equation (5) is transformed into the steady solution of equation (15). Substituting equation (13) into equation (11), there are:  $x = r \cos(\alpha t/\sqrt{2} + \theta)$

In the formula,  $r$  and  $\theta$  are determined by formula (15), order  $t - t\varepsilon\sigma + \theta = \psi$ , Then  $\dot{\psi} = -\varepsilon\sigma + \dot{\theta}$ , by substituting  $\varepsilon\bar{\beta} = \beta$ ,  $\varepsilon\bar{\chi} = \chi$ ,  $\varepsilon\bar{H} = H$ ,  $\varepsilon\bar{h} = h$ , then equation (15) becomes the following average equation:

$$\begin{cases} dr/dt = H \sin(\psi)/\sqrt{2} - \alpha\pi\chi^2/\sqrt{2} - \sqrt{2}\alpha^2 r/4 + \beta(\alpha/(2\beta))^3 r^3/4 \\ r dr/dt = -\sigma r - (\alpha/(2\beta))^2 r^3/64 - 3(\alpha/(2\beta))^4 r^5/1024 - \sqrt{2}H \cos(\psi)/2 \end{cases} \quad (16)$$

For steady motion ( $\dot{r} = 0, \dot{\psi} = 0$ ), the following equation should be satisfied:

$$\begin{cases} H \sin(\psi)/\sqrt{2} - \alpha\pi\chi^2/\sqrt{2} - \sqrt{2}\alpha^2r/4 + \beta(\alpha/(2\beta))^3 r^3/4 = 0 \\ -\sigma r - (\alpha/(2\beta))^2 r^3/64 - 3(\alpha/(2\beta))^4 r^5/1024 - \sqrt{2}H \cos(\psi)/2 = 0 \end{cases} \quad (17)$$

Equations (16) and (17) are combined to eliminate variables  $\psi$ , and the bifurcation equation of equation (4) after translation is obtained as follows:

$$\begin{aligned} & \left[ -\alpha\pi\chi^2r/\sqrt{2} - \alpha^2r/2\sqrt{2} + \beta(\alpha/2\beta)^3 r^3/4 \right]^2 \\ & + \left[ -\sigma r - (\alpha/2\beta)^2 r^3/64 - 3(\alpha/2\beta)^4 r^5/1024 \right]^2 - H^2/2 = 0 \end{aligned} \quad (18)$$

Equation (18) is a bifurcation equation after replacing equation (3) with periodic motion equation of sediment particles, which reflects the nonlinear characteristics of sediment particles with equilibrium position  $P'(2\beta/\alpha, 0)$  as origin and periodic motion amplitude  $r$  as variable.

#### 4. The numerical analysis

Take  $u_* = 0.02 \text{ m/s}$ ,  $\rho_p = 2.65 \times 10^3 \text{ kg/m}^3$ ,  $\bar{D} = 1e^{-2} \text{ cm}$ ,  $d_p = 1.5e^{-3} \text{ cm}$ ,  $\nu = 1.31e^{-6}$ ,  $H = 0.2\omega_0$ , and substitute into equation (18), obtain the relationship curve between the amplitude of sediment particle motion and the coordination parameters, as shown in figure 1:

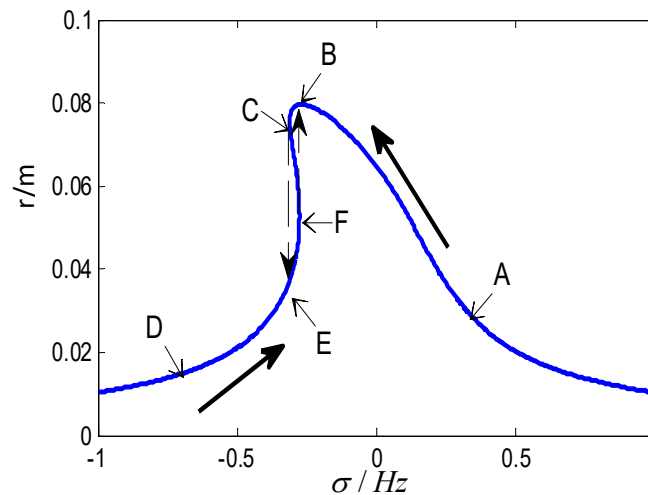


Figure 1. amplitude frequency response curve of sediment particle motion

In figure 1, the abscissa is the coordination parameter  $\sigma$ , and the ordinate is the amplitude  $r$  of the periodic movement. When  $\sigma$  increasing from small, the  $\sigma$  changes along the curve DEFBA, from point D to point F, the displacement amplitude  $r$  increases suddenly, jumping from point F to point B; when  $\sigma$  decreases gradually,  $r$  changes along the curve ABCED, and increases rapidly from point A to point B. when reaching point B, the amplitude of displacement motion  $r$  reaches the maximum, and then  $r$  decreases suddenly with the

continuous decrease of  $\sigma$  at the point C, jump from point C to point E. It can be seen that the value of amplitude  $r$  jumps with the change of  $\sigma$ , which is the performance of nonlinear mutation. This shows that under the condition of a certain amplitude of periodic excitation, the change of turbulence frequency will lead to the sudden change of the movement form of sediment particles, which indicates that the movement of sediment particles is abrupt.

Now we analyze the response relationship between the amplitude of single particle sediment periodic motion and particle size under different coordination parameters  $\sigma$ . Take  $u_* = 0.02 \text{ m/s}$ ,

$\rho_p = 2.65 \times 10^3 \text{ kg/m}^3$ ,  $\bar{D} = 1e^{-2} \text{ cm}$ ,  $\nu = 1.31e^{-6}$ ,  $H = 0.2\omega_0$  into equation (18), The response curves of amplitude and particle size of single particle sediment under different coordination parameters  $\sigma$  are calculated. The result is shown in figure 2:

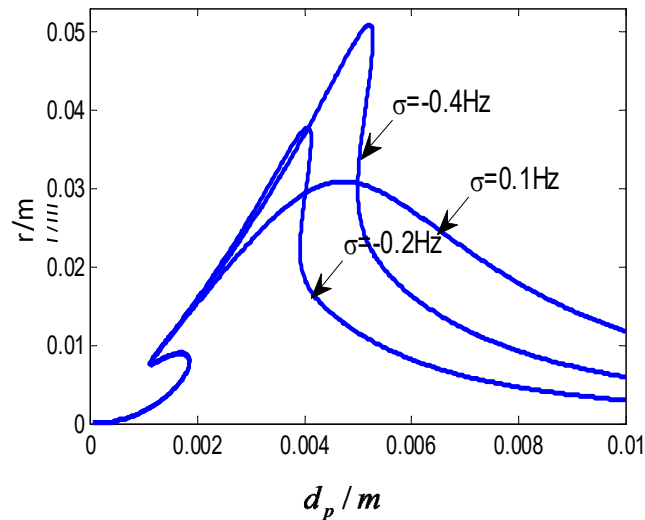


Figure 2. response curve of sediment particle motion amplitude and particle size

In figure 2, the abscissa is the particle size of sediment particles  $d_p$ , and the ordinate is the amplitude  $r$  of periodic movement. It can be seen that in at least three groups of sediment particles, for a certain size of sediment particles, there are three different amplitudes  $r$ . The results show that under the condition of certain periodic excitation amplitude, the change of sediment particle size will also lead to the sudden change of sediment particle movement form, which indicates that the movement of sediment particles has mutation.

## 5. Conclusion

Starting from the motion equation of single particle sediment, the nonlinear characteristics of vertical motion of single particle sediment are emphatically analyzed. The stability conditions of steady periodic solution of single particle sediment vertical motion are analyzed by using nonlinear dynamic method. It is proved that there is jump mutation phenomenon in vertical movement of sediment near bed surface.

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