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Timoshenko Beams and the Hamiltonian System

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Abstract. The significance of the transition from Lagrangian system to Hamiltonian system lies in that it has entered the form of symplectic geometry from the traditional Euclidean geometry and broken through the traditional concept, so that the dual mixed variable method has entered into the vast field of applied mechanics. Thus the Hamiltonian system method is very significant for engineering mechanics system and even mathematical physics method. It provides a powerful tool for discussing this kind of problems and improves the solution of elasticity to a new platform.

1. Introduction

In elastic mechanics, the displacement method or force method is traditionally used to discuss the governing equations and constitutive relations, which belongs to the single variable method and inevitably leads to higher-order partial differential equations. Due to the complexity of the constitutive relations in the computation of the Timoshenko beams, the solution of the problem in the case of single variable has certain limitations, and it is generally difficult to get a closed solution space. Therefore, a new theoretical method is needed to solve the classical Timoshenko problems [1-3]. In terms of numerical methods, scholars have begun to find new solutions to improve the calculation accuracy and efficiency. However, the single variable method is very difficult to deal with the complex mixed boundary problems, and only when the symplectic structure is satisfied in the numerical calculation can the accuracy and the efficiency be fully improved.

Zhong opened up a new research field in the application of Hamiltonian system in elastic mechanics, and made great achievements [4]. Because the eigensolution expansion method is used, the problem is transformed into a Hamiltonian operator matrix. In addition, the study of elasticity in symplectic system takes the primal variable and its dual variable as the basic variable, which makes the separated variable method be implemented smoothly, thus forming a unique direct method [5]. Steele and Kim [6] also established the hybrid variational principle method and discussed the elastic dynamic problems.

Generally, the basic assumptions of the Euler Bernoulli beam include the following two points:

(1) The cross section of the vertical beam axis before the deformation of the beam is still plane after the deformation (rigid section assumption);

(2) The plane after the deformation of the cross section is still perpendicular to the axis after the deformation (straight normal assumption).

The classical beam theory ignores the transverse shear deformation of the beam, which is only suitable for the long beam; when the effective length of the beam is short or composite beam, it is

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obviously inappropriate to ignore the shear deformation [7-9]. Therefore, the Timoshenko beam theory puts forward a modification to the beams with shear deformation that cannot be ignored. It only retains the first assumption and satisfies the stress-strain relationship of the shear.

2. Dynamic equation of Timoshenko beam

In addition to the transverse distributed load, the load acting on the beam also has the inertia force generated by the deflection of the beam:

$$\begin{aligned} \tilde{q}_I(x) &= -\rho A \frac{\partial^2 \tilde{w}}{\partial t^2} \\ \tilde{m}_t(x) &= -\rho I \frac{\partial^2 \tilde{\theta}}{\partial t^2} \end{aligned} \tag{1}$$

So the dynamic equations can be expressed as

$$\begin{cases} \left(\tilde{F}_s + \dfrac{\partial \tilde{F}_s}{\partial x} dx \right) - \tilde{F}_s + \left(\tilde{q} + \tilde{q}_I \right) dx = 0 \\ \left(\tilde{M} + \dfrac{\partial \tilde{M}}{\partial x} dx \right) - \tilde{M} + \tilde{F}_s dx + \left(\tilde{m} + \tilde{m}_I \right) dx = 0 \end{cases} \tag{2}$$

By using the following relationships

$$\begin{cases} \tilde{M} = EI \cdot \tilde{k},\ \tilde{F}_s = kGA \cdot \tilde{\gamma} \\ \tilde{\kappa}(x) = \dfrac{d\tilde{\theta}}{dx},\ \tilde{\gamma} = \dfrac{d\tilde{w}}{dx} - \tilde{\theta} \end{cases} \tag{3}$$

One has

$$\begin{cases} \dfrac{\partial}{\partial x}\left[kGA\left(\dfrac{d\tilde{w}}{dx} - \tilde{\theta} \right) \right] + \tilde{q} = \rho A \dfrac{\partial^2 \tilde{w}}{\partial t^2} \\ \dfrac{\partial}{\partial x}\left(EI \dfrac{d\tilde{\theta}}{dx} \right) + kGA\left(\dfrac{d\tilde{w}}{dx} - \tilde{\theta} \right) + \tilde{m} = \rho I \dfrac{\partial^2 \tilde{\theta}}{\partial t^2} \end{cases} \tag{4}$$

The initial conditions are

$$\begin{aligned} \tilde{w} &= w_0(x) \\ \tilde{\theta} &= \theta_0(x) \end{aligned} \quad (t = 0) \tag{5}$$

In the frequency domain, Eq. (4) is simplified to

$$\begin{cases} \dfrac{\partial}{\partial x}\left[kGA\left(\dfrac{dw}{dx} - \theta \right) \right] + q = \rho A \omega^2 w = 0 \\ \dfrac{\partial}{\partial x}\left(EI \dfrac{d\theta}{dx} \right) + kGA\left(\dfrac{dw}{dx} - \theta \right) + m + \rho I \omega^2 \theta = 0 \end{cases} \tag{6}$$

Simulate horizontal coordinate as time coordinate, and take

$$\mathbf{q} = \left(w \quad \theta \right)^T \quad \dot{\mathbf{q}} = \left(\dot{w} \quad \dot{\theta} \right)^T \tag{7}$$

The Lagrangian density function is

$$L(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{K}_{22}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\mathbf{K}_{21}\mathbf{q} + \frac{1}{2}\mathbf{q}^T\mathbf{K}_{11}\mathbf{q} - \mathbf{g}^T\mathbf{q} \tag{8}$$

where

$$\mathbf{K}_{11} = \begin{pmatrix} -\rho A\omega^2 & 0 \\ 0 & kGA - \rho I\omega^2 \end{pmatrix},\ \mathbf{K}_{22} = \begin{pmatrix} kGA & 0 \\ 0 & EI \end{pmatrix},\ \mathbf{K}_{21} = \begin{pmatrix} 0 & -kGA \\ 0 & 0 \end{pmatrix} \tag{9}$$

On the variation of the equation, the differential equation is obtained as

$$\frac{d}{dx}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{10}$$

Substituting formula (9) into the above formula, we have

$$\mathbf{K}_{22}\ddot{\mathbf{q}} + (\mathbf{K}_{21} - \mathbf{K}_{12})\dot{\mathbf{q}} - \mathbf{K}_{11}\mathbf{q} + \mathbf{g} = 0 \tag{11}$$

and

$$\mathbf{g} = (q,\ m) \tag{12}$$

Introducing Hamiltonian density function

$$\mathbf{H}(\mathbf{q},\mathbf{p}) = \mathbf{p}^T\dot{\mathbf{q}} - L(\mathbf{q},\mathbf{p}) = \mathbf{p}^T\mathbf{A}\mathbf{q} - \frac{1}{2}\mathbf{q}^T\mathbf{B}\mathbf{q} + \frac{1}{2}\mathbf{p}^T\mathbf{D}\mathbf{p} + \mathbf{h}_\mathbf{q}^T\mathbf{p} - \mathbf{h}_\mathbf{q}^T\mathbf{q} \tag{13}$$

we get the following Hamiltonian dual equations

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{\partial H}{\partial \mathbf{p}} = \mathbf{A}\mathbf{q} + \mathbf{D}\mathbf{p} + \mathbf{h}_\mathbf{q} \\ \dot{\mathbf{p}} = \dfrac{\partial L}{\partial \mathbf{q}} = -\dfrac{\partial M}{\partial \mathbf{q}} = \mathbf{B}\mathbf{q} - \mathbf{A}^T\mathbf{p} + \mathbf{h}_\mathbf{p} \end{cases} \tag{14}$$

Hamiltonian equations can be simply written as

$$\dot{\mathbf{v}} = \mathbf{H}\mathbf{v} + \mathbf{h} \tag{15}$$

where

$$\mathbf{H} = \begin{pmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{B} & \mathbf{-A}^T \end{pmatrix},\ \mathbf{h} = \begin{pmatrix} \mathbf{h}_\mathbf{q} \\ \mathbf{h}_\mathbf{p} \end{pmatrix} \tag{16}$$

3. Numerical example

To obtain the dependence of the transverse shear stiffness we have to solve Eq. (8). The solution of the spectral problemwas made numerically by using proposed method, and the values of transverse shear stiffness is given in Fig. 1.

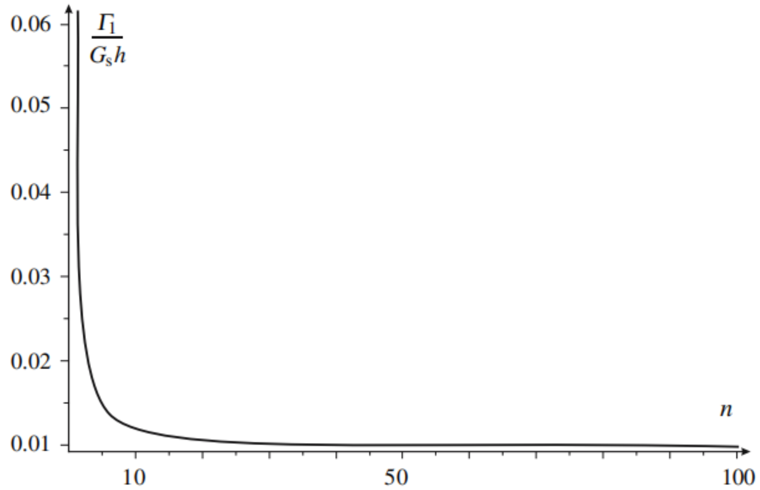

Figure 1. Transverse shear stiffness.

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