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Quantized Weyl algebras, the double centralizer property, and a new first fundamental theorem for $U_q(\mathfrak{gl}_n)$

Gail Letzter[1], **Siddhartha Sahi**[2,*] and **Hadi Salmasian**[3]

[1] Mathematics Research Group, National Security Agency, United States of America
[2] Department of Mathematics, Rutgers University, Piscataway, NJ, United States of America
[3] Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada

E-mail: gletzter@verizon.net, sahi@math.rutgers.edu and hadi.salmasian@uottawa.ca

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Abstract

Let $\mathscr{P} := \mathscr{P}_{m\times n}$ denote the quantized coordinate ring of the space of $m\times n$ matrices, equipped with natural actions of the quantized enveloping algebras $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$. Let $\mathscr{L}$ and $\mathscr{R}$ denote the images of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$ in $\mathrm{End}(\mathscr{P})$, respectively. We define a q-analogue of the algebra of polynomial-coefficient differential operators inside $\mathrm{End}(\mathscr{P})$, henceforth denoted by $\mathscr{P}\mathscr{D}$, and we prove that $\mathscr{L}\cap\mathscr{P}\mathscr{D}$ and $\mathscr{R}\cap\mathscr{P}\mathscr{D}$ are mutual centralizers inside $\mathscr{P}\mathscr{D}$. Using this, we establish a new First Fundamental theorem of invariant theory for $U_q(\mathfrak{gl}_n)$. We also compute explicit formulas in terms of q-determinants for generators of the algebras $\mathscr{L}_\mathfrak{h}\cap\mathscr{P}\mathscr{D}$ and $\mathscr{R}_\mathfrak{h}\cap\mathscr{P}\mathscr{D}$, where $\mathscr{L}_\mathfrak{h}$ and $\mathscr{R}_\mathfrak{h}$ denote the images of the Cartan subalgebras of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$ in $\mathrm{End}(\mathscr{P})$, respectively. Our algebra $\mathscr{P}\mathscr{D}$ and the algebra $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ that is defined in (Shklyarov *et al* 2004 *Int. J. Math.* **15** 855–94) are related by extension of scalars, but we give a new construction of $\mathscr{P}\mathscr{D}$ using deformed twisted tensor products.

Keywords: double centralizer theorem, quantized Weyl algebras, first fundamental theorem, quantum groups

* Author to whom any correspondence should be addressed.

1. Introduction

The First Fundamental theorem (FFT) is one of the pinnacles of invariant theory with a history as old as Hermann Weyl's influential book, *The Classical Groups* [Wy39]. In its original form, the FFT for the group GL_n describes the generators of the subalgebra of GL_n-invariants in the polynomial algebra $\mathcal{P}(V^{\oplus k} \oplus (V^*)^{\oplus l})$, where $V := \mathbb{C}^n$ denotes the standard GL_n-module.

It was pointed out by Howe [Ho95, section 2.3] that the FFT has an equivalent formulation as a double centralizer property, which we now recall. Let $\mathrm{Mat}_{m\times n}$ denote the vector space of complex $m \times n$ matrices. Then $\mathrm{Mat}_{m\times n}$ has a natural $\mathrm{GL}_m \times \mathrm{GL}_n$-module structure by left and right matrix multiplication. We equip the algebra $\mathcal{P} := \mathcal{P}(\mathrm{Mat}_{m\times n})$ of polynomials on $\mathrm{Mat}_{m\times n}$ and the algebra $\mathcal{PD} := \mathcal{PD}(\mathrm{Mat}_{m\times n})$ of polynomial-coefficient differential operators on $\mathrm{Mat}_{m\times n}$ with their canonical $\mathrm{GL}_m \times \mathrm{GL}_n$-module structures. Recall that $\mathcal{P}$ is a $\mathcal{PD}$-module. The (infinitesimal) actions of the Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ on $\mathcal{P}$ are given by certain differential operators of order one, which are usually called *polarization operators*. It follows that there exists a homomorphism of algebras $\phi : U_{m,n} \to \mathcal{PD}$, where $U_{m,n} := U(\mathfrak{gl}_m) \otimes U(\mathfrak{gl}_n)$ is the tensor product of the universal enveloping algebras of $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$, such that the diagram

$$\begin{tikzcd} U_{m,n} \otimes \mathcal{P} \arrow[rr, "x\otimes f\mapsto x\cdot f"] \arrow[dr, "x\otimes f\mapsto\phi(x)\otimes f"'] & & \mathcal{P} \\ & \mathcal{PD} \otimes \mathcal{P} \arrow[ur, "D\otimes f\mapsto D\cdot f"'] & \end{tikzcd} \tag{1}$$

commutes. The *operator commutant version* of the FFT, according to [Ho95, theorem 2.3.3], states that the subalgebra $\mathcal{PD}^{\mathrm{GL}_m}$ of GL_m-invariants in $\mathcal{PD}$ is generated by the image of $U(\mathfrak{gl}_n)$. Since $\mathcal{PD}^{\mathrm{GL}_m} = \mathcal{PD}^{U(\mathfrak{gl}_m)}$, the latter assertion is equivalent to the following: the images of $U(\mathfrak{gl}_m)$ and $U(\mathfrak{gl}_n)$ in $\mathcal{PD}$ are mutual centralizers.

In [LZZ11, section 6] the authors extend the original form of the FFT to the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ by considering a q-analogue of $\mathcal{P}(V^{\oplus k} \oplus (V^*)^l)$ that carries a $U_q(\mathfrak{gl}_n)$-action, and then describing the generators of the subalgebra of invariants. It is then natural to ask if the operator commutant version of the FFT also has a q-analogue. It turns out that in the quantized setting, the situation for the operator commutant FFT is more subtle than in the classical case. One major issue is how to quantize the Weyl algebra $\mathcal{PD}$ and, more importantly, the map $\phi : U_{m,n} \to \mathcal{PD}$. Indeed we provide some justification that the latter map cannot be fully quantized (see proposition 3.14.4). Nevertheless, our first main result (theorem A) is a positive answer to the above question.

From now on let $\Bbbk := \mathbb{C}(q)$ be the field of rational functions in a parameter q. For the operator commutant FFT in the quantized setting we need a quantized Weyl algebra $\mathscr{PD} := \mathscr{PD}_{m\times n}$. The $\Bbbk$-algebra $\mathscr{PD}$ that we consider is closely related to the algebra $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ of [SSV04, BKV06] (see corollary 3.16.1). We give a different construction of $\mathscr{PD}$ as the *deformed twisted tensor product* of $\mathscr{P} := \mathscr{P}_{m\times n}$, the quantized coordinate ring of $\mathrm{Mat}_{m\times n}$, and $\mathscr{D} := \mathscr{D}_{m\times n}$, the quantized algebra of constant-coefficient differential operators on $\mathrm{Mat}_{m\times n}$ (see section 3 for precise definitions). The construction of $\mathscr{P}$ and $\mathscr{D}$ is analogous to the FRT construction [KS97, section 9.1]. Concretely, the algebra $\mathscr{PD}$ is generated by $2mn$ generators $t_{i,j}$ and $\partial_{i,j}$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, modulo the relations that are described in section 3 (see definition 3.7.3). From now on we set

$$U_L := U_q(\mathfrak{gl}_m) \quad , \quad U_R := U_q(\mathfrak{gl}_n) \quad , \quad U_{LR} := U_L \otimes U_R.$$

Both $\mathscr{P}$ and $\mathscr{PD}$ are U_{LR}-module algebras (the explicit formulas for the U_{LR}-action on generators are given in remark 3.6.2). Furthermore, $\mathscr{P}$ is naturally a $\mathscr{PD}$-module. In particular, we have homomorphisms of associative algebras

$$\phi_U : U_{LR} \to \mathrm{End}_{\Bbbk}(\mathscr{P}) \quad \text{and} \quad \phi_{PD} : \mathscr{PD} \to \mathrm{End}_{\Bbbk}(\mathscr{P}).$$

Since $\mathscr{P}$ is a faithful $\mathscr{PD}$-module (see proposition 3.11.4), we can identify $\mathscr{PD}$ with $\phi_{PD}(\mathscr{PD})$. Using the latter identification, we set

$$\mathscr{L} := \phi_U(U_L \otimes 1), \quad \mathscr{R} := \phi_U(1 \otimes U_R), \quad \mathscr{L}_\bullet := \mathscr{L} \cap \mathscr{PD}, \quad \mathscr{R}_\bullet := \mathscr{R} \cap \mathscr{PD}.$$

Note that in general we have $\mathscr{L}_\bullet \subsetneq \mathscr{L}$ and $\mathscr{R}_\bullet \subsetneq \mathscr{R}$. In fact if $m \leqslant n$ then the restriction of ϕ_U to $U_L \otimes 1$ yields an isomorphism $U_L \cong \mathscr{L}$ but one can show that $\phi_U^{-1}(\mathscr{L}_\bullet)$ is properly contained in the locally finite part of U_L (see proposition 3.14.3 and example 10.2.2). Set

$$\mathsf{L}_{i,j} := \sum_{r=1}^{n} t_{i,r}\partial_{j,r} \text{ for } 1 \leqslant i,j \leqslant m \text{ and } \mathsf{R}_{i,j} := \sum_{r=1}^{m} t_{r,i}\partial_{r,j} \text{ for } 1 \leqslant i,j \leqslant n. \tag{2}$$

The $\mathsf{L}_{i,j}$ and the $\mathsf{R}_{i,j}$ are natural analogues in $\mathscr{PD}$ of the polarization operators of the (non-quantized) Weyl algebra $\mathcal{PD}$. We have $\mathsf{L}_{i,j} \in \mathscr{L}_\bullet$ and $\mathsf{R}_{i,j} \in \mathscr{R}_\bullet$ (see corollary 6.1.6). Of course, similar inclusions hold in the non-quantized case. However, the situation with the preimages of the $\mathsf{L}_{i,j}$ and the $\mathsf{R}_{i,j}$ in U_L and U_R is more complicated. For example, the root vector $E_{\varepsilon_i - \varepsilon_j}$ of U_L (respectively, U_R), where $i < j$, does not lie in the preimage of $\mathsf{L}_{i,j}$ (respectively, $\mathsf{R}_{i,j}$).

Henceforth we adopt the following notation: for subsets $\mathcal{Y}, \mathcal{Z}$ of an algebra $\mathcal{X}$, we set

$$\mathcal{Y}^{\mathcal{Z}} := \{y \in \mathcal{Y} : yz = zy \quad \text{for all } z \in \mathcal{Z}\}. \tag{3}$$

Our first main theorem is the following.

Theorem A. *Let $\mathscr{L}$, $\mathscr{R}$, $\mathscr{L}_\bullet$, and $\mathscr{R}_\bullet$ be the subalgebras of $\mathrm{End}_{\Bbbk}(\mathscr{P})$ defined above. We identify $\mathscr{PD}$ with $\phi_{PD}(\mathscr{PD}) \subseteq \mathrm{End}_{\Bbbk}(\mathscr{P})$. Then the following statements hold.*

(i) $\mathscr{PD}^{\mathscr{R}_\bullet} = \mathscr{PD}^{\mathscr{R}} = \mathscr{L}_\bullet$. Furthermore, $\mathscr{L}_\bullet$ is generated by the $\mathsf{L}_{i,j}$ for $1 \leqslant i,j \leqslant m$.
(ii) $\mathscr{PD}^{\mathscr{L}_\bullet} = \mathscr{PD}^{\mathscr{L}} = \mathscr{R}_\bullet$. Furthermore, $\mathscr{R}_\bullet$ is generated by the $\mathsf{R}_{i,j}$ for $1 \leqslant i,j \leqslant n$.

Let us elucidate the relation between theorem A and the literature on Howe duality and the FFT in the quantized setting. Quantized analogues of $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-duality have been established in [Zh02] and [NYM93], but these works do not consider the double centralizer property inside a quantized Weyl algebra. To compare our results with those of Lehrer–Zhang–Zhang [LZZ11], we briefly explain their formulation of the FFT for $U_q(\mathfrak{gl}_n)$. In [LZZ11, section 6] the authors define a q-analogue of the algebra $\mathcal{P}(V^{\oplus k} \oplus (V^*)^{\oplus l})$, which they call $\mathcal{A}_{k,l}$ (this algebra tacitly depends on n as well). The algebra $\mathcal{A}_{k,l}$ is isomorphic to a twisted tensor product of $\mathscr{P}_{k\times n}$ and $\mathscr{D}_{l\times n}$, but the twisting is only with respect to the universal R-matrix of $U_q(\mathfrak{gl}_n)$. In particular, in the special case $k = l$ the relations on the generators of $\mathcal{A}_{k,k}$ are not symmetric with respect to their indices. Because of this asymmetry, $\mathcal{A}_{k,k}$ does not appear to be the desired object for proving a double centralizer statement. The twisting that we consider to define $\mathscr{PD}$ uses the universal R-matrices of both U_L and U_R. In addition, unlike $\mathcal{A}_{k,l}$ whose relations are homogeneous, the relation (R6) of $\mathscr{PD}$ is not homogeneous. From this viewpoint, $\mathscr{PD}$ resembles the classical Weyl algebra more than $\mathcal{A}_{k,k}$. We remark that recently, Jakobsen [Ja23] considered an embedding of the quantized enveloping algebras into what is called the Hayashi-Weyl algebra [Ha94]. This is different from our approach, in that the Hayashi-Weyl algebra quantizes the (localization of the) enveloping algebra of the Heisemberg Lie algebra, whereas our algebra quantizes the usual Weyl algebra.

Our second main result (theorem B below) is a new FFT for $U_q(\mathfrak{gl}_n)$, in the spirit of the aforementioned result of [LZZ11], for a family of algebras $\mathscr{A}_{k,l,n}$ where k,l,n are positive integers. The latter algebras generalize $\mathscr{P}\mathscr{D}$ and indeed we have $\mathscr{A}_{m,m,n} = \mathscr{P}\mathscr{D}_{m\times n}$. An explicit presentation of $\mathscr{A}_{k,l,n}$ by generators and relations is given in proposition 3.9.2. To state theorem B, we need some notation. For integers $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant n$ there exists an embedding of associative algebras

$$\boldsymbol{e} = \boldsymbol{e}^{m\times n}_{a\times b} : \mathscr{P}\mathscr{D}_{a\times b} \hookrightarrow \mathscr{P}\mathscr{D}_{m\times n} \tag{4}$$

that is defined as follows. We relabel the generators of $\mathscr{P}\mathscr{D}_{a\times b}$ by setting

$$\tilde{t}_{i,j} := t_{a+1-i,b+1-j} \text{ and } \tilde{\partial}_{i,j} := \partial_{a+1-i,b+1-j} \quad \text{for } 1 \leqslant i \leqslant a,\ 1 \leqslant j \leqslant b. \tag{5}$$

We relabel the generators of $\mathscr{P}\mathscr{D}_{m\times n}$ similarly, with a and b replaced by m and n respectively. The map $\boldsymbol{e}$ of (4) is uniquely determined by the assignments $\boldsymbol{e}(\tilde{t}_{i,j}) := \tilde{t}_{i,j}$ and $\boldsymbol{e}(\tilde{\partial}_{i,j}) := \tilde{\partial}_{i,j}$ (see proposition 3.9.4). Concretely, the map $\boldsymbol{e}$ identifies the $a\times b$ matrices $[t_{i,j}]$ and $[\partial_{i,j}]$ formed by the generators of $\mathscr{P}\mathscr{D}_{a\times b}$ with the intersections of the lowest a rows and the rightmost b columns in the analogous $m\times n$ matrices formed by the generators of $\mathscr{P}\mathscr{D}_{m\times n}$. Now fix integers $k,l,n \geqslant 1$ and set $m := \max\{k,l\}$. We define $\mathscr{A}_{k,l,n}$ to be the subalgebra of $\mathscr{P}\mathscr{D}_{m\times n}$ that is generated by the $\tilde{t}_{i,j}$ and the $\tilde{\partial}_{i',j}$, where $1 \leqslant i \leqslant k$, $1 \leqslant i' \leqslant l$, and $1 \leqslant j \leqslant n$. The U_R-action on $\mathscr{P}\mathscr{D}_{m\times n}$ leaves $\mathscr{A}_{k,l,n}$ invariant and thus $\mathscr{A}_{k,l,n}$ is a U_R-module algebra. The standard degree filtration of $\mathscr{A}_{k,l,n}$ (corresponding to setting $\deg \tilde{t}_{i,j} = \deg \tilde{\partial}_{i',j} = 1$) is U_R-stable, hence the associated graded algebra $\mathrm{gr}(\mathscr{A}_{k,l,n})$ is also a U_R-module algebra. Let ϵ_R be the counit of U_R and denote the subalgebra of U_R-invariants in $\mathscr{A}_{k,l,n}$ by $(\mathscr{A}_{k,l,n})_{(\epsilon_R)}$, that is,

$$(\mathscr{A}_{k,l,n})_{(\epsilon_R)} := \{D \in \mathscr{A}_{k,l,n} : x\cdot D = \epsilon_R(x) D \text{for } x \in U_R\}.$$

We denote the subalgebra of U_R-invariants in $\mathrm{gr}(\mathscr{A}_{k,l,n})$ by $\left(\mathrm{gr}(\mathscr{A}_{k,l,n})\right)_{(\epsilon_R)}$ as well. For $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$ we define elements $\tilde{\mathsf{L}}_{i,j} \in \mathscr{A}_{k,l,n}$ by the formula

$$\tilde{\mathsf{L}}_{i,j} := \sum_{r=1}^{n} \tilde{t}_{i,r}\tilde{\partial}_{j,r} = \sum_{r=1}^{n} t_{m-i+1,r}\partial_{m-j+1,r}. \tag{6}$$

By the same formula we can define analogous elements in $\mathrm{gr}(\mathscr{A}_{k,l,n})$. By a slight abuse of the symbol $\mathrm{gr}(\cdot)$, we denote these elements of $\mathrm{gr}(\mathscr{A}_{k,l,n})$ by $\mathrm{gr}(\tilde{\mathsf{L}}_{i,j})$.

Theorem B. *The algebras* $(\mathscr{A}_{k,l,n})_{(\epsilon_R)}$ *and* $\left(\mathrm{gr}(\mathscr{A}_{k,l,n})\right)_{(\epsilon_R)}$ *are generated by the* $\tilde{\mathsf{L}}_{i,j}$ *and the* $\mathrm{gr}(\tilde{\mathsf{L}}_{i,j})$ *respectively, for* $1 \leqslant i \leqslant k$ *and* $1 \leqslant j \leqslant l$.

It would be interesting to relate theorem B to the quantized FFT of [LZZ11, theorem 6.10] for example by a deformation argument. However, we are unable to establish such a connection.

Our third main theorem (theorem C below) explicitly describes the images in $\mathscr{P}\mathscr{D} = \mathscr{P}\mathscr{D}_{m\times n}$ of the Cartan subalgebras of U_L and U_R. To state theorem C we need to define certain elements of $\mathscr{P}\mathscr{D}$ that are constructed using q-determinants. Let $\mathbf{i} := (i_1,\ldots,i_r)$ and $\mathbf{j} :=$

$(j_1,\dots,j_r)$ be r-tuples of integers satisfying $1 \leqslant i_1 < \cdots < i_r \leqslant m$ and $1 \leqslant j_1 < \cdots < j_r \leqslant n$. Define quantum minors $M_{\mathbf{j}}^{\mathbf{i}} \in \mathscr{P}$ and $\overline{M}_{\mathbf{j}}^{\mathbf{i}} \in \mathscr{D}$ by

$$M_{\mathbf{j}}^{\mathbf{i}} := \sum_{\sigma} (-q)^{\ell(\sigma)} t_{i_{\sigma(1)},j_1} \cdots t_{i_{\sigma(r)},j_r} \quad \text{and} \quad \overline{M}_{\mathbf{j}}^{\mathbf{i}} := \sum_{\sigma} \left(-q^{-1}\right)^{\ell(\sigma)} \partial_{i_{\sigma(1)},j_1} \cdots \partial_{i_{\sigma(r)},j_r}, \tag{7}$$

where the summations are over permutations in r letters, and $\ell(\sigma)$ denotes the length of σ (in the sense of Coxeter groups). For $a,b,r \geqslant 1$ define $\mathbf{D}(r,a,b) \in \mathscr{P}\mathscr{D}_{a\times b}$ by

$$\mathbf{D}(r,a,b) := \sum_{\mathbf{i}} \sum_{\mathbf{j}} M_{\mathbf{j}}^{\mathbf{i}} \overline{M}_{\mathbf{j}}^{\mathbf{i}}, \tag{8}$$

where the summation indices $\mathbf{i} := (i_1,\dots,i_r)$ and $\mathbf{j} := (j_1,\dots,j_r)$ satisfy $1 \leqslant i_1 < \cdots < i_r \leqslant a$ and $1 \leqslant j_1 < \cdots < j_r \leqslant b$. We also set $\mathbf{D}(0,a,b) = 1$. For $0 \leqslant r \leqslant n$ and $1 \leqslant k \leqslant n$ we define $\mathbf{D}_{k,r} \in \mathscr{P}\mathscr{D}_{m\times n}$ by

$$\mathbf{D}_{k,r} := \boldsymbol{e}_{m\times k}^{m\times n}(\mathbf{D}(r,m,k)).$$

Similarly, for $0 \leqslant r \leqslant m$ and $1 \leqslant k \leqslant m$ we define $\mathbf{D}'_{k,r} \in \mathscr{P}\mathscr{D}_{m\times n}$ by

$$\mathbf{D}'_{k,r} := \boldsymbol{e}_{k\times n}^{m\times n}(\mathbf{D}(r,k,n)).$$

Note that $\mathbf{D}_{k,r} = \mathbf{D}'_{k,r} = 0$ when $r > \min\{k,m,n\}$ and $\mathbf{D}_{k,0} = \mathbf{D}'_{k,0} = 1$. Set

$$\mathbf{R}_a := \sum_{r=0}^{a} \left(q^2-1\right)^r \mathbf{D}_{a,r} \text{ for } 1 \leqslant a \leqslant n \quad \text{and} \quad \mathbf{L}_b := \sum_{r=0}^{b} \left(q^2-1\right)^r \mathbf{D}'_{b,r} \text{ for } 1 \leqslant b \leqslant m. \tag{9}$$

Furthermore, let $U_{\mathfrak{h},L}$ and $U_{\mathfrak{h},R}$ denote the Cartan subalgebras of U_L and U_R, respectively (see section 3.1). We set

$$\mathscr{L}_{\mathfrak{h}} := \phi_U(U_{\mathfrak{h},L} \otimes 1), \quad \mathscr{L}_{\mathfrak{h},\bullet} := \mathscr{P}\mathscr{D} \cap \mathscr{L}_{\mathfrak{h}}, \quad \mathscr{R}_{\mathfrak{h}} := \phi_U(1 \otimes U_{\mathfrak{h},R}),$$
$$\mathscr{R}_{\mathfrak{h},\bullet} := \mathscr{P}\mathscr{D} \cap \mathscr{R}_{\mathfrak{h}}. \tag{10}$$

Theorem C. *The following statements hold.*

(i) $\mathscr{R}_{\mathfrak{h},\bullet}$ *is generated by* $\mathbf{R}_1,\dots,\mathbf{R}_n$.
(ii) $\mathscr{L}_{\mathfrak{h},\bullet}$ *is generated by* $\mathbf{L}_1,\dots,\mathbf{L}_m$.

In the proofs of our theorems we borrow at least two key ideas from [LZZ11]. First, we use the bialgebra structure of $\mathscr{P}_{n\times n}$ to define a map $\Gamma_{k,l,n}$ from $\mathscr{P}_{k\times l}$ onto the subalgebra of U_R-invariants in $\mathrm{gr}(\mathscr{A}_{k,l,n})$ (assuming $n \geqslant \max\{k,l\}$). The map $\Gamma_{k,l,n}$, given in definition 7.1.3, is similar to the map introduced in [LZZ11, lemma 6.11]. Second, we define a new product on $\mathscr{P}_{k\times l}$ such that the map $\Gamma_{k,l,n}$ becomes an isomorphism of algebras (see definition 8.2.4). This product is analogous to the one defined in [LZZ11, lemma 6.13]. However our product is given by a more complicated (and asymmetric) formula, because it needs to be simutaneously compatible with *two* universal R-matrices. As a consequence, establishing the desired properties of this product requires new ideas (see section 8). Because of this, and the fact that unlike [LZZ11] the generators $\mathsf{L}_{i,j}$ (respectively, $\mathsf{R}_{i,j}$) or even their graded analogues are not weight vectors for the Cartan subalgebras of the two copies of U_L (respectively, U_R) that act on $\mathscr{P}\mathscr{D}$, the proofs of theorems A and B become substantially more complicated than the analogous results of [LZZ11]. See section 9.2 for more details.

The results of this paper were obtained as part of a project on Capelli operators for quantum symmetric spaces. From this standpoint, it is natural to ask if one can define quantized Weyl algebras in the latter setting and then realize the action of (a large subalgebra of) the quantized enveloping algebra via elements of this Weyl algebra. We address this question and its connection to Capelli operators in upcoming work [LSS22a, LSS22b].

The structure of this paper is as follows. In section 2 we review the required background material on Hopf algebras and twisted tensor products. In section 3 we construct and study the quantized Weyl algebra $\mathscr{P}\mathscr{D}$ and its variations, namely $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$, $\mathscr{A}_{k,l,n}$ and $\mathscr{A}_{k,l,n}^{\mathrm{gr}}$. The main goal of section 4 is to prove that under ϕ_U the elements $K_{\lambda_{L,a}} \otimes 1$ and $1 \otimes K_{\lambda_{R,b}}$, defined in (75), of the Cartan subalgebra of U_{LR} are mapped into $\mathscr{P}\mathscr{D}$. In section 5 we compute explicit formulas for $\phi_U(K_{\lambda_{L,a}} \otimes 1)$ and $\phi_U(1 \otimes K_{\lambda_{R,b}})$. Section 6 is devoted to some general properties of the polarization operators $\mathsf{L}_{i,j}$, $\mathsf{R}_{i,j}$ and their variants. In section 7 we define the map $\Gamma_{k,l,n}$ and establish some of its properties. Section 8 defines the new product on $\mathscr{P}_{k\times l}$ and establishes its properties. Theorems A and B are proved in section 9 and theorem C is proved in section 10. Finally, appendix lists the commonly used notation in the paper.

2. Hopf algebras and deformed twisted tensor products

Throughout this section $\mathbb{K}$ will denote an arbitrary field and H will be a Hopf algebra over $\mathbb{K}$. We denote the coproduct, counit, and antipode of H by Δ, ϵ, and S. The opposite and co-opposite of H are denoted by H^{op} and H^{cop} (we use the same notation for bialgebras as well). Throughout the paper, our notation for specific Hopf algebras H will remain consistent with these choices.

If A is an associative algebra and V is an A-module, a subspace $W \subseteq V$ is called A-*stable* if $A \cdot W \subseteq W$. Finally, an associative algebra A is called an H-*module algebra* if it is equipped by an H-module structure such that the product of A yields an H-module homomorphism $A \otimes A \to A$.

2.1. Local finiteness modulo an ideal

Given a two-sided ideal I of H (considered as an associative algebra), we set $E(x,I) := \{\mathrm{ad}_y(x) + I : y \in H\}$ for $x \in H$, where $\mathrm{ad}_y(x) := \sum y_1 x S(y_2)$ is the left adjoint action of H (we use the Sweedler notation $\Delta(y) = \sum y_1 \otimes y_2$ for the coproduct). We set

$$\mathscr{F}(H,I) := \{x \in H : \dim_{\mathbb{K}} E(x,I) < \infty\}. \tag{11}$$

For $I = 0$ this is the locally finite part of H (in the sense of [JL94]), which we will denote by $\mathscr{F}(H)$. We have $E(xx',I) \subseteq E(x,I)E(x',I)$ for $x,x' \in H$. Consequently, $\mathscr{F}(H,I)$ is a subalgebra of H.

2.2. The finite dual of H

Given a finite dimensional left H-module V, by the *right dual* of V we mean the dual space V^* equipped with the H-action defined by $\langle x \cdot v^*, v\rangle := \langle v^*, S^{-1}(x) \cdot v\rangle$ for $v^* \in V^*$ and $v \in V$, where $\langle \cdot,\cdot\rangle : V^* \otimes V \to \mathbb{K}$ is the canonical pairing. The *matrix coefficients* of V are the linear functionals $\mathsf{m}_{v^*,v} \in H^*$ defined by

$$\mathsf{m}_{v^*,v}(x) := \langle v^*, x \cdot v\rangle \quad \text{for } x \in H,\ v \in V,\ v^* \in V^*.$$

Indeed $\mathsf{m}_{v^*,v} \in H^\circ$, where H° denotes the finite dual of H (for the definition of H° see [KS97, section 1.2.8]). Recall that H° has a canonical Hopf algebra structure. The product of H° is given by

$$\lambda\mu(x) := \sum \lambda(x_1)\mu(x_2) \quad \text{for } \lambda,\mu \in H^\circ \text{and } x \in H, \tag{12}$$

where $\Delta(x) = \sum x_1 \otimes x_2$. Given two finite dimensional H-modules V and W, we have

$$\mathsf{m}_{v^*\otimes w^*, v\otimes w} = \mathsf{m}_{v^*,v}\mathsf{m}_{w^*,w} \quad \text{for } v,w \in V\text{and } v^*, w^* \in W.$$

Let Δ° denote the coproduct of H°, so that $\Delta^\circ(\lambda) = \sum \lambda_1 \otimes \lambda_2$ for $\lambda \in H^\circ$, where $\sum \lambda_1 \otimes \lambda_2$ is uniquely determined by

$$\langle \lambda, xy\rangle = \sum \langle \lambda_1, x\rangle \langle \lambda_2, y\rangle \quad \text{for } x,y \in H. \tag{13}$$

If $\{v_i\}_{i=1}^d$ is a basis of V and $\{v_i^*\}_{i=1}^d$ is the dual basis of V^*, then $\Delta^\circ(\mathsf{m}_{v^*,v}) = \sum_{i=1}^d \mathsf{m}_{v^*,v_i} \otimes \mathsf{m}_{v_i^*,v}$. The following remark will be used in section 3.

Remark 2.2.1. Let $H^\bullet \subseteq H^\circ$ be a sub-bialgebra. Then $H^\bullet$ is an H-module algebra with respect to *right translation*, where the action is defined by $\langle x\cdot\lambda, y\rangle := \langle \lambda, yx\rangle$ for $\lambda \in H^\bullet$ and $x,y \in H$. If H is equipped with a $\mathbb{K}$-linear map $x \mapsto x^\natural$ that yields an isomorphism of Hopf algebras $H \to H^{\mathrm{op}}$, then $H^\bullet$ has another H-module algebra structure defined by $\langle x\cdot\lambda, y\rangle := \langle \lambda, x^\natural y\rangle$, which we call *left translation*. Given any homomorphism of associative algebras $\tau : H \to H$, we can define τ-twisted left and right translation actions of H on $H^\bullet$, given respectively by the formulas

$$\langle x\cdot\lambda, y\rangle := \langle \lambda, \tau(x)^\natural y\rangle \quad \text{and} \quad \langle x\cdot\lambda, y\rangle := \langle \lambda, y\tau(x)\rangle.$$

When $H^\bullet$ is equipped with either one of the two τ-twisted actions, the following statements hold.

(i) If $\tau : H \to H$ is a homomorphism of coalgebras, then $H^\bullet$ is an H-module algebra.
(ii) If $\tau : H \to H$ is an anti-homomorphism of coalgebras, then $H^\bullet$ is an H^{cop}-module algebra.

2.3. The isotypic component of the trivial H-module

For any H-module V we set

$$V_{(\epsilon)} := \{v \in V : h\cdot v = \epsilon(h)v\},$$

where as before ϵ denotes the counit of H. Let $\psi : H \to \mathrm{End}_{\mathbb{K}}(V)$ be the algebra homomorphism corresponding to this module structure (hence $h\cdot v = \psi(h)v$ for $h \in H$ and $v \in V$). We equip $\mathrm{End}_{\mathbb{K}}(V)$ with an H-module structure, defined by $h\cdot T := \sum \psi(h_1)T\psi(S(h_2))$ for $h \in H$ and $T \in \mathrm{End}_{\mathbb{K}}(V)$, where $\Delta(h) = \sum h_1 \otimes h_2$.

Lemma 2.3.1. $\mathrm{End}_{\mathbb{K}}(V)_{(\epsilon)} = \mathrm{End}_{\mathbb{K}}(V)^{\psi(H)}$*, where the right hand side is defined as in (3).*

Proof. The inclusion $\supseteq$ follows from

$$h\cdot T = \sum \psi(h_1)T\psi(S(h_2)) = T\sum \psi(h_1)\psi(S(h_2)) = T\psi\left(\sum h_1 S(h_2)\right) = \epsilon(h)T,$$

for $T \in \mathrm{End}_{\mathbb{K}}(V)^{\psi(H)}$. For the inclusion $\subseteq$ note that if $T \in \mathrm{End}_{\mathbb{K}}(V)_{(\epsilon)}$ then

$$\psi(h)T = \sum \psi(h_1)\epsilon(h_2)T = \sum \psi(h_1)T\psi(S(h_2))\psi(h_3)$$
$$= \sum \epsilon(h_1)T\psi(h_2) = T\psi\left(\sum \epsilon(h_1)h_2\right) = T\psi(h)\,.$$

□

2.4. Braided triples, twisted tensor products and their deformations

Let $\mathcal{C}$ be a full subcategory of the category of H-modules that is closed under direct sums and tensor products. To ensure that $\mathcal{C}$ is a monoidal category we assume that the trivial H-module (the one-dimensional vector space $\mathbb{K}$ equipped with the action $h \mapsto \epsilon(h)$ for $h \in H$) belongs to $\mathrm{Obj}(\mathcal{C})$.

Assume that $\mathcal{C}$ is *braided*. The braiding $\check{R}$ of $\mathcal{C}$ is a natural family of H-module isomorphisms

$$\check{R}_{V,W} : V \otimes W \to W \otimes V \quad \text{for } V, W \in \mathrm{Obj}(\mathcal{C})$$

that satisfies the usual hexagon axioms (see for example [EGNO15, definition 8.1.1]). Henceforth we call $(H, \mathcal{C}, \check{R})$ a *braided triple*.

Let A and B be two H-module algebras, with products m_A and m_B. Assume that $A, B \in \mathrm{Obj}(\mathcal{C})$.

Lemma 2.4.1. *The map $\check{R}_{B,A} : B \otimes A \to A \otimes B$ satisfies the following relations:*

(i) $\check{R}_{B,A}(1 \otimes a) = a \otimes 1$ *for* $a \in A$ *and* $\check{R}_{B,A}(b \otimes 1) = 1 \otimes b$ *for* $b \in B$.
(ii) $\check{R}_{B,A} \circ (\mathrm{id}_B \otimes m_A) = (m_A \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \check{R}_{B,A})(\check{R}_{B,A} \otimes \mathrm{id}_A)$.
(iii) $\check{R}_{B,A} \circ (m_B \otimes \mathrm{id}_A) = (\mathrm{id}_A \otimes m_B)(\check{R}_{B,A} \otimes \mathrm{id}_B)(\mathrm{id}_B \otimes \check{R}_{B,A})$.

Proof. This is well known, but we supply a proof because we did not find a reference.

(i) Equip $\mathbb{K}$ with the canonical H-module structure induced by the counit $\epsilon : H \to \mathbb{K}$. It is well known (see for example [EGNO15, exercise 8.1.6]) that one has a commuting triangle

$$\begin{tikzcd}
\mathbb{K} \otimes A \arrow[rr, "\check{R}_{\mathbb{K},A}"] \arrow[dr] & & A \otimes \mathbb{K} \arrow[dl] \\
& A &
\end{tikzcd}$$

where the lower sides of the triangle are the canonical left and right unit isomorphisms. It follows immediately that $\check{R}_{\mathbb{K},A}(1 \otimes a) = a \otimes 1$. Furthermore, by naturality of $\check{R}$ the diagram

$$\begin{tikzcd}
\mathbb{K} \otimes A \arrow[r, "\check{R}_{\mathbb{K},A}"] \arrow[d, "1_B \otimes \mathrm{id}_A"'] & A \otimes \mathbb{K} \arrow[d, "\mathrm{id}_A \otimes 1_B"] \\
B \otimes A \arrow[r, "\check{R}_{B,A}"] & A \otimes B
\end{tikzcd}$$

is commutative, hence $\check{R}_{B,A}(1 \otimes a) = a \otimes 1$. The other relation is proved similarly.

(ii) Consider the diagram of maps below:

$$\begin{tikzcd}
A \otimes B \otimes A \arrow[dr, "\mathrm{id}_A \otimes \check{R}_{B,A}"'] & B \otimes A \otimes A \arrow[l, "\check{R}_{B,A} \otimes \mathrm{id}_A"'] \arrow[r, "\mathrm{id}_B \otimes m_A"] \arrow[d, "\check{R}_{B,A\otimes A}"] & B \otimes A \arrow[d, "\check{R}_{B,A}"] \\
 & A \otimes A \otimes B \arrow[r, "m_A \otimes \mathrm{id}_B"'] & A \otimes B
\end{tikzcd}$$

The square is commutative by naturality of $\check{R}$ (because m_A and m_B are H-module homomorphisms) and the triangle is commutative by the hexagon axiom. The assertion of (ii) follows from comparing two maps $B \otimes A \otimes A \to A \otimes B$ in the diagram: one is obtained by composition of the top and the right edges of the square, the other is obtained by the outer edges of the triangle and the bottom edge of the square.

(iii) Similar to (ii). □

Definition 2.4.2. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ be H-module algebras. We denote the products of A and B by m_A and m_B, respectively. The $\check{R}$*-twisted tensor product* of A and B, denoted by $A \otimes_{\check{R}} B$, is the vector space $A \otimes B$ equipped with the binary operation

$$(m_A \otimes m_B) \circ \left(\mathrm{id}_A \otimes \check{R}_{B,A} \otimes \mathrm{id}_B\right). \tag{14}$$

Proposition 2.4.3. *$A \otimes_{\check{R}} B$ is an associative algebra.*

Proof. This is well known and proved for example in [VV94, proposition 2.2]. □

From lemma 2.4.1(i) it follows that for $a, a' \in A$ and $b, b' \in B$ the product (14) satisfies

$$(a \otimes 1)(a' \otimes b)(1 \otimes b') = (aa' \otimes bb'). \tag{15}$$

Remark 2.4.4. The vector space $A \otimes_{\check{R}} B = A \otimes B$ carries two module structures: as the outer tensor product over $\mathbb{K}$ of H-modules A and B, it is an $H \otimes H$-module. As the inner tensor product of A and B, it is an H-module. Of course the latter H-module structure is obtained from the former one via restriction along the coproduct map $H \to H \otimes H$.

Proposition 2.4.5. *$A \otimes_{\check{R}} B$ is an H-module algebra with the H-module structure of remark 2.4.4.*

Proof. Given $a, a' \in A$ and $b, b' \in B$, if we write $\check{R}_{B,A}(b \otimes a') = \sum a'' \otimes b''$ then for $x \in H$ we have

$$x \cdot ((a \otimes b)(a' \otimes b')) = x \cdot \sum aa'' \otimes b''b = \sum (x_1 \cdot (aa'')) \otimes (x_2 \cdot (b''b)).$$

Since A and B are H-module algebras, from the latter equalities and (15) it follows that

$$\begin{aligned} x \cdot ((a \otimes b)(a' \otimes b')) &= \sum (x_1 \cdot a)(x_2 \cdot a'') \otimes (x_3 \cdot b'')(x_4 \cdot b) \\ &= \sum ((x_1 \cdot a) \otimes 1)((x_2 \cdot a'') \otimes (x_3 \cdot b''))(1 \otimes (x_4 \cdot b)). \end{aligned}$$

Since $\check{R}_{A,B}$ is an H-module isomorphism, from the latter equalities and (15) we have

$$\begin{aligned} x\cdot((a\otimes b)(a'\otimes b')) &= \sum((x_1\cdot a)\otimes 1)\left(\check{R}_{B,A}((x_2\cdot b)\otimes(x_3\cdot a'))\right)(1\otimes(x_4\cdot b')) \\ &= \sum((x_1\cdot a)\otimes(x_2\cdot b))((x_3\cdot a')\otimes(x_4\cdot b')) \\ &= \sum(x_1\cdot(a\otimes b))(x_2\cdot(a'\otimes b')). \end{aligned}$$

Thus $A\otimes_{\check{R}} B$ is an H-module algebra. □

Let $E_A\subseteq A$ and $E_B\subseteq B$ be subspaces that generate A and B, respectively. Thus $A\cong T(E_A)/I_A$ and $B\cong T(E_B)/I_B$, where $T(X)$ denotes the tensor algebra on X, and I_A and I_B denote the corresponding ideals of relations. We assume that E_A and E_B are H-stable.

Remark 2.4.6. The H-module structures on E_A and E_B equip $T(E_A)$ and $T(E_B)$ with canonical H-module algebra structures. It is straightforward to verify that the maps $T(E_A)\to A$ and $T(E_B)\to B$ are H-module homomorphisms. In particular, I_A and I_B are H-stable subspaces of $T(E_A)$ and $T(E_B)$, respectively.

Consider the linear map

$$\gamma_{A,B}: E_A\otimes E_B\to T(E_A\oplus E_B)\ ,\ \gamma_{A,B}(a\otimes b):=ab.$$

Note that we can express $\gamma_{A,B}$ as the composition

$$E_A\otimes E_B\xrightarrow{\mathsf{i}_A\otimes\mathsf{i}_B} T(E_A\oplus E_B)\otimes T(E_A\oplus E_B)\xrightarrow{a\otimes b\mapsto ab} T(E_A\oplus E_B), \tag{16}$$

where $\mathsf{i}_A: E_A\to T(E_A\oplus E_B)$ and $\mathsf{i}_B: E_B\to T(E_A\oplus E_B)$ are canonical embeddings. For the next lemma, recall that the H-module structure on $E_A\oplus E_B$ induces a canonical H-module algebra structure on $T(E_A\oplus E_B)$.

Lemma 2.4.7. *$\gamma_{A,B}$ is an H-module homomorphism.*

Proof. Since i_A, i_B and the product of $T(E_A\oplus E_B)$ are H-module homomorphisms, the assertion follows from the description of $\gamma_{A,B}$ in (16). □

By the universal property of tensor algebras the map $E_A\oplus E_B\to A\otimes_{\check{R}} B$ given by the assignment $a\oplus b\mapsto a\otimes 1+1\otimes b$ induces a homomorphism of algebras

$$\pi: T(E_A\oplus E_B)\to A\otimes_{\check{R}} B.$$

Proposition 2.4.8. *Let A, B, E_A and E_B be as above. Then π induces an isomorphism of algebras*

$$T(E_A\oplus E_B)/I_{A,B}\cong A\otimes_{\check{R}} B,$$

where $I_{A,B}$ denotes the two-sided ideal of $T(E_A\oplus E_B)$ generated by I_A, I_B and the relations

$$ba-\gamma_{A,B}\circ\check{R}_{B,A}(b\otimes a)\quad \textit{for } a\in E_A,\ b\in E_B. \tag{17}$$

Proof. We have $\pi(E_A\oplus E_B)=E_{A,B}$ where $E_{A,B}:=(E_A\otimes 1)\oplus(1\otimes E_B)$. Furthermore, $A\otimes_{\check{R}} B$ is generated as an algebra by $E_{A,B}$. Thus π is a surjection. Next we prove that π is an injection. From the definition of the product of $A\otimes_{\check{R}} B$ it follows that $I_{A,B}\subseteq\ker\pi$. Thus, to complete the proof it suffices to verify the reverse inclusion.

By the relations (17), every element of $T(E_A \oplus E_B)/I_{A,B}$ can be expressed as a linear combination of products of elements of E_A and E_B in which elements of E_A occur before elements of E_B. Thus,

$$T(E_A \oplus E_B) = T(E_A)T(E_B) + I_{A,B}. \tag{18}$$

Now choose $\{a_\alpha\}_{\alpha\in\mathcal{I}_A} \subseteq T(E_A)$ and $\{b_\beta\}_{\beta\in\mathcal{I}_B} \subseteq T(E_B)$ such that $\{I_A + a_\alpha\}_{\alpha\in\mathcal{I}_A}$ is a basis of $T(E_A)/I_A \cong A$ and $\{I_B + b_\beta\}_{\beta\in\mathcal{I}_B}$ is a basis of $T(E_B)/I_B \cong B$. From (18) it follows that the elements $I_{A,B} + a_\alpha b_\beta$ for $\alpha \in I_A$ and $\beta \in I_B$ constitute a spanning set of $T(E_A \oplus E_B)/I_{A,B}$.

From (15) it follows that $\pi(a_\alpha) = \overline{a}_\alpha \otimes 1$ for some $\overline{a}_\alpha \in A$ and $\pi(b_\beta) = 1 \otimes \overline{b}_\beta$ for some $\overline{b}_\beta \in B$. Also, $\pi(I_{A,B} + a_\alpha b_\beta) = \overline{a}_\alpha \otimes \overline{b}_\beta$. The sets $\{\overline{a}_\alpha\}_{\alpha\in\mathcal{I}_A}$ and $\{\overline{b}_\beta\}_{\beta\in\mathcal{I}_B}$ are bases of A and B, respectively. Thus π maps a spanning set of $T(E_A \oplus E_B)/I_{A,B}$ bijectively onto the basis $\{a_\alpha \otimes b_\beta : \alpha \in \mathcal{I}_A, \beta \in \mathcal{I}_B\}$ of $A \otimes_{\check{R}} B$. It follows that $\ker\pi \subseteq I_{A,B}$.

□

Let $\psi : E_B \times E_A \to \mathbb{K}$ be an H-invariant bilinear form, that is

$$\sum \psi(x_1 \cdot b, x_2 \cdot a) = \epsilon(x)\psi(b,a) \quad \text{for } a \in E_A,\ b \in E_B,\ x \in H, \tag{19}$$

where as before $\epsilon : H \to \mathbb{K}$ denotes the counit of H. Let $I_{A,B,\psi}$ denote the two-sided ideal of $T(E_A \oplus E_B)$ that is generated by I_A, I_B, and relations of the form

$$ba - \gamma_{A,B} \circ \check{R}_{B,A}(b \otimes a) - \psi(b,a) \quad \text{for } a \in E_A,\ b \in E_B. \tag{20}$$

Definition 2.4.9. We call the algebra $A \otimes_{\check{R},\psi} B := T(E_A \oplus E_B)/I_{A,B,\psi}$ the *ψ-deformed $\check{R}$-twisted tensor product* of A and B relative to E_A and E_B.

The H-module structure of $E_A \oplus E_B$ equips $T(E_A \oplus E_B)$ with a canonical H-module algebra structure. We have the following statement.

Proposition 2.4.10. *The canonical H-module algebra structure on $T(E_A \oplus E_B)$ descends to an H-module algebra structure on $A \otimes_{\check{R},\psi} B$.*

Proof. It suffices to verify that $I_{A,B,\psi}$ is an H-stable subspace. By remark 2.4.6 the subspaces I_A and I_B of $T(E_A \oplus E_B)$ are H-stable. Thus it remains to show that the relations (20) span an H-submodule of $T(E_A \oplus E_B)$.

Any $x \in H$ acts on $T^0(E_A \oplus E_B) \cong \mathbb{K}$ by $\epsilon(x)$. Next let $a \in E_A$, $b \in E_B$ and $x \in H$. By (19) and lemma 2.4.7 we have

$$\begin{aligned} &x \cdot \big(ba - \gamma_{A,B} \circ \check{R}_{B,A}(b \otimes a) - \psi(b,a)\big) \\ &\quad = \sum \big((x_1 \cdot b)(x_2 \cdot a) - \gamma_{A,B} \circ \check{R}_{B,A}(x_1 \cdot b \otimes x_2 \cdot a) - \psi(x_1 \cdot b, x_2 \cdot a)\big), \end{aligned}$$

which is a sum of relations of the form (20). □

By the universal property of tensor algebras the maps i_A and i_B induce embeddings of associative algebras $\mathsf{i}_A : T(E_A) \to T(E_A \oplus E_B)$ and $\mathsf{i}_B : T(E_B) \to T(E_A \oplus E_B)$. The latter maps induce H-equivariant homomorphisms of associative algebras

$$A \cong T(E_A)/I_A \xrightarrow{\bar{\mathsf{i}}_A} A \otimes_{\check{R},\psi} B \quad \text{and} \quad B \cong T(E_B)/I_B \xrightarrow{\bar{\mathsf{i}}_B} A \otimes_{\check{R},\psi} B.$$

By tensoring the latter maps and then composing with the products of the algebras $T(E_A \oplus E_B)$ and $A \otimes_{\check{R},\psi} B$ we obtain the following commutative diagram:

$$\begin{tikzcd} T(E_A)\otimes T(E_B) \arrow[r, "\mathrm{i}_A\otimes \mathrm{i}_B"] \arrow[d] & T(E_A\oplus E_B)\otimes T(E_A\oplus E_B) \arrow[r, "a\otimes b\mapsto ab"] \arrow[d] & T(E_A\oplus E_B) \arrow[d] \\ A\otimes B \arrow[r, "\bar{\mathrm{i}}_A\otimes \bar{\mathrm{i}}_B"] & (A\otimes_{\check{R},\psi} B)\otimes (A\otimes_{\check{R},\psi} B) \arrow[r, "a\otimes b\mapsto ab"] & A\otimes_{\check{R},\psi} B \end{tikzcd} \tag{21}$$

In the above diagram the vertical maps are the canonical quotients.

Remark 2.4.11. From remark 2.4.6 it follows that $\mathrm{i}_A\otimes \mathrm{i}_B$ and $\bar{\mathrm{i}}_A\otimes\bar{\mathrm{i}}_B$ are $H\otimes H$-module homomorphisms. Also, the products of $T(E_A\oplus E_B)$ and $A\otimes_{\check{R},\psi} B$ are H-module homomorphisms (see proposition 2.4.10). Thus the composition of the bottom horizontal maps in (21), which is given by

$$A\otimes B\to A\otimes_{\check{R},\psi} B\ ,\ a\otimes b\mapsto I_{A,B,\psi}+ab, \tag{22}$$

is an H-module homomorphism (recall that an $H\otimes H$-module homomorphism is also an H-module homomorphism by restriction along the coproduct map $H\to H\otimes H$).

2.5. Locally finite braided triples and their products

Given a braided triple $(H,\mathcal{C},\check{R})$ we set

$$R_{V,W} := \sigma_{W,V}\circ \check{R}_{V,W}, \tag{23}$$

where

$$\sigma_{W,V}: W\otimes V\to V\otimes W \quad , \quad w\otimes v\mapsto v\otimes w$$

is the standard flip map.

Definition 2.5.1. We say a braided triple $(H,\mathcal{C},\check{R})$ is *locally finite* if it satisfies the following conditions.

(i) Every $V\in \mathrm{Obj}(\mathcal{C})$ is a sum of its finite dimensional submodules that belong to $\mathrm{Obj}(\mathcal{C})$.
(ii) For finite dimensional modules $V,W\in\mathrm{Obj}(\mathcal{C})$ there exist $\omega_{V,W},\overline{\omega}_{V,W}\in H\otimes H$ such that

$$R_{V,W}(v\otimes w)=\omega_{V,W}\cdot(v\otimes w)\quad \text{and}\quad R^{-1}_{V,W}(v\otimes w)=\overline{\omega}_{V,W}\cdot(v\otimes w)\ \text{ for } v\otimes w\in V\otimes W.$$

Let us briefly explain the idea behind definition 2.5.1. The braiding of the category of modules of the Hopf algebra $U_q(\mathfrak{gl}_n)$ is given by a formal series that does not belong to $U_q(\mathfrak{gl}_n)\otimes U_q(\mathfrak{gl}_n)$ (see section 3.2). Thus, in order to give a rigorous proof of proposition 2.6.2 below, we need to be able to replace this formal series locally by a finite 2-tensor.

Lemma 2.5.2. *Let $(H,\mathcal{C},\check{R})$ be a locally finite braided triple and let $V_1,\dots,V_r,W_1,\dots,W_r\in$ $\mathrm{Obj}(\mathcal{C})$ be finite dimensional modules. Let $\omega_{V,W},\overline{\omega}_{V,W}\in H\otimes H$ be chosen as in definition 2.5.1(ii), where $V:=\bigoplus_{i=1}^r V_i$ and $W:=\bigoplus_{i=1}^r W_i$. Then*

$$R_{V_i,W_j}(v_i\otimes w_j)=\omega_{V,W}\cdot(v_i\otimes w_j)\quad \textit{and}\quad R^{-1}_{V_i,W_j}(v_i\otimes w_j)=\overline{\omega}_{V,W}\cdot(v_i\otimes w_j),$$

for all $v_i\in V_i$ and $w_j\in W_j$ where $1\leqslant i,j\leqslant r$.

Proof. This follows from naturality of $\check{R}$ with respect to the canonical maps $V_i\otimes W_j\hookrightarrow V\otimes W$. $\square$

We need to work with the braided triples associated to tensor products of Hopf algebras $U_q(\mathfrak{gl}_n)$. To this end, we need the following definition.

Definition 2.5.3. Let $(H,\mathcal{C},\check{R})$ and $(H',\mathcal{C}',\check{R}')$ be locally finite braided triples. Set $H'':= H\otimes H'$ and equip H'' with the canonical tensor product Hopf algebra structure. Furthermore, let $\mathcal{C}''$ be the full subcategory of the category of H''-modules whose objects are direct sums of modules of the form $V\otimes V'$ where $V\in\mathrm{Obj}(\mathcal{C})$ and $V'\in\mathrm{Obj}(\mathcal{C}')$. We define a braiding on $\mathcal{C}''$ as follows. First, for $V,W\in\mathrm{Obj}(\mathcal{C})$ and $V',W'\in\mathrm{Obj}(\mathcal{C}')$ we set

$$\check{R}''_{V\otimes V',W\otimes W'} := \left(\check{R}_{V,W}\right)_{13}\circ\left(\check{R}'_{V',W'}\right)_{24}. \tag{24}$$

Here $\left(\check{R}'_{V',W'}\right)_{24}$ means that $\check{R}'_{V',W'}$ acts on the 2[nd] and the 4[th] components of $(V\otimes V')\otimes(W\otimes W')$, resulting in a map

$$(V\otimes V')\otimes(W\otimes W')\to(V\otimes W')\otimes(W\otimes V'),$$

and $\left(\check{R}_{V,W}\right)_{13}$ is defined analogously. Next for $\widetilde{V}:=\bigoplus_i V_i\otimes V'_i$ and $\widetilde{W}:=\bigoplus_j W_j\otimes W'_j$ we define

$$\check{R}''_{\widetilde{V},\widetilde{W}} := \bigoplus_{i,j}\check{R}''_{V_i\otimes V'_i,W_j\otimes W'_j}.$$

Proposition 2.5.4. *Let H'', $\mathcal{C}''$ and $\check{R}''$ be as in definition 2.5.3. Then $(H'',\mathcal{C}'',\check{R}'')$ is a locally finite braided triple.*

Proof. It is trivial to check that $(H'',\mathcal{C}'',\check{R}'')$ meets the condition of definition 2.5.1(i). Next we show that $(H'',\mathcal{C}'',\check{R}'')$ is a braided triple. It is straightforward to check that $\mathcal{C}''$ is closed with respect to tensor products and arbitrary direct sums. The hexagon axioms for $\check{R}''$ follow from those for $\check{R}$ and $\check{R}'$. Next we verify naturality of $\check{R}''$. Since $\check{R}''$ is defined by expansion on direct sums of modules, it suffices to prove commutativity of the diagram

$$\begin{tikzcd}
(V\otimes V')\otimes(W\otimes W') \arrow[r, "\check{R}_{V\otimes V',W\otimes W'}"] \arrow[d, "f_V\otimes f_W"'] & (W\otimes W')\otimes(V\otimes V') \arrow[d, "f_W\otimes f_V"] \\
(\underline{V}\otimes\underline{V}')\otimes(\underline{W}\otimes\underline{W}') \arrow[r, "\check{R}_{\underline{V}\otimes\underline{V}',\underline{W}\otimes\underline{W}'}"'] & (\underline{W}\otimes\underline{W}')\otimes(\underline{V}\otimes\underline{V}')
\end{tikzcd}$$

for all choices of $V,W,\underline{V},\underline{W}\in\mathrm{Obj}(\mathcal{C})$, $V',W',\underline{V}',\underline{W}'\in\mathrm{Obj}(\mathcal{C}')$, $f_V\in\mathrm{Mor}_{\mathcal{C}''}(V\otimes V',\underline{V}\otimes\underline{V}')$ and $f_W\in\mathrm{Mor}_{\mathcal{C}''}(W\otimes W',\underline{W}\otimes\underline{W}')$. The subtlety here is that we cannot assume that f_V and f_W can be decomposed into tensor products of maps on the tensor components. Since $\mathcal{C}''$ satisfies the condition of definition 2.5.1(i), using naturality of $\check{R}$ and $\check{R}'$ we can assume that all of the modules in the commutative diagram are finite dimensional. Using lemma 2.5.2 for the triples $(H,\mathcal{C},\check{R})$ and $(H',\mathcal{C}',\check{R}')$ it follows that $R_{V\otimes V',W\otimes W'}$ and $R_{\underline{V}\otimes\underline{V}',\underline{W}\otimes\underline{W}'}$ are given by the left action of the same 2-tensor in $H''\otimes H''$. Commutativity of the diagram follows immediately.

Finally, we show that the condition of definition 2.5.1(ii) holds. If $R_{V,W}$ and $R'_{V',W'}$ are equal to the actions of $\omega_{V,W}\in H\otimes H$ and $\omega'_{V',W'}\in H'\otimes H'$ respectively, then $R''_{V\otimes V',W\otimes W'}$ is equal to the action of

$$(\omega_{V,W})_{13}\left(\omega'_{V',W'}\right)_{24}\in H''\otimes H''.$$

This verifies definition 2.5.1(ii) for pairs of H''-modules in $\mathcal{C}''$ of the form $V\otimes V'$ and $W\otimes W'$. Using lemma 2.5.2 for the triples $(H,\mathcal{C},\check{R})$ and $(H',\mathcal{C}',\check{R}')$, the claim follows for general H''-modules in $\mathcal{C}''$. A similar argument can be given for $\left(R''_{V\otimes V',W\otimes W'}\right)^{-1}$. □

2.6. A result on H''-stable subalgebras of $A\otimes_{\check{R}''}B$

We continue with the notation of definition 2.5.3. Let A and B be two H''-module algebras such that $A,B\in\mathrm{Obj}(\mathcal{C}'')$. Recall from remark 2.4.4 and proposition 2.4.5 that $A\otimes_{\check{R}''}B$ is an H''-module algebra and also an $H''\otimes H''$-module (but not necessarily an $H''\otimes H''$-module algebra). For $\omega\in H\otimes H$ let

$$T_\omega : A\otimes_{\check{R}''}B\to A\otimes_{\check{R}''}B$$

denote the linear endomorphism obtained by the action of ω_{13}. Here by definition $(x\otimes y)_{13} := x\otimes 1\otimes y\otimes 1$ for $x,y\in H$.

Lemma 2.6.1. *Let $(H,\mathcal{C},\check{R})$ and $(H',\mathcal{C}',\check{R}')$ be locally finite braided triples and let $(H'',\mathcal{C}'',\check{R}'')$ be defined as in definition 2.5.3. Let $A,B\in\mathrm{Obj}(\mathcal{C}'')$ be H''-module algebras. Let $V''_A\subseteq A$ and $V''_B\subseteq B$ be finite dimensional H''-submodules such that $V''_A,V''_B\in\mathrm{Obj}(\mathcal{C}'')$. We decompose V''_A and V''_B as*

$$V''_A=\bigoplus_{i\in\Omega_A}V_{i,A}\otimes V'_{i,A}\quad \textit{and}\quad V''_B=\bigoplus_{j\in\Omega_B}V_{j,B}\otimes V'_{j,B},$$

where $V_{i,A},V_{j,B}\in\mathrm{Obj}(\mathcal{C})$ and $V'_{i,A},V'_{j,B}\in\mathrm{Obj}(\mathcal{C}')$ for $i\in\Omega_A$ and $j\in\Omega_B$ (here Ω_A and Ω_B are index sets). Let $\omega_{V,W},\overline{\omega}_{V,W}\in H\otimes H$ be chosen as in definition 2.5.1(ii) where

$$V:=\bigoplus_{i\in\Omega_A}V_{i,A}\quad \textit{and}\quad W:=\bigoplus_{j\in\Omega_B}V_{j,B}.$$

For $x,y\in H$ let $\omega_{x,y}\in H\otimes H$ be defined by $\omega_{x,y}:=\overline{\omega}_{V,W}(y\otimes x)\omega_{V,W}$. Then

$$\left(\left(\check{R}''_{B,A}\right)^{-1}\circ T_{x\otimes y}\circ\check{R}''_{B,A}\right)\Big|_{V''_B\otimes V''_A}=T_{\omega_{x,y}}\quad \textit{for } 1\leqslant i\leqslant r.$$

Proof. By naturality of $\check{R}''$ it follows that

$$\left(\check{R}''_{B,A}\right)|_{V''_B\otimes V''_A}=\check{R}''_{V''_B,V''_A}\quad\text{and}\quad \left(\check{R}''_{B,A}\right)^{-1}|_{V''_A\otimes V''_B}=\left(\check{R}''_{V''_B,V''_A}\right)^{-1}.$$

Thus from (24) it follows that

$$\left(\left(\check{R}''_{B,A}\right)^{-1}\circ T_{x\otimes y}\circ\check{R}''_{B,A}\right)|_{V_{j,B}\otimes V'_{j,B}\otimes V_{i,A}\otimes V'_{i,A}}=\left(\check{R}_{V_{j,B},V_{i,A}}\right)^{-1}_{13}\circ T_{x\otimes y}\circ\left(\check{R}_{V_{j,B},V_{i,A}}\right)_{13}.$$

The relation (23) and lemma 2.5.2 for the triple $(H,\mathcal{C},\check{R})$ imply the assertion of the lemma. □

Proposition 2.6.2. *Let $(H,\mathcal{C},\check{R})$ and $(H',\mathcal{C}',\check{R}')$ be locally finite braided triples and let $(H'',\mathcal{C}'',\check{R}'')$ be defined as in definition 2.5.3. Let $A,B\in\mathrm{Obj}(\mathcal{C}'')$ be H''-module algebras. Let $\mathcal{E}$ be a subspace of $A\otimes_{\check{R}''}B$ and let $\mathcal{A}$ denote the subalgebra of $A\otimes_{\check{R}''}B$ that is generated by $\mathcal{E}$. Finally, let $\overline{H}$ be a sub-bialgebra of H.*

(i) *Assume that for every pair of finite dimensional modules* $V, W \in \mathrm{Obj}(\mathcal{C})$*, we can choose* $\omega_{V,W}, \overline{\omega}_{V,W} \in \overline{H} \otimes H$ *that satisfy definition* 2.5.1*(ii). If* $\mathcal{E}$ *is stable under the action of the subalgebra* $H \otimes 1 \otimes \overline{H} \otimes 1$ *of* $H'' \otimes H''$*, then so is* $\mathcal{A}$*.*

(ii) *Assume that for every pair of finite dimensional modules* $V, W \in \mathrm{Obj}(\mathcal{C})$*, we can choose* $\omega_{V,W}, \overline{\omega}_{V,W} \in H \otimes \overline{H}$ *that satisfy definition* 2.5.1*(ii). If* $\mathcal{E}$ *is stable under the action of the subalgebra* $\overline{H} \otimes 1 \otimes H \otimes 1$ *of* $H'' \otimes H''$*, then so is* $\mathcal{A}$*.*

Proof. For $r \geqslant 0$ set $\mathcal{A}_r := \mathrm{Span}\left\{w^{(1)} \cdots w^{(r)} \,:\, w^{(i)} \in \mathcal{E} \text{for } 1 \leqslant i \leqslant r\right\}$, so that $\mathcal{A} := \sum_{r \geqslant 0} \mathcal{A}_r$. We prove by induction on r that under the assumptions of (i) (respectively, of (ii)), the subspace $\mathcal{A}_r$ is $(H \otimes 1 \otimes \overline{H} \otimes 1)$-stable (respectively, $(\overline{H} \otimes 1 \otimes H \otimes 1)$-stable). For $r = 0$ this is trivial and for $r = 1$ this follows from the assumption on $\mathcal{E}$. Next assume $r > 1$.

Choose any $\alpha, \beta \in A \otimes_{\check{R}''} B$. We can express α and β as finite sums $\alpha = \sum a \otimes b$ and $\beta = \sum a' \otimes b'$ where $a, a' \in A$ and $b, b' \in B$. For each pair (b, a') that occurs in these summations we also express $\check{R}''_{B,A}(b \otimes a')$ as a summation, that is,

$$\check{R}''_{B,A}(b \otimes a') = \sum a'' \otimes b'', \tag{25}$$

where of course the $a'' \in A$ and the $b'' \in B$ depend on b and a'. For $h, h' \in H$ we have

$$\begin{aligned}(h \otimes 1 \otimes h' \otimes 1) \cdot \alpha\beta &= (h \otimes 1 \otimes h' \otimes 1) \cdot \left(\sum (a \otimes b)(a' \otimes b')\right) \\ &= (h \otimes 1 \otimes h' \otimes 1) \cdot \left(\sum aa'' \otimes b''b'\right) \\ &= \sum ((h \otimes 1) \cdot (aa'')) \otimes ((h' \otimes 1) \cdot (b''b')).\end{aligned}$$

Since both A and B are H''-module algebras, from the above calculation and (15) we obtain

$$\begin{aligned}(h \otimes 1 \otimes h' \otimes 1) \cdot \alpha\beta &= \sum ((h_1 \otimes 1) \cdot a)\left((h_2 \otimes 1) \cdot a''\right) \otimes \left((h'_1 \otimes 1) \cdot b''\right)\left((h'_2 \otimes 1) \cdot b'\right) \\ &= \sum (((h_1 \otimes 1) \cdot a) \otimes 1)\left((h_2 \otimes 1 \otimes h'_1 \otimes 1) \cdot (a'' \otimes b'')\right)\left(1 \otimes \left((h'_2 \otimes 1) \cdot b'\right)\right),\end{aligned} \tag{26}$$

where $\Delta(h) = \sum h_1 \otimes h_2$ and $\Delta(h') = \sum h'_1 \otimes h'_2$. From lemma 2.6.1 it follows that for each pair (h_2, h'_1) that occurs on the right hand side of (26) there exists a two-tensor $\omega_{h_2,h'_1} = \sum u_{h_2,h'_1} \otimes u'_{h_2,h'_1}$ in $H \otimes H$ such that

$$\begin{aligned}(h_2 \otimes 1 \otimes h'_1 \otimes 1) \cdot \sum (a'' \otimes b'') &= \check{R}''_{B,A}\left(\left(\omega_{h_2,h'_1}\right)_{13} \cdot (b \otimes a')\right) \\ &= \check{R}''_{B,A}\left(\left(\left(u_{h_2,h'_1} \otimes 1\right) \cdot b\right) \otimes \left(\left(u'_{h_2,h'_1} \otimes 1\right) \cdot a'\right)\right).\end{aligned} \tag{27}$$

Note that by lemma 2.6.1 the two-tensors ω_{h_2,h'_1} are of the form

$$\omega_{h_2,h'_1} = \overline{\omega}_{V,W}(h'_1 \otimes h_2)\,\omega_{V,W}, \tag{28}$$

where $\omega_{V,W}, \overline{\omega}_{V,W} \in H \otimes H$ satisfy the constraint of definition 2.5.1(ii) for suitable H-modules V, W. By comparing (26) and (27) and then using (15) we obtain

$$\begin{aligned}(h\otimes 1\otimes h'\otimes 1)\cdot\alpha\beta &= \sum\left(((h_1\otimes 1)\cdot a)\otimes\left(\left(u_{h_2,h_1'}\otimes 1\right)\cdot b\right)\right)\left(\left(\left(u'_{h_2,h_1'}\otimes 1\right)\cdot a'\right)\otimes((h_2'\otimes 1)\cdot b')\right)\\ &= \sum\left(\left(h_1\otimes 1\otimes u_{h_2,h_1'}\otimes 1\right)\cdot\alpha\right)\left(\left(u'_{h_2,h_1'}\otimes 1\otimes h_2'\otimes 1\right)\cdot\beta\right).\end{aligned}$$

In what follows we complete the proofs of part (i) and part (ii).

(i) It suffices to prove that for $h\in H$ and $h'\in\overline{H}$ and $w^{(1)},\dots,w^{(r)}\in\mathcal{E}$ we have

$$(h\otimes 1\otimes h'\otimes 1)\cdot\left(w^{(1)}\cdots w^{(r)}\right)\in\mathcal{A}_r. \tag{29}$$

We set

$$\alpha := w^{(1)}\quad\text{and}\quad\beta := w^{(2)}\cdots w^{(r)}.$$

Since $\Delta(\overline{H})\subseteq\overline{H}\otimes\overline{H}$, we can assume that $h_1',h_2'\in\overline{H}$, hence by considering the first component on both sides of (28) we obtain $u_{h_2,h_1'}\in\overline{H}$. Thus, by the induction hypothesis we have

$$\left(\left(h_1\otimes 1\otimes u_{h_2,h_1'}\otimes 1\right)\cdot\alpha\right)\in\mathcal{E}\subseteq\mathcal{A}_1.\quad\text{and}\quad\left(\left(u'_{h_2,h_1'}\otimes 1\otimes h_2'\otimes 1\right)\cdot\beta\right)\in\mathcal{A}_{r-1}. \tag{30}$$

The inclusion (29) follows from $\mathcal{A}_1\mathcal{A}_{r-1}=\mathcal{A}_r$.

(ii) It suffices to prove (29) for $h\in\overline{H}$ and $h'\in H$. We define α and β as in (i). As $h\in\overline{H}$, we can assume that $h_1,h_2\in\overline{H}$, hence $u'_{h_2,h_1'}\in\overline{H}$. Again the induction hypothesis implies (30) and the inclusion (29) follows from $\mathcal{A}_1\mathcal{A}_{r-1}=\mathcal{A}_r$. □

2.7. *Braidings and matrix coefficients*

Let $(H,\mathcal{C},\check{R})$ be a locally finite braided triple. As in (23) we set $R_{V,W}:=\sigma_{W,V}\circ\check{R}_{V,W}$ for $V,W\in\mathrm{Obj}(\mathcal{C})$. Let $H^\circ_{\mathcal{C}}\subseteq H^\circ$ denote the $\mathbb{Z}$-span of matrix coefficients of the H-modules that belong to $\mathrm{Obj}(\mathcal{C})$. Since $\mathcal{C}$ is closed under direct sums and tensor products, indeed $H^\circ_{\mathcal{C}}$ is a sub-bialgebra of H°.

Let $f\in H^\circ_{\mathcal{C}}$ be a sum of matrix coefficients of $V_1,\dots,V_N\in\mathrm{Obj}(\mathcal{C})$, that is

$$f=\sum_{i=1}^{N}\mathsf{m}_{v_i^*,v_i},$$

where $v_i\in V_i$ and $v_i^*\in V_i^*$. Similarly, let $g\in H^\circ_{\mathcal{C}}$ be a sum of matrix coefficients of $W_1,\dots,W_{N'}\in\mathrm{Obj}(\mathcal{C})$, that is $g=\sum_{i=1}^{N'}\mathsf{m}_{w_i^*,w_i}$. Choose $\omega_{V,W},\varpi_{V,W}\in H\otimes H$ that satisfy the condition of definition 2.5.1(ii) for $V:=\bigoplus_{i=1}^{N}V_i$ and $W:=\bigoplus_{i=1}^{N'}W_i$. We define

$$R(f\otimes g):=\omega_{V,W}\cdot(f\otimes g)\quad\text{and}\quad R^{-1}(f\otimes g):=\varpi_{V,W}\cdot(f\otimes g), \tag{31}$$

where the actions of $\omega_{V,W}$ and $\varpi_{V,W}$ are by right translation on tensor components. This means that for example if $\omega_{V,W}=\sum r\otimes r'\in H\otimes H$, then

$$(\omega_{V,W}\cdot(f\otimes g))(h\otimes h'):=\sum f(hr)\,g(h'r').$$

Remark 2.7.1. (i) For $x,y\in H$ we have

$$\left(\omega_{V,W}\cdot\left(\mathsf{m}_{v_i^*,v_i}\otimes\mathsf{m}_{w_j^*,w_j}\right)\right)(x\otimes y)=\langle v_i^*\otimes w_j^*,(x\otimes y)\cdot R_{V,W}(v_i\otimes w_j)\rangle, \tag{32}$$

hence the left hand side of (32) is independent of the choice of $\omega_{V,W}$. Summing over i and j it follows that $R(f\otimes g)$ is independent of the choice of $\omega_{V,W}$ as well. The latter observation and lemma 2.5.2 imply that $R(f\otimes g)$ does not depend on how f and g are expressed as sums of matrix coefficients. An analogous statement holds for $R^{-1}(f\otimes g)$.

(ii) From lemma 2.5.2 it also follows that the formulas (31) extend to linear maps

$$R, R^{-1} : H^\circ_{\mathcal{C}} \otimes H^\circ_{\mathcal{C}} \to H^\circ_{\mathcal{C}} \otimes H^\circ_{\mathcal{C}}.$$

Indeed the latter maps R and R^{-1} are mutual inverses.

We define

$$\langle f\otimes g, R\rangle := (R(f\otimes g))(1\otimes 1). \tag{33}$$

In the rest of this subsection $\Delta^\circ(f) = \sum f_1\otimes f_2$ and $\Delta^\circ(g) = \sum g_1\otimes g_2$.

Lemma 2.7.2. *Let $f,g\in H^\circ_{\mathcal{C}}$. Then the following relations hold.*

(i) $R(f\otimes g) = \sum f_1\otimes g_1\langle f_2\otimes g_2, R\rangle$.
(ii) $f\otimes g = \sum\langle R^{-1}(f_1\otimes g_1), R\rangle f_2\otimes g_2$.

Proof. It suffices to verify the assertion when f and g are matrix coefficients of finite dimensional H-modules $V,W\in \mathrm{Obj}(\mathcal{C})$. Suppose that $f := \mathsf{m}_{v^*,v}$ and $g := \mathsf{m}_{w^*,w}$. If $R_{V,W}(v\otimes w) = \sum\tilde{v}\otimes\tilde{w}$ then by (32) we have $R(f\otimes g) = \sum\mathsf{m}_{v^*,\tilde{v}}\otimes\mathsf{m}_{w^*,\tilde{w}}$. Choose dual bases $\{v_i\}$ and $\{v_i^*\}$ for V and V^* and dual bases $\{w_j\}$ and $\{w_j^*\}$ for W and W^*. Using the coproduct and counit identities of H° we have

$$\begin{aligned}\sum f_1\otimes g_1\langle f_2\otimes g_2, R\rangle &= \sum\mathsf{m}_{v^*,v_i}\otimes\mathsf{m}_{w^*,w_j}\left(\langle v_i^*,\tilde{v}\rangle\langle w_j^*,\tilde{w}\rangle\right)\\ &= \sum\langle v_i^*,\tilde{v}\rangle\mathsf{m}_{v^*,v_i}\otimes\langle w_j^*,\tilde{w}\rangle\mathsf{m}_{w^*,w_j} = \sum\mathsf{m}_{v^*,\tilde{v}}\otimes\mathsf{m}_{w^*,\tilde{w}} = R(f\otimes g).\end{aligned}$$

This proves (i). For (ii) note that if $R^{-1}_{V,W}(v\otimes w) = \sum\tilde{v}\otimes\tilde{w}$ then $R^{-1}(f\otimes g) = \sum\mathsf{m}_{v^*,\tilde{v}}\otimes\mathsf{m}_{w^*,\tilde{w}}$. Next we write $R^{-1}_{V,W}(v_i\otimes w_j) = \sum\tilde{v}^{i,j}\otimes\tilde{w}^{i,j}$ for each pair of indices i,j. Then

$$\begin{aligned}\sum\langle R^{-1}(f_1\otimes g_1), R\rangle f_2\otimes g_2 &= \sum\langle\mathsf{m}_{v^*,\tilde{v}^{i,j}}\otimes\mathsf{m}_{w^*,\tilde{w}^{i,j}}, R\rangle\mathsf{m}_{v_i^*,v}\otimes\mathsf{m}_{w_j^*,w}\\ &= \sum\langle v^*, v_i\rangle\mathsf{m}_{v_i^*,v}\otimes\langle w^*,w_j\rangle\mathsf{m}_{w_j^*,w} = \mathsf{m}_{v^*,v}\otimes\mathsf{m}_{w^*,w} = f\otimes g.\end{aligned}$$

□

Fix finite dimensional H-modules $V,W\in\mathrm{Obj}(\mathcal{C})$. Let $\{v_i\}_{i=1}^d$ and $\{w_i\}_{i=1}^{d'}$ be bases of V and W. Also, let $\{v_i^*\}_{i=1}^d$ and $\{w_i^*\}_{i=1}^{d'}$ be the dual bases of V^* and W^*. We denote the matrix entries of $R_{V,W}$ in the basis $v_i\otimes w_j$ by R_{ij}^{kl}, so that

$$R_{V,W}(v_i\otimes w_j) = \sum_{k,l} R_{ij}^{kl} v_k\otimes w_l.$$

Set $\mathsf{t}^V_{a,b} := \mathsf{m}_{v_a^*,v_b}$ and $\mathsf{t}^W_{a,b} := \mathsf{m}_{w_a^*,w_b}$. Then $R_{ij}^{kl} = \langle\mathsf{t}^V_{k,i}\otimes\mathsf{t}^W_{l,j}, R\rangle$, so that as in [Ja96, lemma 7.12] we have the well known relations

$$\sum_{k,l}\langle\mathsf{t}^V_{k,i}\otimes\mathsf{t}^W_{l,j}, R\rangle\mathsf{t}^W_{a,l}\mathsf{t}^V_{b,k} = \sum_{k,l}\langle\mathsf{t}^V_{b,k}\otimes\mathsf{t}^W_{a,l}, R\rangle\mathsf{t}^V_{k,i}\mathsf{t}^W_{l,j}\quad\text{for all } i,j,a,b. \tag{34}$$

From (34) it follows that

$$\sum g_1 f_1 \langle f_2 \otimes g_2, R\rangle = \sum f_2 g_2 \langle f_1 \otimes g_1, R\rangle \quad \text{for } f, g \in H_{\mathcal{C}}^{\circ}. \tag{35}$$

3. The *q*-Weyl algebra $\mathscr{P}\mathscr{D}$

In this section we construct $\mathscr{P}\mathscr{D}$ as a deformed twisted tensor product of the algebras $\mathscr{P}$ and $\mathscr{D}$ with respect to the univeral R-matrix of U_{LR}. Recall that $\Bbbk := \mathbb{C}(q)$.

3.1. The algebra $U_q(\mathfrak{gl}_n)$

For $n \in \mathbb{N}$, the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ is the $\Bbbk$-algebra generated by E_i, F_i for $1 \leqslant i \leqslant n-1$ and $K_{\varepsilon_i}^{\pm 1}$ for $1 \leqslant i \leqslant n$, that satisfy the relations $K_{\varepsilon_i} K_{\varepsilon_i}^{-1} = K_{\varepsilon_i}^{-1} K_{\varepsilon_i} = 1$, $K_{\varepsilon_i} K_{\varepsilon_j} = K_{\varepsilon_j} K_{\varepsilon_i}$,

$$K_{\varepsilon_i} E_j K_{\varepsilon_i}^{-1} = q^{[\![i,j]\!] - [\![i,j+1]\!]} E_j \ , \ K_{\varepsilon_i} F_j K_{\varepsilon_i}^{-1} = q^{-[\![i,j]\!] + [\![i,j+1]\!]} F_j \ , \ E_i F_j - F_j E_i = [\![i,j]\!] \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

where $K_i := K_{\varepsilon_i} K_{\varepsilon_{i+1}}^{-1}$ and

$$[\![a,b]\!] := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b, \end{cases} \tag{36}$$

as well as the quantum Serre relations. For $\lambda := \sum_{i=1}^n m_i \varepsilon_i \in \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ we set

$$K_\lambda := \prod_{i=1}^n K_{\varepsilon_i}^{m_i}. \tag{37}$$

The *Cartan subalgebra* of $U_q(\mathfrak{gl}_n)$ is the subalgebra spanned by the K_λ for $\lambda \in \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$. We denote the Cartan subalgebras of $U_L \cong U_q(\mathfrak{gl}_m)$ and $U_R \cong U_q(\mathfrak{gl}_n)$ by $U_{\mathfrak{h},L}$ and $U_{\mathfrak{h},R}$, respectively. Following [KS97] for the choice of the coproduct Δ on $U_q(\mathfrak{gl}_n)$, we set

$$\Delta(E_i) := E_i \otimes K_i + 1 \otimes E_i \quad , \quad \Delta(F_i) := F_i \otimes 1 + K_i^{-1} \otimes F_i \quad , \quad \Delta(K_{\varepsilon_i}) := K_{\varepsilon_i} \otimes K_{\varepsilon_i}.$$

The counit and antipode of $U_q(\mathfrak{gl}_n)$ are given by

$$\epsilon(E_i) = \epsilon(F_i) = 0 \ , \ \epsilon\left(K_{\varepsilon_i}^{\pm 1}\right) = 1 \ , \ S(E_i) = -E_i K_i^{-1} \ , \ S(F_i) = -K_i F_i \ , \ S(K_{\varepsilon_i}) = K_{\varepsilon_i}^{-1}.$$

3.2. The universal R-matrix of $U_q(\mathfrak{gl}_n)$

Recall that a Hopf algebra H is called quasitriangular if it has a *universal R-matrix*, i.e. if there exists an invertible 2-tensor $\mathcal{R} \in H \otimes H$ satisfying

$$\Delta^{\mathrm{cop}} = \mathcal{R}\Delta\mathcal{R}^{-1} \quad , \quad (\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \quad , \quad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{38}$$

Strictly speaking, $U_q(\mathfrak{gl}_n)$ is not quasitriangular because the formal series that is usually called the universal R-matrix of $U_q(\mathfrak{gl}_n)$ indeed belongs to a topological tensor product $U_h(\mathfrak{gl}_n) \widehat{\otimes} U_h(\mathfrak{gl}_n)$ where $U_h(\mathfrak{gl}_n)$ denotes the h-adic Drinfeld–Jimbo quantum group. However, it turns out that the setting of braided triples is a rigorous way to work with this universal R-matrix.

Let $\mathcal{C}^{(n)}$ denote the full subcategory of the category of $U_q(\mathfrak{gl}_n)$-modules whose objects are direct sums of irreducible finite dimensional $U_q(\mathfrak{gl}_n)$-modules with highest weight of the form $q^{\sum_{i=1}^n \lambda_i \varepsilon_i}$, where $\lambda_1 \geq \cdots \geqslant \lambda_n$ are integers. Such modules are sometimes called modules of type $(1, \ldots, 1)$. We define a braiding on $\mathcal{C}^{(n)}$ as follows. First we fix a formal series description of the universal R-matrix for $U_q(\mathfrak{gl}_n)$. For more details see [VY20, theorem 3.108] or [KS97, section 8.3.2].

Definition 3.2.1. Given $n \in \mathbb{N}$, the standard root vectors of $U_q(\mathfrak{gl}_n)$ are

$$E_{\varepsilon_i - \varepsilon_j} := (-1)^{j-i-1} \left[E_i, [\ldots, E_{j-1}]_{q^{-1}}\right]_{q^{-1}} \quad \text{and} \quad F_{\varepsilon_i - \varepsilon_j} := (-1)^{j-i-1} \left[F_{j-1}, [\ldots, F_i]_q\right]_q,$$

where $1 \leqslant i < j \leqslant n$ and $[x,y]_{q^{\pm 1}} := xy - q^{\pm 1} yx$. We set

$$\mathcal{R}^{(n)} := \left(e^{h \sum_{i=1}^n H_i \otimes H_i}\right) \prod_{i=1}^{\binom{n}{2}} \mathrm{Exp}_q \left(\left(q - q^{-1}\right) E_{\beta_i} \otimes F_{\beta_i}\right), \tag{39}$$

with the conventions $e^{hH_i} := K_{\varepsilon_i}$, $e^h := q$, $\mathrm{Exp}_q(x) := \sum_{r \geqslant 0} q^{\binom{r}{2}} \frac{x^r}{[r]_q!}$ and $\beta_{i + \frac{j(j-1)}{2}} := \varepsilon_i - \varepsilon_{j+1}$ for $1 \leqslant i \leqslant j \leqslant n-1$. Also, set

$$\underline{\mathcal{R}}^{(n)} := \left(\mathcal{R}^{(n)}\right)_{21}^{-1}.$$

In what follows, we need $\mathcal{R}^{(n)}$ to define $\mathscr{P}$ and $\mathscr{D}$, and we need $\underline{\mathcal{R}}^{(n)}$ to define $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ and $\mathscr{P}\mathscr{D}$. The formal series $\mathcal{R}^{(n)}$ and $\underline{\mathcal{R}}^{(n)}$ equip the category $\mathcal{C}^{(n)}$ with two braidings which we describe below. Given $V, W \in \mathrm{Obj}(\mathcal{C}^{(n)})$, the formal series (39) defines a linear map

$$\mathcal{R}^{(n)}_{V,W} : V \otimes W \to V \otimes W.$$

To give sense to the action of $\mathcal{R}^{(n)}$ on $V \otimes W$ we make the following two observations. First, for any $v \otimes w \in V \otimes W$ all but finitely many terms of $\mathrm{Exp}_q\left((q - q^{-1}) E_{\beta_i} \otimes F_{\beta_i}\right)$ vanish on $v \otimes w$. Second, if $v \in V$ and $w \in W$ are weight vectors of weights q^μ and q^ν with $\mu := \sum_{i=1}^n \mu_i \varepsilon_i$ and $\nu := \sum_{i=1}^n \nu_i \varepsilon_i$ respectively, then the action of $e^{h \sum_{i=1}^n H_i \otimes H_i}$ on $v \otimes w$ is by multiplication by the scalar $q^{\langle \mu, \nu \rangle}$ where

$$\langle \mu, \nu \rangle := \sum_{i=1}^n \mu_i \nu_i. \tag{40}$$

By a similar reasoning, the action of $\underline{\mathcal{R}}^{(n)}$ yields linear maps

$$\underline{\mathcal{R}}^{(n)}_{V,W} : V \otimes W \to V \otimes W.$$

It is well known that by setting

$$\check{\mathcal{R}}^{(n)}_{V,W} := \sigma_{V,W} \circ \mathcal{R}^{(n)}_{V,W} \quad \text{and} \quad \check{\underline{\mathcal{R}}}^{(n)}_{V,W} := \sigma_{V,W} \circ \underline{\mathcal{R}}^{(n)}_{V,W} \tag{41}$$

we obtain braidings on $\mathcal{C}^{(n)}$, which we will denote by $\check{\mathcal{R}}^{(n)}$ and $\check{\underline{\mathcal{R}}}^{(n)}$

Proposition 3.2.2. *Set $H := U_q(\mathfrak{gl}_n)$, $\mathcal{C} := \mathcal{C}^{(n)}$, and $\check{R} := \check{\mathcal{R}}^{(n)}$ or $\check{\underline{\mathcal{R}}}^{(n)}$. Then $(H, \mathcal{C}, \check{R})$ is a locally finite braided triple.*

Proof. The only assertion that we need to prove is the property of definition 2.5.1(ii). Fix finite dimensional $V, W \in \mathrm{Obj}\left(\mathcal{C}^{(n)}\right)$. We construct an element of $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$ that acts on $V \otimes W$ as $\mathcal{R}^{(n)}_{V,W}$. Since $E^{N_i}_{\beta_i} \otimes F^{N_i}_{\beta_i}$ vanishes on $V \otimes W$ when N_i is sufficiently large, the exponential factor

$$\mathrm{Exp}_q\left(\left(q - q^{-1}\right) E_{\beta_i} \otimes F_{\beta_i}\right)$$

can be replaced by a finite sum. Next we provide a finite two-tensor that replaces $e^{h\sum_{i=1}^n H_i \otimes H_i}$. Let $\mu^{(1)}, \ldots, \mu^{(N)}$ be the distinct weights of W. Choose $\nu \in \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ such that the values $\langle \nu, \mu^{(i)} \rangle$ are mutually distinct numbers. For $1 \leqslant i \leqslant N$ define $T_i \in U_q(\mathfrak{gl}_n)$ by

$$T_i := \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \left(\frac{K_\nu - q^{\langle \nu, \mu^{(j)} \rangle}}{q^{\langle \nu, \mu^{(i)} \rangle} - q^{\langle \nu, \mu^{(j)} \rangle}} \right).$$

Then T_i acts by 0 or 1 on the $\mu^{(j)}$-weight space of W, depending on if $j \neq i$ or $j = i$ respectively. It follows that the action of $\sum_{i=1}^N K_{\mu^{(i)}} \otimes T_i$ on $V \otimes W$ is identical to the action of $e^{h\sum_{i=1}^n H_i \otimes H_i}$. Thus the action of $\mathcal{R}^{(n)}$ on $V \otimes W$ is identical to the action of a (finite) two-tensor in $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$. Analogous constructions can be given for $\left(\mathcal{R}^{(n)}_{V,W}\right)^{-1}$ and $\left(\underline{\mathcal{R}}^{(n)}_{V,W}\right)^{\pm 1}$. $\square$

As in section 2.2 let $U_q(\mathfrak{gl}_n)^\circ$ be the finite dual of $U_q(\mathfrak{gl}_n)$.

Definition 3.2.3. Let $U_q(\mathfrak{gl}_n)^\bullet \subseteq U_q(\mathfrak{gl}_n)^\circ$ denote the sub-bialgebra that is spanned by matrix coefficients of objects of $\mathcal{C}^{(n)}$.

For $f, g \in U_q(\mathfrak{gl}_n)^\bullet$ we define $\langle f \otimes g, \mathcal{R}^{(n)} \rangle$ and $\langle f \otimes g, \underline{\mathcal{R}}^{(n)} \rangle$ as in (33). For finite dimensional $V \in \mathrm{Obj}\left(\mathcal{C}^{(n)}\right)$ the right dual V^* also belongs to $\mathrm{Obj}\left(\mathcal{C}^{(n)}\right)$. From this and the fact that S and S^{-1} are conjugate by an element of the Cartan subalgebra it follows that if $f \in U_q(\mathfrak{gl}_n)^\bullet$ then $f \circ S^{\pm 1} \in U_q(\mathfrak{gl}_n)^\bullet$. It is well known (see for example [KS97, section 8.1.1]) that

$$\left(\mathcal{R}^{(n)}\right)_{21} = \left(\underline{\mathcal{R}}^{(n)}\right)^{-1} = \left(1 \otimes S^{-1}\right)\left(\underline{\mathcal{R}}^{(n)}\right) \quad \text{and} \quad (S \otimes 1)\left(\mathcal{R}^{(n)}\right) = \left(\mathcal{R}^{(n)}\right)^{-1} \tag{42}$$

Consequently, for $f, g \in U_q(\mathfrak{gl}_n)^\bullet$ we have

$$\left\langle f \otimes \left(g \circ S^{-1}\right), \underline{\mathcal{R}}^{(n)} \right\rangle = \left\langle g \otimes f, \mathcal{R}^{(n)} \right\rangle \quad \text{and} \quad \left\langle (f \circ S) \otimes g, \mathcal{R}^{(n)} \right\rangle = \left\langle g \otimes f, \underline{\mathcal{R}}^{(n)} \right\rangle. \tag{43}$$

3.3. *The involution $x \mapsto x^\natural$*

It is well known (for example see [No96, section 1.4]) that there exists a unique $\mathbb{k}$-linear isomorphism of Hopf algebras

$$U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)^{\mathrm{op}}, \ x \mapsto x^\natural, \tag{44}$$

such that

$$E_i^\natural := qK_iF_i \quad , \quad F_i^\natural := q^{-1}E_iK_i^{-1} \quad , \quad K_{\varepsilon_i}^\natural := K_{\varepsilon_i}. \tag{45}$$

Lemma 3.3.1. *$S(x^\natural) = S^{-1}(x)^\natural$ for $x \in U_q(\mathfrak{gl}_n)$.*

Proof. Both sides are automorphisms of the algebra $U_q(\mathfrak{gl}_n)$. Therefore it suffices to verify that they agree on the E_i, the F_i, and the K_{ε_i}. This is a straightforward calculation. □

By the canonical duality between $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_n)^\circ$, the map (44) induces an isomorphism of Hopf algebras $U_q(\mathfrak{gl}_n)^\circ \to (U_q(\mathfrak{gl}_n)^\circ)^{\mathrm{cop}}$. We denote the latter map by $u \mapsto u^\natural$ as well, so that

$$\langle u^\natural, x\rangle = \langle u, x^\natural\rangle \quad \text{for } u \in U_q(\mathfrak{gl}_n)^\circ \text{ and } x \in U_q(\mathfrak{gl}_n). \tag{46}$$

3.4. The algebras $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$

From now on we denote the standard positive system of the root system of $\mathfrak{gl}_n$ by

$$\Delta_n^+ := \{\varepsilon_i - \varepsilon_j : 1 \leqslant i < j \leqslant n\}$$

Let $V^{(n)}$ denote the irreducible $U_q(\mathfrak{gl}_n)$-module of highest weight $q^{-\varepsilon_n}$ (all highest weights are considered with respect to Δ_n^+). Thus $V^{(n)} \cong \Bbbk^n$ as a vector space and the homomorphism of algebras $U_q(\mathfrak{gl}_n) \to \mathrm{End}_{\Bbbk}\left(V^{(n)}\right)$ is uniquely determined by the assignments

$$K_{\varepsilon_i} \mapsto \mathsf{1} + \left(q^{-1} - 1\right)\mathsf{E}_{i,i} \quad , \quad E_i \mapsto \mathsf{E}_{i+1,i} \quad , \quad F_i \mapsto \mathsf{E}_{i,i+1},$$

where the $\mathsf{E}_{i,j}$ are the elementary matrix units associated to the standard basis $\{e_i\}_{i=1}^n$ of $V^{(n)}$ and $\mathsf{1} := \sum_{i=1}^n \mathsf{E}_{i,i}$. Using (39) the R-matrix of $V^{(n)} \otimes V^{(n)}$ can be computed directly, and we obtain

$$\mathcal{R}^{(n)}_{V^{(n)},V^{(n)}} = \sum_{1\leqslant i\leqslant n} q\mathsf{E}_{i,i}\otimes\mathsf{E}_{i,i} + \sum_{1\leqslant i\neq j\leqslant n}\mathsf{E}_{i,i}\otimes\mathsf{E}_{j,j} + \left(q-q^{-1}\right)\sum_{1\leqslant j<i\leqslant n}\mathsf{E}_{i,j}\otimes\mathsf{E}_{j,i}.$$

For $1 \leqslant i,j \leqslant n$ let $t_{i,j}$ denote the matrix coefficient $\mathsf{m}_{e_i^*,e_j}$ of $V^{(n)}$. By (35) the $t_{i,j}$ satisfy the following relations:

(R1) $t_{k,i}t_{k,j} = qt_{k,j}t_{k,i}$, $t_{i,k}t_{j,k} = qt_{j,k}t_{i,k}$ for $i<j$.
(R2) $t_{i,l}t_{k,j} = t_{k,j}t_{i,l}$, $t_{i,j}t_{k,l} - t_{k,l}t_{i,j} = (q-q^{-1})t_{i,l}t_{k,j}$ for $i<k$ and $j<l$.

Similarly, let $\check{V}^{(n)}$ denote the irreducible $U_q(\mathfrak{gl}_n)$-module with highest weight q^{ε_1}. Again $\check{V}^{(n)} \cong \Bbbk^n$ as vector spaces, but the map $U_q(\mathfrak{gl}_n) \to \mathrm{End}_{\Bbbk}\left(\check{V}^{(n)}\right)$ is uniquely determined by the assignments

$$K_{\varepsilon_i} \mapsto \mathsf{1} + (q-1)\mathsf{E}_{i,i} \quad , \quad E_i \mapsto \mathsf{E}_{i,i+1} \quad , \quad F_i \mapsto \mathsf{E}_{i+1,i}.$$

Indeed $\check{V}^{(n)} \cong \left(V^{(n)}\right)^*$. The R-matrix of $\check{V}^{(n)} \otimes \check{V}^{(n)}$ is

$$\mathcal{R}^{(n)}_{\check{V}^{(n)},\check{V}^{(n)}} = \sum_{1\leqslant i\leqslant n} q\mathsf{E}_{i,i}\otimes\mathsf{E}_{i,i} + \sum_{1\leqslant i\neq j\leqslant n}\mathsf{E}_{i,i}\otimes\mathsf{E}_{j,j} + \left(q-q^{-1}\right)\sum_{1\leqslant i<j\leqslant n}\mathsf{E}_{i,j}\otimes\mathsf{E}_{j,i}.$$

If $\partial_{i,j}$ for $1 \leqslant i,j \leqslant n$ denotes the matrix coefficient $\mathsf{m}_{e_i^*,e_j}$ of $\check{V}^{(n)}$, then again from (35) it follows that the $\partial_{i,j}$ satisfy relations similar to those between the $t_{i,j}$, with q replaced by q^{-1}. Equivalently,

(R1′) $\partial_{k,j}\partial_{k,i}=q\partial_{k,i}\partial_{k,j}$, $\partial_{j,k}\partial_{i,k}=q\partial_{i,k}\partial_{j,k}$ for $i<j$.
(R2′) $\partial_{k,j}\partial_{i,l}=\partial_{i,l}\partial_{k,j}, \partial_{k,l}\partial_{i,j}-\partial_{i,j}\partial_{k,l}=(q-q^{-1})\partial_{k,j}\partial_{i,l}$ for $i<k$ and $j<l$.

Definition 3.4.1. Let $\mathscr{P}_{n\times n}$ denote the subalgebra of $U_q(\mathfrak{gl}_n)^\circ$ generated by the $t_{i,j}$, for $1\leqslant i,j\leqslant n$. Similarly, let $\mathscr{D}_{n\times n}$ denote the subalgebra of $U_q(\mathfrak{gl}_n)^\circ$ generated by the $\partial_{i,j}$, for $1\leqslant i,j\leqslant n$.

It is well known (for example see [Ta92]) that the relations (R1)–(R2) yield a presentation of $\mathscr{P}_{n\times n}$ by generators and relations. Since $\mathscr{D}_{n\times n}\cong\mathscr{P}_{n\times n}^{\mathrm{op}}$, a similar statement holds for $\mathscr{D}_{n\times n}$. From section 2.2 it follows that both $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$ are bialgebras with the coproducts satisfying

$$t_{i,j}\mapsto\sum_k t_{i,k}\otimes t_{k,j}\quad\text{and}\quad\partial_{i,j}\mapsto\sum_k\partial_{i,k}\otimes\partial_{k,j},$$

and the counits satisfying $t_{i,j},\partial_{i,j}\mapsto[\![i,j]\!]$. Henceforth we denote the coproducts of $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$ by $\Delta_{\mathscr{P}}$ and $\Delta_{\mathscr{D}}$, respectively. In the proof of lemma 3.4.2 we use the relations

$$t_{i,j}(K_{\varepsilon_k})=[\![i,j]\!]\,q^{-[\![i,k]\!]}\quad,\quad t_{i,j}(E_k)=[\![i,k+1]\!]\,[\![j,k]\!]\quad,\quad t_{i,j}(F_k)=[\![i,k]\!]\,[\![j,k+1]\!]\,.$$

Lemma 3.4.2. *$t_{i,j}^\natural=t_{j,i}$ and $\partial_{i,j}^\natural=\partial_{j,i}$, where $t_{i,j}^\natural$ and $\partial_{i,j}^\natural$ are defined by (46).*

Proof. We only give the proof for the $t_{i,j}$, as the argument for the $\partial_{i,j}$ is similar. The assertion follows if we verify that

$$\langle t_{i,j},x^\natural\rangle=\langle t_{j,i},x\rangle\quad\text{for }x\in U_q(\mathfrak{gl}_n)\,.\tag{47}$$

It suffices to check (47) when x is a generator of $U_q(\mathfrak{gl}_n)$, because if (47) holds for $x,y\in U_q(\mathfrak{gl}_n)$ then

$$\langle t_{i,j},(xy)^\natural\rangle=\langle t_{i,j},y^\natural x^\natural\rangle=\sum_a\langle t_{i,a},y^\natural\rangle\langle t_{a,j},x^\natural\rangle=\sum_a\langle t_{a,i},y\rangle\langle t_{j,a},x\rangle=\langle t_{j,i},xy\rangle,$$

hence (47) also holds for xy. When x is one of the standard generators of $U_q(\mathfrak{gl}_n)$, checking (47) is a direct calculation. For example for $x=E_k$ we have

$$\langle t_{i,j},E_k^\natural\rangle=q\sum_a\langle t_{i,a},K_k\rangle\langle t_{a,j},F_k\rangle.$$

The right hand side vanishes unless $j=k+1$ and $i=k$, in which case we have $\langle t_{i,j},E_k^\natural\rangle=1$. It follows immediately that $\langle t_{i,j},E_k^\natural\rangle=\langle t_{j,i},E_k\rangle$ for all i,j,k. □

According to remark 2.2.1, the canonical $U_q(\mathfrak{gl}_n)\otimes U_q(\mathfrak{gl}_n)$-module structure of $U_q(\mathfrak{gl}_n)$ by left and right translation equips both $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$ with $U_q(\mathfrak{gl}_n)\otimes U_q(\mathfrak{gl}_n)$-module algebra structures. Our next goal is to describe the latter actions explicitly (all of the actions are from the left side).

Let $\mathcal{R}_{\mathscr{D}}$ be the action of $U_q(\mathfrak{gl}_n)$ on $\mathscr{D}_{n\times n}$ by right translation, as in remark 2.2.1. We have

$$\mathcal{R}_{\mathscr{D}}(x)\,u=\sum\langle u_2,x\rangle u_1\quad\text{for }x\in U_q(\mathfrak{gl}_n)\,,\ u\in\mathscr{D}_{n\times n},$$

where as usual $\Delta(u) = \sum u_1 \otimes u_2$. Similarly, let $\mathcal{L}_{\mathscr{D}}$ be the action of $U_q(\mathfrak{gl}_n)$ on $\mathscr{D}_{n\times n}$ by left translation. Thus

$$\mathcal{L}_{\mathscr{D}}(x)u = \sum \langle u_1, x^\natural \rangle u_2 \quad \text{for } x \in U_q(\mathfrak{gl}_n),\ u \in \mathscr{D}_{n\times n}.$$

By remark 2.2.1 both $\mathcal{L}_{\mathscr{D}}$ and $\mathcal{R}_{\mathscr{D}}$ equip $\mathscr{D}_{n\times n}$ with $U_q(\mathfrak{gl}_n)$-module algebra structures.

Next we define the left and right $U_q(\mathfrak{gl}_n)$-actions on $\mathscr{P}_{n\times n}$. For $c \in \Bbbk$ let ξ_c denote the unique automorphism of $U_q(\mathfrak{gl}_n)$ defined by

$$\xi_c(E_i) := cE_i \quad , \quad \xi_c(F_i) := c^{-1}F_i \quad , \quad \xi_c(K_{\varepsilon_i}) = K_{\varepsilon_i}.$$

Lemma 3.4.3. *$t_{i,j} \circ \xi_c = c^{i-j} t_{i,j}$ and $\partial_{i,j} \circ \xi_c = c^{j-i}\partial_{i,j}$.*

Proof. We only give the proof of the assertion for the $t_{i,j}$. In this case we need to verify the equality

$$t_{i,j}(\xi_c(x)) = c^{i-j} t_{i,j}(x) \tag{48}$$

for $x \in U_q(\mathfrak{gl}_n)$. This is a straightforward calculation in the special case where x is one of the standard generators of $U_q(\mathfrak{gl}_n)$. To complete the proof of (48) note that if (48) holds for x and x', then it also holds for xx' because

$$t_{i,j}(\xi_c(xx')) = \sum_{a=1}^{n} t_{i,a}(\xi_c(x))\, t_{a,j}(\xi_c(x')) = \sum_{a=1}^{n} c^{(i-a)+(a-j)} t_{i,a}(x)\, t_{a,j}(x') = c^{i-j} t_{i,j}(xx').$$

□

The map

$$\Xi : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)^{\mathrm{op,cop}} \ ,\ x \mapsto \xi_{-1/q}(S(x))$$

is an isomorphism of Hopf algebras. Thus, the pullback of Ξ induces an isomorphism of Hopf algebras $U_q(\mathfrak{gl}_n)^\circ \to (U_q(\mathfrak{gl}_n)^\circ)^{\mathrm{op,cop}}$, given by $u \mapsto u \circ \Xi$. Set

$$\iota(u) := u \circ \Xi \quad \text{for } u \in U_q(\mathfrak{gl}_n)^\circ. \tag{49}$$

Lemma 3.4.4. *We have*

$$\iota(t_{i,j}) = \partial_{j,i} \quad \text{for } 1 \leqslant i,j \leqslant n. \tag{50}$$

In particular, the restriction of ι to $\mathscr{P}_{n\times n}$ is an isomorphism of bialgebras $\iota : \mathscr{P}_{n\times n} \to \mathscr{D}_{n\times n}^{\mathrm{op,cop}}$.

Proof. We need to verify $t_{i,j}(\Xi(x)) = \partial_{j,i}(x)$ for $x \in U_q(\mathfrak{gl}_n)$. It suffices to check the latter relation for the standard generators of $U_q(\mathfrak{gl}_n)$, and this special case follows from a direct calculation. □

Next note that the map

$$\underline{\iota} : \mathscr{P}_{n\times n} \to \mathscr{D}_{n\times n}^{\mathrm{op}} \ ,\ \underline{\iota}(u) := \iota(u)^\natural$$

is an isomorphism of bialgebras (indeed $\underline{\iota}(t_{i,j}) = \partial_{i,j}$). Let $\mathcal{R}_{\mathscr{D},\tau}$ and $\mathcal{L}_{\mathscr{D},\tau}$ denote the τ-twists of $\mathcal{R}_{\mathscr{D}}$ and $\mathcal{L}_{\mathscr{D}}$ (see remark 2.2.1), where we set $\tau(x) := S^{-1}(x)^\natural$ for $x \in U_q(\mathfrak{gl}_n)$. For $u \in \mathscr{P}_{n\times n}$ and $x \in U_q(\mathfrak{gl}_n)$ set

$$\mathcal{R}_{\mathscr{P}}(x)u := \underline{\iota}^{-1}(\mathcal{R}_{\mathscr{D},\tau}(x)\underline{\iota}(u)) \quad \text{and} \quad \mathcal{L}_{\mathscr{P}}(x)u := \underline{\iota}^{-1}(\mathcal{L}_{\mathscr{D},\tau}(x)\underline{\iota}(u)).$$

By remark 2.2.1(ii), $\mathcal{R}_{\mathscr{D},\tau}$ and $\mathcal{L}_{\mathscr{D},\tau}$ equip $\mathscr{D}_{n\times n}$ with $U_q(\mathfrak{gl}_n)^{\mathrm{cop}}$-module algebra structures. It follows that $\mathcal{R}_{\mathscr{P}}$ and $\mathcal{L}_{\mathscr{P}}$ equip $\mathscr{P}_{n\times n}$ with $U_q(\mathfrak{gl}_n)$-module structures. By a direct calculation

$$\mathcal{R}_{\mathscr{P}}(x)u = \sum \langle \iota(u_2), S^{-1}(x)\rangle u_1 \quad \text{and} \quad \mathcal{L}_{\mathscr{P}}(x)u = \sum \langle \iota(u_1), S^{-1}(x)^{\natural}\rangle u_2,$$

for $x \in U_q(\mathfrak{gl}_n)$ and $u \in \mathscr{P}_{n\times n}$. Using (49), lemma 3.3.1 and the relation $S^2 = \xi_{q^2}$ we obtain

$$\mathcal{R}_{\mathscr{P}}(x)u = \sum \langle u_2 \circ \xi_{-1/q}, x\rangle u_1 \quad \text{and} \quad \mathcal{L}_{\mathscr{P}}(x)u = \sum \langle (u_1 \circ \xi_{-q})^{\natural}, x\rangle u_2.$$

3.5. The algebras $\mathscr{P}$ and $\mathscr{D}$

Our next goal is to extend the constructions of section 3.4 to the $m\times n$ case.

Definition 3.5.1. Let m and n be positive integers and set $N := \max\{m,n\}$. We define the algebra $\mathscr{P} := \mathscr{P}_{m\times n}$ (respectively, $\mathscr{D} := \mathscr{D}_{m\times n}$) to be the subalgebra of $\mathscr{P}_{N\times N}$ (respectively, $\mathscr{D}_{N\times N}$) that is generated by the $t_{i,j}$ (respectively, the $\partial_{i,j}$) where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Note that by restricting the $U_q(\mathfrak{gl}_N)\otimes U_q(\mathfrak{gl}_N)$-module algebra structures on $\mathscr{P}_{N\times N}$ and $\mathscr{D}_{N\times N}$ we obtain U_{LR}-module algebra structures on $\mathscr{P}$ and $\mathscr{D}$. Let us describe these U_{LR}-module algebras more precisely. For convenience we first assume that $m \leqslant n$. Then the subalgebra of $U_q(\mathfrak{gl}_n)$ generated by $E_i, F_i, K_{\varepsilon_j}^{\pm 1}$ for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant m$ is isomorphic to $U_q(\mathfrak{gl}_m) \cong U_L$. With this identification of U_L with a subalgebra of $U_q(\mathfrak{gl}_n)$ we have the following lemma.

Lemma 3.5.2. *Suppose that $m \leqslant n$ and we identify U_L with a subalgebra of $U_q(\mathfrak{gl}_n)$ as above. Then $\mathcal{L}_{\mathscr{D}}(x)\mathscr{D} \subseteq \mathscr{D}$ and $\mathcal{L}_{\mathscr{P}}(x)\mathscr{P} \subseteq \mathscr{P}$ for $x \in U_L$.*

Proof. We only prove this for $\mathcal{L}_{\mathscr{D}}$ (for $\mathcal{L}_{\mathscr{P}}$ the proof is similar). Since $\mathscr{D}_{n\times n}$ is a U_L-module algebra, it suffices to check that if $i \leqslant m$ and x is one of the standard generators of U_L then $\mathcal{L}_{\mathscr{D}}(x)\partial_{i,j} \in \mathscr{D}$. If $x = E_k$ for $1 \leqslant k \leqslant m-1$ then

$$\mathcal{L}_{\mathscr{D}}(E_k)\partial_{i,j} = \sum_{a=1}^{n} \langle \partial_{i,a}, E_k^{\natural}\rangle \partial_{a,j} = \sum_{a=1}^{n} \langle \partial_{i,a}, qK_kF_k\rangle \partial_{a,j} = q\sum_{a=1}^{n}\sum_{b=1}^{n} \langle \partial_{i,b}, K_k\rangle\langle \partial_{b,a}, F_k\rangle \partial_{a,j}. \tag{51}$$

We have $\langle \partial_{b,a}, F_k\rangle = \langle e_b^*, \mathsf{E}_{k+1,k}e_a\rangle = [\![a,k]\!][\![b,k+1]\!]$. In particular, $\langle \partial_{b,a}, F_k\rangle = 0$ unless $a \leqslant m-1$. It follows that the right hand side of (51) is a linear combination of the $\partial_{a,j}$ where $a \leqslant m-1$, hence it lies in $\mathscr{D}$. The calculations for the cases $x = F_k$ for $1 \leqslant k \leqslant m-1$ and $x = K_{\varepsilon_k}^{\pm 1}$ for $1 \leqslant k \leqslant m$ are similar. ☐

Lemma 3.5.2 implies that $\mathscr{P}$ and $\mathscr{D}$ are U_{LR}-stable subspaces of $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$, where we consider

$$U_{LR} = U_q(\mathfrak{gl}_m)\otimes U_q(\mathfrak{gl}_n) \cong U_L \otimes U_R$$

as a subalgebra of $U_q(\mathfrak{gl}_n)\otimes U_q(\mathfrak{gl}_n)$ via the aforementioned embedding $U_L \hookrightarrow U_q(\mathfrak{gl}_n)$. Thus, $\mathscr{P}$ and $\mathscr{D}$ inherit U_{LR}-module algebra structures from $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$.

Henceforth we mostly drop the symbols $\mathcal{L}_{\mathscr{P}}$, $\mathcal{L}_{\mathscr{D}}$, $\mathcal{R}_{\mathscr{P}}$ and $\mathcal{R}_{\mathscr{D}}$ from our notation. Instead, we use the notation

$$(x\otimes y)\cdot u$$

to denote the action of $x\otimes y\in U_{LR}$ on $u\in\mathscr{P}$ (or $u\in\mathscr{D}$). The actions of $x\otimes y\in U_L\otimes U_R$ on $u\in\mathscr{P}$ and on $v\in\mathscr{D}$ are given explicitly by the formulas

$$(x\otimes y)\cdot u=\sum\langle\iota(u_1),S^{-1}(x)^{\natural}\rangle\langle\iota(u_3),S^{-1}(y)\rangle u_2=\sum\langle(u_1\circ\xi_{-q})^{\natural},x\rangle\langle u_3\circ\xi_{-1/q},y\rangle u_2 \tag{52}$$

and

$$(x\otimes y)\cdot v=\sum\langle v_1,x^{\natural}\rangle\langle v_3,y\rangle v_2=\sum\langle v_1^{\natural},x\rangle\langle v_3,y\rangle v_2, \tag{53}$$

with $(\Delta_{\mathscr{P}}\otimes 1)\circ\Delta_{\mathscr{P}}(u)=\sum u_1\otimes u_2\otimes u_3$ and $(\Delta_{\mathscr{D}}\otimes 1)\circ\Delta_{\mathscr{D}}(v)=\sum v_1\otimes v_2\otimes v_3$ in Sweedler notation, where $\Delta_{\mathscr{P}}$ and $\Delta_{\mathscr{D}}$ denote the coproducts of $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$, respectively.

Remark 3.5.3. In the case $m>n$ the construction of the U_{LR}-action is the same, except that we embed $U_q(\mathfrak{gl}_m)\otimes U_q(\mathfrak{gl}_n)$ in $U_q(\mathfrak{gl}_m)\otimes U_q(\mathfrak{gl}_m)$. However, formulas (52) and (53) remain the same.

Definition 3.5.4. The map $\phi_U: U_{LR}\to\mathrm{End}_{\Bbbk}(\mathscr{P})$ is the homomorphism of algebras induced by the action (52).

3.6. U_{LR}-module decomposition of $\mathscr{P}$ and $\mathscr{D}$

For any integer partition λ satisfying $\ell(\lambda)\leqslant n$, where $\ell(\lambda)$ denotes the length of λ, let V_λ denote the irreducible finite dimensional U_R-module of type $(1,\dots,1)$ with highest weight $q^{\sum_i\lambda_i\varepsilon_i}$ (with respect to Δ_n^+). If λ satisfies $\ell(\lambda)\leqslant m$ we use the same notation V_λ to denote the analogously defined module of U_L.

The algebras $\mathscr{P}$ and $\mathscr{D}$ are naturally graded by degree of monomials. For $d\geqslant 0$ let $\mathscr{P}^{(d)}$ (respectively, $\mathscr{D}^{(d)}$) denote the graded component of degree d of $\mathscr{P}$ (respectively, $\mathscr{D}$). Furthermore, let $\Lambda_{d,r}$ be the set of integer partitions λ such that $\ell(\lambda)\leqslant d$ and $|\lambda|=r$, where $|\lambda|$ denotes the size of λ. The following proposition is well known and its proof can be found for example in [NYM93, Ta92, Zh02].

Proposition 3.6.1. *Set $d:=\min\{m,n\}$. We have isomorphisms of U_{LR}-modules*

$$\mathscr{P}^{(r)}\cong\bigoplus_{\lambda\in\Lambda_{d,r}}V_\lambda^*\otimes V_\lambda^*\quad\textit{and}\quad\mathscr{D}^{(r)}\cong\bigoplus_{\lambda\in\Lambda_{d,r}}V_\lambda\otimes V_\lambda.$$

Remark 3.6.2. The action of $U_L\otimes U_R$ on the generators of $\mathscr{P}$ and $\mathscr{D}$ can be computed explicitly. For the subalgebra $U_R\cong 1\otimes U_R$ of U_{LR}, the action is given by

$$\begin{aligned} E_k\cdot\partial_{i,j}&=[\![k+1,j]\!]\,\partial_{i,k}, & F_k\cdot\partial_{i,j}&=[\![k,j]\!]\,\partial_{i,k+1}, & K_{\varepsilon_k}\cdot\partial_{i,j}&=q^{[\![k,j]\!]}\partial_{i,j},\\ E_k\cdot t_{i,j}&=-[\![k,j]\!]\,q^{-1}t_{i,k+1}, & F_k\cdot t_{i,j}&=-[\![k+1,j]\!]\,qt_{i,k}, & K_{\varepsilon_k}\cdot t_{i,j}&=q^{-[\![k,j]\!]}t_{i,j}, \end{aligned}$$

where $1\leqslant k\leqslant n-1$, $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$. For $U_L\cong U_L\otimes 1$ the formulas are similar but the action occurs in the first index (thus, they are obtained by replacing $\partial_{i,j}$ by $\partial_{j,i}$ and $t_{i,j}$ by $t_{j,i}$).

3.7. The algebras $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ and $\mathscr{P}\mathscr{D}$

For $n\geqslant 1$ let $\mathcal{R}^{(n)}$ and $\underline{\mathcal{R}}^{(n)}$ be as in definition 3.2.1.

Definition 3.7.1. We set $\mathcal{C}_L := \mathcal{C}^{(m)}$ and $\mathcal{C}_R := \mathcal{C}^{(n)}$. Furthermore, we set

$$\mathcal{R}_L := \mathcal{R}^{(m)} \quad , \quad \underline{\mathcal{R}}_L := \underline{\mathcal{R}}^{(m)} \quad , \quad \mathcal{R}_R := \mathcal{R}^{(n)} \quad , \quad \underline{\mathcal{R}}_R := \underline{\mathcal{R}}^{(n)}.$$

We define braidings $\check{\mathcal{R}}_L$ and $\check{\underline{\mathcal{R}}}_L$ on $\mathcal{C}_L$, and $\check{\mathcal{R}}_R$ and $\check{\underline{\mathcal{R}}}_L$ on $\mathcal{C}_R$ as in (41).

From proposition 3.2.2 it follows that $(U_L, \mathcal{C}_L, \check{\underline{\mathcal{R}}}_L)$ and $(U_R, \mathcal{C}_R, \check{\underline{\mathcal{R}}}_R)$ are locally finite braided triples. Thus the product (in the sense of definition 2.5.3) of $(U_L, \mathcal{C}_L, \check{\underline{\mathcal{R}}}_L)$ and $(U_R, \mathcal{C}_R, \check{\underline{\mathcal{R}}}_R)$ is also a locally finite braided triple of the form $(U_{LR}, \mathcal{C}_{LR}, \check{\underline{\mathcal{R}}}_{LR})$, where $\check{\underline{\mathcal{R}}}_{LR}$ is defined as in (24). Furthermore, proposition 3.6.1 implies that $\mathscr{P}, \mathscr{D} \in \mathrm{Obj}(\mathcal{C}_{LR})$.

Recall that $\mathscr{D}^{(1)}$ and $\mathscr{P}^{(1)}$ are U_{LR}-modules. Let $\psi_\circ : \mathscr{D}^{(1)} \times \mathscr{P}^{(1)} \to \Bbbk$ be the U_{LR}-invariant $\Bbbk$-bilinear form, in the sense of (19), that is defined by

$$\psi_\circ(\partial_{i,j}, t_{k,l}) := [\![i,k]\!]\,[\![j,l]\!] \quad \text{for } 1 \leqslant i,k \leqslant m, \text{and } 1 \leqslant j,l \leqslant n.$$

Definition 3.7.2. We define the algebras $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ and $\mathscr{P}\mathscr{D}$ by

$$\mathscr{P}\mathscr{D}^{\mathrm{gr}} := \mathscr{P} \otimes_{\check{\mathcal{R}}} \mathscr{D} \quad \text{and} \quad \mathscr{P}\mathscr{D} := \mathscr{P} \otimes_{\check{\mathcal{R}},\psi_\circ} \mathscr{D}, \tag{54}$$

according to definition 2.4.2 and definition 2.4.9, with $A := \mathscr{P}$, $E_A := \mathscr{P}^{(1)}$, $B := \mathscr{D}$, $E_B := \mathscr{D}^{(1)}$, $\psi := \psi_\circ$, and $\check{\mathcal{R}} := \check{\underline{\mathcal{R}}}_{LR}$.

It turns out that there is an equivalent description of $\mathscr{P}\mathscr{D}$ and $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ by generators and relations. Recall the notation $[\![a,b]\!]$ that was defined in (36). We set

$$[\![a,b]\!]_q := \begin{cases} q & \text{if } a = b, \\ q - q^{-1} & \text{if } a \neq b. \end{cases}$$

Definition 3.7.3. The algebra $\mathscr{P}\mathscr{D}$ is generated by $2mn$ generators $t_{i,j}$ and $\partial_{i,j}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, modulo the relations (R1), (R2), (R1$'$), (R2$'$) of section 3.4 and the relations

$$\partial_{\bar{a}_1,\bar{a}_2} t_{a_1,a_2} = [\![a_1,\bar{a}_1]\!]\,[\![a_2,\bar{a}_2]\!] + \sum_{b_1 \geqslant a_1} \sum_{\bar{b}_1 \geqslant \bar{a}_1} \sum_{b_2 \geqslant a_2} \sum_{\bar{b}_2 \geqslant \bar{a}_2} (\diamondsuit_1 + \square_1)(\diamondsuit_2 + \square_2)\, t_{b_1,b_2} \partial_{\bar{b}_1,\bar{b}_2}, \tag{55}$$

where

$$\diamondsuit_i := [\![a_i,\bar{a}_i]\!]\,\big[\![b_i,\bar{b}_i]\!\big]\,[\![a_i,b_i]\!]_q \quad \text{and} \quad \square_i := (1 - [\![a_i,\bar{a}_i]\!])\,[\![a_i,b_i]\!]\,\big[\![\bar{a}_i,\bar{b}_i]\!\big].$$

The algebra $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ is also generated by $2mn$ generators $t_{i,j}$ and $\partial_{i,j}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ modulo the same relations, except that $[\![a_1,\bar{a}_1]\!][\![a_2,\bar{a}_2]\!]$ does not occur on the right hand side of (55).

Remark 3.7.4. The relation (55) of $\mathscr{P}\mathscr{D}$ can be written more explicitly as the relations (R3)–(R6) below:

(R3) $\partial_{c,b} t_{d,a} = t_{d,a} \partial_{c,b}$ if $b \neq a$ and $c \neq d$.

(R4) $\partial_{c,b} t_{c,a} = q t_{c,a} \partial_{c,b} + \sum_{c' > c} (q - q^{-1}) t_{c',a} \partial_{c',b}$ if $b \neq a$.

(R5) $\partial_{c,a} t_{d,a} = q t_{d,a} \partial_{c,a} + \sum_{a' > a} (q - q^{-1}) t_{d,a'} \partial_{c,a'}$ if $c \neq d$.

(R6) $\partial_{c,d} t_{c,d} = 1 + \sum_{c' \geqslant c} \sum_{d' \geqslant d} q^{[\![c',c]\!] + [\![d',d]\!]} (q - q^{-1})^{2 - [\![c',c]\!] - [\![d',d]\!]} t_{c',a'} \partial_{c',a'}$.

For $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ the relation (55) has the same explicit form, except that (R6) should be replaced by

$$(\mathrm{R}6') \quad \partial_{c,d}t_{c,d} = \sum_{c'\geqslant c}\sum_{d'\geqslant d} q^{[\![c',c]\!]+[\![d',d]\!]}(q-q^{-1})^{2-[\![c',c]\!]-[\![d',d]\!]} t_{c',d'}\partial_{c',d'}.$$

Proposition 3.7.5. *definition 3.7.2 and definition 3.7.3 are equivalent.*

Proof. We just need to explain how to compute the mixed relations (17) and (20). As a U_{LR}-module,

$$\mathscr{D}^{(1)} \cong \check{V}^{(m)} \otimes \check{V}^{(n)} \quad \text{and} \quad \mathscr{P}^{(1)} \cong V^{(m)} \otimes V^{(n)},$$

where the isomorphisms are $\partial_{i,j} \mapsto e_i \otimes e_j$ and $t_{i,j} \mapsto e_i \otimes e_j$. By a direct calculation using definition 3.2.1 we obtain

$$(\underline{\mathcal{R}}_L)_{\check{V}^{(m)},V^{(m)}} = q\sum_{1\leqslant i\leqslant m} \mathsf{E}_{i,i}\otimes\mathsf{E}_{i,i} + \sum_{1\leqslant i\neq j\leqslant m} \mathsf{E}_{i,i}\otimes\mathsf{E}_{j,j} + \left(q-q^{-1}\right)\sum_{1\leqslant i<j\leqslant m} \mathsf{E}_{j,i}\otimes\mathsf{E}_{j,i}.$$

The formula for $(\underline{\mathcal{R}}_R)_{\check{V}^{(n)},V^{(n)}}$ is similar, with m replaced by n. The mixed relations (20) of $\mathscr{P}\mathscr{D}$ and (17) of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ can now be computed explicitly based on definitions 2.4.2 and 2.4.9. □

3.8. *Bases of monomials for $\mathscr{P}\mathscr{D}$ and $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$*

Consider the monomials

$$t_{1,1}^{a_{1,1}}\cdots t_{1,n}^{a_{1,n}}\cdots t_{m,1}^{a_{m,1}}\cdots t_{m,n}^{a_{m,n}}\partial_{m,n}^{b_{m,n}}\cdots\partial_{m,1}^{b_{m,1}}\cdots\partial_{1,n}^{b_{1,n}}\cdots\partial_{1,1}^{b_{1,1}}, \quad a_{i,j},b_{i,j}\in\mathbb{Z}^{\geqslant 0}. \tag{56}$$

The expression (56) makes sense both as an element of $\mathscr{P}\mathscr{D}$ and an element of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$.

Proposition 3.8.1. *The monomials (56) form a $\Bbbk$-basis of $\mathscr{P}\mathscr{D}$.*

Proof. By a standard straightening argument we can show that by using the relations (R1), (R1′), (R2), (R2′) and (R3)–(R6) any product of the $t_{i,j}$ and the $\partial_{i,j}$ can be expressed as a linear combination of the monomials (56). The fact that the latter monomials are indeed linearly independent follows from Bergman's Diamond Lemma and some straightforward (although tedious) computations. This was also pointed out in [SSV04, section 10]. In [LSS22a] we give a more conceptual proof of this assertion using the theory of PBW deformations of quadratic algebras. □

Proposition 3.8.2. *The algebra $\mathscr{P}$ has a basis consisting of monomials (56) where $b_{i,j}=0$ for all i,j. The algebra $\mathscr{D}$ has a basis consisting of monomials (56) where $a_{i,j}=0$ for all i,j.*

Proof. This follows from proposition 3.8.1. It is also proved for example in [NYM93, theorem 1.4]. □

By proposition 2.4.8 the algebra $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ is a quotient of the free algebra on $2mn$ generators $t_{i,j}$ and $\partial_{i,j}$. Note that by a slight abuse of notation we use the same notation for generators of $\mathscr{P}\mathscr{D}$ and $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$. In the next proposition we describe a basis for this quotient.

Proposition 3.8.3. *The monomials (56) form a basis of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$.*

Proof. This follows from the vector space decomposition $\mathscr{P}\mathscr{D}^{\mathrm{gr}} = \mathscr{P}\otimes\mathscr{D}$ and proposition 3.8.2. □

Remark 3.8.4. From the results of this subsection it follows that $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ has two realizations:

(i) According to definition 3.7.2 we have $\mathscr{P}\mathscr{D}^{\mathrm{gr}} \cong \mathscr{P} \otimes \mathscr{D}$ as a $\mathbb{k}$-vector space. Thus, $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ is generated by $2mn$ generators $t_{i,j} \otimes 1$ and $1 \otimes \partial_{i,j}$.
(ii) By proposition 2.4.8 we can realize $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ as a quotient of the free $\mathbb{k}$-algebra generated by $2mn$ generators: the $t_{i,j}$ and the $\partial_{i,j}$.

Definition 3.8.5. Define a total order $\prec$ on the set of pairs (i,j) with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ as follows: we set $(i,j) \prec (i',j')$ if either $i+j < i'+j'$, or $i+j = i'+j'$ and $i < i'$.

Remark 3.8.6. The algebra $\mathscr{P}\mathscr{D}$ has another basis consisting of monomials of the form

$$\left(\prod_{i,j} t_{i,j}^{a_{i,j}}\right)\left(\prod_{i,j} \partial_{i,j}^{b_{i,j}}\right), \tag{57}$$

where the $\partial_{i,j}$ (respectively, the $t_{i,j}$) occur in ascending (respectively, descending) order relative to the total order $\prec$. This can be deduced from proposition 3.8.1. Indeed by an elementary argument one can show that any monomial of the $t_{i,j}$ of total degree d that is sorted in the order given in proposition 3.8.1 can be expressed as a linear combination of monomials of the $t_{i,j}$ of total degree d that are sorted in the order given in (57). A similar assertion holds for monomials in the $\partial_{i,j}$. Thus the monomials of the form (57) span $\mathscr{P}\mathscr{D}$. A dimension counting argument implies that the latter monomials also form a basis.

By an analogous reasoning we can also show that $\mathscr{P}\mathscr{D}$ has a basis that consists of the monomials

$$t_{m,n}^{a_{m,n}} \cdots t_{m,1}^{a_{m,1}} \cdots t_{1,n}^{a_{1,1}} \cdots t_{1,1}^{a_{1,1}} \partial_{1,1}^{b_{1,1}} \cdots \partial_{1,n}^{b_{1,n}} \cdots \partial_{m,1}^{b_{m,1}} \cdots \partial_{m,n}^{b_{m,n}}, \quad a_{i,j}, b_{i,j} \in \mathbb{Z}^{\geqslant 0}. \tag{58}$$

Here the $\partial_{i,j}$ (respectively, the $t_{i,j}$) are sorted according to the lexicographic order (respectively, the reverse lexicographic order) on indices.

3.9. The algebras $\mathscr{A}_{k,l,n}$ and $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$

In this subsection we consider two families of algebras, the $\mathscr{A}_{k,l,n}$ and the $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$, that slightly generalize $\mathscr{P}\mathscr{D}$ and $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$.

Definition 3.9.1. Fix integers $k,l,n \geqslant 1$ and set $m := \max\{k,l\}$. Let $\tilde{t}_{i,j}$ and $\tilde{\partial}_{i,j}$ be as in (5) where $a = m$ and $b = n$. We define $\mathscr{A}_{k,l,n}$ (respectively, $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$) to be the subalgebra of $\mathscr{P}\mathscr{D} = \mathscr{P}\mathscr{D}_{m\times n}$ (respectively, $\mathscr{P}\mathscr{D}^{\mathrm{gr}} \cong \mathscr{P} \otimes \mathscr{D}$) that is generated by the $\tilde{t}_{i,j}$ and the $\tilde{\partial}_{i',j}$ (respectively, the $\tilde{t}_{i,j} \otimes 1$ and the $1 \otimes \tilde{\partial}_{i',j}$) where

$$1 \leqslant i \leqslant k \quad , \quad 1 \leqslant i' \leqslant l \quad \text{and} \quad 1 \leqslant j \leqslant n. \tag{59}$$

Proposition 3.9.2. *The algebras $\mathscr{A}_{k,l,n}$ and $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ have the following presentations:*

(i) $\mathscr{A}_{k,l,n}$ is isomorphic to the quotient of the free $\mathbb{k}$-algebra generated by the symbols $t_{i,j}$ and $\partial_{i',j}$ with i, i', j satisfying (59), modulo the relations (R1), (R2), (R1′), (R2′) of section 3.4 and the relations (R3)–(R6) of remark 3.7.4.
(ii) $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ is isomorphic to the quotient of the free $\mathbb{k}$-algebra generated by the symbols $t_{i,j}$ and $\partial_{i',j}$ with i, i', j satisfying (59), modulo the relations (R1), (R2), (R1′), (R2′) of section 3.4 and the relations (R3)–(R5) and (R6′) of remark 3.7.4.

Proof. (i) Denote the quotient of the free algebra by $\mathscr{F}_{k,l,n}$. Since $\mathscr{A}_{k,l,n}$ is a subalgebra of $\mathscr{P}\mathscr{D} = \mathscr{P}\mathscr{D}_{m\times n}$ for $m := \max\{k,l\}$, from the explicit description of the relations of $\mathscr{P}\mathscr{D}$ it follows that there exists a natural epimorphism $\mathsf{f}_{k,l,n} : \mathscr{F}_{k,l,n} \to \mathscr{A}_{k,l,n}$ that is uniquely defined by the assignments $t_{i,j} \mapsto t_{m-k+i,j}$ and $\partial_{i',j} \mapsto \partial_{m-l+i',j}$. A standard straightening argument proves that every element of $\mathscr{F}_{k,l,n}$ is a linear combination of monomials of the form (56). proposition 3.8.1 implies that $\mathsf{f}_{k,l,n}$ maps the latter monomials to a linearly independent set of elements of $\mathscr{A}_{k,l,n}$. Thus $\mathsf{f}_{k,l,n}$ is an isomorphism.

(ii) Similar to the proof of (i), with proposition 3.8.1 replaced by proposition 3.8.3. □

Definition 3.9.3. For any $1 \leqslant r \leqslant n$ we can identify $U_q(\mathfrak{gl}_r)$ with a Hopf subalgebra of $U_q(\mathfrak{gl}_n)$ via the monomorphism of associative algebras

$$\boldsymbol{\kappa}_{r,n} : U_q(\mathfrak{gl}_r) \to U_q(\mathfrak{gl}_n), \tag{60}$$

defined by $\boldsymbol{\kappa}_{r,n}(E_i) := E_{i+n-r}$, $\boldsymbol{\kappa}_{r,n}(F_i) := F_{i+n-r}$, and $\boldsymbol{\kappa}_{r,n}(K^{\pm 1}_{\varepsilon_i}) := K^{\pm 1}_{\varepsilon_{i+n-r}}$.

In the next proposition we establish the existence of the map (4). Recall from remark 3.8.4(ii) that we consider both $\mathscr{P}\mathscr{D}$ and $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ as algebras generated by $2mn$ generators $t_{i,j}$ and $\partial_{i,j}$.

Proposition 3.9.4. *Fix $1 \leqslant m' \leqslant m$ and $1 \leqslant n' \leqslant n$. Let $\tilde{t}_{i,j}, \tilde{\partial}_{i,j} \in \mathscr{P}\mathscr{D}_{m'\times n'}$ (respectively, $\tilde{t}_{i,j}, \tilde{\partial}_{i,j} \in \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m'\times n'}$) for $1 \leqslant i \leqslant m'$ and $1 \leqslant j \leqslant n'$ be as in (5) for $a := m'$ and $b := n'$, that is*

$$\tilde{t}_{i,j} := t_{m'+1-i,n'+1-j} \quad \textit{and} \quad \tilde{\partial}_{i,j} := \partial_{m'+1-i,n'+1-j}.$$

Also, let $\tilde{t}_{i,j}, \tilde{\partial}_{i,j} \in \mathscr{P}\mathscr{D}_{m\times n}$ (respectively, $\tilde{t}_{i,j}, \tilde{\partial}_{i,j} \in \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m\times n}$) for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ be as in (5) for $a := m$ and $b := n$, that is

$$\tilde{t}_{i,j} := t_{m+1-i,n+1-j} \quad \textit{and} \quad \tilde{\partial}_{i,j} := \partial_{m+1-i,n+1-j}.$$

Then the following assertions hold.

(i) The assignments $\tilde{t}_{i,j} \mapsto \tilde{t}_{i,j}$ and $\tilde{\partial}_{i,j} \mapsto \tilde{\partial}_{i,j}$ for $1 \leqslant i \leqslant m'$ and $1 \leqslant j \leqslant n'$ define unique embeddings of algebras

$$\boldsymbol{e} := \boldsymbol{e}^{m\times n}_{m'\times n'} : \mathscr{P}\mathscr{D}_{m'\times n'} \to \mathscr{P}\mathscr{D}_{m\times n} \quad \textit{and} \quad \boldsymbol{e}^{\mathrm{gr}} := (\boldsymbol{e}^{\mathrm{gr}})^{m\times n}_{m'\times n'} : \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m'\times n'} \to \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m\times n}.$$

(ii) If we identify $U_q(\mathfrak{gl}_{m'}) \otimes U_q(\mathfrak{gl}_{n'})$ with a subalgebra of $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$ via $\boldsymbol{\kappa}_{m',m} \otimes \boldsymbol{\kappa}_{n',n}$ then the maps $\boldsymbol{e}$ and $\boldsymbol{e}^{\mathrm{gr}}$ are $U_q(\mathfrak{gl}_{m'}) \otimes U_q(\mathfrak{gl}_{n'})$-equivariant.

Proof. We only give the details of the proofs of these assertions for $\mathscr{P}\mathscr{D}$. The arguments for $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ are analogous.

(i) From definition 3.7.3 and remark 3.7.4 it follows that the generators $\tilde{t}_{i,j}$ and $\tilde{\partial}_{i,j}$ of $\mathscr{P}\mathscr{D}_{m'\times n'}$ and $\mathscr{P}\mathscr{D}_{m\times n}$ satisfy identical relations. It follows that there exists a homomorphism of algebras $\mathscr{P}\mathscr{D}_{m'\times n'} \to \mathscr{P}\mathscr{D}_{m\times n}$. By proposition 3.8.1 the latter map takes a basis of $\mathscr{P}\mathscr{D}_{m'\times n'}$ to a basis of $\mathscr{P}\mathscr{D}_{m\times n}$, hence it is an injection. The uniqueness assertion is trivial.

(ii) We give the proof for $\boldsymbol{e}$ only, since the proof for $\boldsymbol{e}^{\mathrm{gr}}$ is similar. Since $\mathscr{P}\mathscr{D}_{m'\times n'}$ and $\mathscr{P}\mathscr{D}_{m\times n}$ are module algebras it suffices to verify equivariance for standard generators of $U_q(\mathfrak{gl}_{m'})$ and $U_q(\mathfrak{gl}_{n'})$ on the $\tilde{t}_{i,j}$ and the $\tilde{\partial}_{i,j}$. This can be done using the explicit formulas of remark 3.6.2. □

3.10. The action of $\mathscr{PD}$ on $\mathscr{P}$ and the map ϕ_{PD}

Recall that by proposition 2.4.10, $\mathscr{PD}$ is a U_{LR}-module algebra. We denote the action of $x \in U_{LR}$ on $D \in \mathscr{PD}$ by $x \cdot D$.

Let $\mathscr{I}$ denote the left ideal of $\mathscr{PD}$ that is generated by $\mathscr{D}^{(1)}$. By proposition 3.8.1 we have a U_{LR}-invariant decomposition

$$\mathscr{PD} \cong \mathscr{I} \oplus \mathscr{P}.$$

This decomposition equips $\mathscr{P} \cong \mathscr{PD}/\mathscr{I}$ with a $\mathscr{PD}$-module structure given by

$$\mathscr{PD} \otimes \mathscr{P} \to \mathscr{P}\ ,\ D \otimes (f + \mathscr{I}) \mapsto D \cdot f \quad \text{for } D \in \mathscr{PD} \text{and } f \in \mathscr{P}, \tag{61}$$

where $D \cdot f := (Df) + \mathscr{I}$.

Definition 3.10.1. The map $\phi_{PD} : \mathscr{PD} \to \mathrm{End}_{\Bbbk}(\mathscr{P})$ is the homomorphism of algebras induced by (61).

To simplify our notation, henceforth for $X \in \mathscr{PD}$ and $f \in \mathscr{P}$ we write $X \cdot f$ instead of $\phi_{PD}(X)f$.

Lemma 3.10.2. *The map (61) is a U_{LR}-module homomorphism.*

Proof. This is a consequence of the following general fact: let H be a Hopf algebra, A be an H-module algebra, and $I \subseteq A$ be an H-stable left ideal of A. Then the canonical A-module structure map $A \otimes A/I \to A/I$ is an H-module homomorphism. □

3.11. $\mathscr{P}$ is a faithful $\mathscr{PD}$-module

The goal of this subsection is to provide a purely algebraic proof of proposition 3.11.4. This proposition is also proved in [SSV04, theorem 2.6] using analytic tools.

Lemma 3.11.1. *Let $\mathscr{P}^{(\leqslant k)} := \bigoplus_{i=0}^{k} \mathscr{P}^{(i)}$ for $k \geqslant 0$. Then $\partial_{i,j} \cdot \mathscr{P}^{(\leqslant k)} \subseteq \mathscr{P}^{(\leqslant k-1)}$.*

Proof. Follows by induction on k and the mixed relations (R3)–(R6) in section 3.7. □

Recall the total order $\prec$ on the set of pairs (i,j) with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ from definition 3.8.5.

Lemma 3.11.2. *Assume that $(i_r,j_r) \prec (i,j)$ for $1 \leqslant r \leqslant k$. Then $\partial_{i,j} \cdot (t_{i_1,j_1} \cdots t_{i_k,j_k}) = 0$.*

Proof. We use induction on k. From $(i_1,j_1) \prec (i,j)$ it follows that either $i > i_1$ or $j > j_1$. If $i > i_1$ then by the mixed relations (R3) or (R5) we have

$$\partial_{i,j} t_{i_1,j_1} \cdots t_{i_k,j_k} = c_1 t_{i_1,j_1} \partial_{i,j} t_{i_2,j_2} \cdots t_{i_k,j_k} + \delta_{j,j_1} c_2 \sum_{j'>j} t_{i_1,j'} \partial_{i,j'} t_{i_2,j_2} \cdots t_{i_k,j_k},$$

for some $c_1, c_2 \in \Bbbk$. The claim now follows from the induction hypothesis, because $i + j' > i + j$ and therefore $(i,j) \prec (i,j')$. When $j > j_1$ the argument is similar. □

For $a \in \mathbb{Z}$ we set

$$\mathbf{c}(a) := \begin{cases} \sum_{i=0}^{a} q^{2i} & \text{if } a \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{62}$$

Lemma 3.11.3. *Assume that* $(i_r, j_r) \prec (i,j)$ *for* $1 \leqslant r \leqslant k$*. Then*

$$\partial_{i,j} \cdot \left(t_{i,j}^a t_{i_1,j_1} \cdots t_{i_k,j_k}\right) = \mathbf{c}\,(a-1)\, t_{i,j}^{a-1} t_{i_1,j_1} \cdots t_{i_k,j_k} \quad \textit{for } a \geqslant 1.$$

Proof. The mixed relation (R6) implies

$$\begin{aligned}\partial_{i,j} t_{i,j}^a t_{i_1,j_1} \cdots t_{i_k,j_k} = {} & t_{i,j}^{a-1} t_{i_1,j_1} \cdots t_{i_k,j_k} + q^2 t_{i,j} \partial_{i,j} t_{i,j}^{a-1} t_{i_1,j_1} \cdots t_{i_k,j_k} \\ & + \left(q^2-1\right) \sum_{i'>i} t_{i',j} \partial_{i',j} t_{i,j}^{a-1} t_{i_1,j_1} \cdots t_{i_k,j_k} + \left(q^2-1\right) \sum_{j'>j} t_{i,j'} \partial_{i,j'} t_{i,j}^{a-1} t_{i_1,j_1} \cdots t_{i_k,j_k} \\ & + \left(q-q^{-1}\right)^2 \sum_{i'>i, j'>j} t_{i',j'} \partial_{i',j'} t_{i,j}^{a-1} t_{i_1,j_1} \cdots t_{i_k,j_k}.\end{aligned}$$

Since $\min\{i'+j, i+j', i'+j'\} > i+j$, by lemma 3.11.2 the sums on the second and the third line lie in the ideal $\mathscr{I}$. The assertion follows by induction on a. □

Proposition 3.11.4. $\mathscr{P}$ *is a faithful* $\mathscr{P}\mathscr{D}$*-module.*

Proof. Let $D \in \mathscr{P}\mathscr{D}$ and assume that $D \neq 0$. Then $D = \sum_{d \geqslant 0} D_d$ where each D_d is a linear combination of monomials of the form (57) with $\sum_{i,j} b_{i,j} = d$. Set $d_\circ := \min\{d : D_d \neq 0\}$. By lemma 3.11.1, for $f \in \mathscr{P}^{(d_\circ)}$ we have $D \cdot f = D_{d_\circ} \cdot f$. Let $\mathcal{T}$ denote the set of all the mn-tuples $\mathsf{b} := (b_{i,j})$ for which a monomial of the form (57) occurs in $D_{d_\circ}$ with a nonzero coefficient. We sort the components of the $\mathsf{b} := (b_{i,j})$ according to $\prec$ on the pairs (i,j). In other words, we assume that $\mathsf{b} := (b_{1,1}, b_{1,2}, b_{2,1}, \ldots, b_{m-1,n}, b_{m,n-1}, b_{m,n})$. Let $\tilde{\mathsf{b}} := (\tilde{b}_{i,j})$ be the minimum of $\mathcal{T}$ in the reverse lexicographic order. Thus, we have

$$\tilde{b}_{m,n} = \min\{b_{m,n} : (b_{i,j}) \in \mathcal{T}\},$$

then also $\tilde{b}_{m-1,n} = \min\{b_{m-1,n} : (b_{i,j}) \in \mathcal{T} \text{and } b_{m,n} = \tilde{b}_{m,n}\}$, and so on. From lemmas 3.11.2 and 3.11.3 it follows that $D_{d_\circ} \cdot \prod_{i,j} t^{\tilde{b}_{i,j}} \neq 0$. □

3.12. Two U_{LR}-actions on $\mathscr{P}\mathscr{D}$ are identical

By proposition 3.11.4 the map ϕ_{PD} is an injection and consequently we can consider $\mathscr{P}\mathscr{D}$ as a subalgebra of $\mathrm{End}_{\Bbbk}(\mathscr{P})$. Thus according to lemma 2.3.1 there exists another action of U_{LR} on elements of $\mathscr{P}\mathscr{D}$. We temporarily denote this action by $x \bullet D$ for $x \in U_{LR}$ and $D \in \mathscr{P}\mathscr{D}$. In the following proposition, we show that the latter action is identical to the action that is defined in the beginning of section 3.10.

Proposition 3.12.1. $x \bullet D = x \cdot D$ *for* $x \in U_{LR}$ *and* $D \in \mathscr{P}\mathscr{D}$*.*

Proof. By lemma 3.10.2, for $f \in \mathscr{P}$ we have

$$(x \bullet D) \cdot f = \sum x_1 \cdot (D \cdot (S(x_2) \cdot f)) = \sum (x_1 \cdot D) \cdot (x_2 \cdot (S(x_3) \cdot f))$$
$$= \sum (x_1 \cdot D) \cdot (\epsilon(x_2) f) = \sum (x_1 \epsilon(x_2) \cdot D) \cdot f = (x \cdot D) \cdot f.$$

Since $\mathscr{P}$ is a faithful module over $\mathrm{End}_{\Bbbk}(\mathscr{P})$, it follows that $x \bullet D = x \cdot D$. □

Henceforth we only use the notation $x \cdot D$ to denote the U_{LR}-action on $\mathscr{P}\mathscr{D}$.

3.13. *The maps $\mathsf{P}_{k,l,n}$*

Recall that $\mathscr{P}\mathscr{D}^{\mathrm{gr}} \cong \mathscr{P} \otimes \mathscr{D}$ as a vector space. Let

$$\mathsf{P} : \mathscr{P}\mathscr{D}^{\mathrm{gr}} \to \mathscr{P}\mathscr{D} \tag{63}$$

be the linear map uniquely defined by $\mathsf{P}(a \otimes b) := ab$ for $a \in \mathscr{P}$ and $b \in \mathscr{D}$.

Proposition 3.13.1. *The map P is an isomorphism of U_{LR}-modules.*

Proof. The map P is identical to the map (22) when $A := \mathscr{P}$, $B := \mathscr{D}$ and $H := U_{LR}$. Hence by remark 2.4.11 it is a homomorphism of U_{LR}-modules. From proposition 3.8.1 and proposition 3.8.3 it follows that P maps a basis of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ to a basis of $\mathscr{P}\mathscr{D}$, hence it is indeed an isomorphism of U_{LR}-modules. □

Definition 3.13.2. Given $k, l, n \geqslant 1$, we set $m := \max\{k, l\}$ and define the map

$$\mathsf{P}_{k,l,n} : \mathscr{A}^{\mathrm{gr}}_{k,l,n} \to \mathscr{A}_{k,l,n}\ ,\ D \mapsto \mathsf{P}(D), \tag{64}$$

where $\mathsf{P} : \mathscr{P}\mathscr{D}^{\mathrm{gr}} \to \mathscr{P}\mathscr{D}$ is as in (63).

For $r, s \in \mathbb{Z}^{\geqslant 0}$ we set

$$\mathscr{P}\mathscr{D}^{\mathrm{gr},(r,s)} := \mathscr{P}^{(r)} \otimes \mathscr{D}^{(s)} \quad \text{and} \quad \mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n} := \mathscr{A}^{\mathrm{gr}}_{k,l,n} \cap \mathscr{P}\mathscr{D}^{\mathrm{gr},(r,s)}, \tag{65}$$

so that

$$\mathscr{P}\mathscr{D}^{\mathrm{gr}} = \bigoplus_{r,s \geqslant 0} \mathscr{P}\mathscr{D}^{\mathrm{gr},(r,s)} \quad \text{and} \quad \mathscr{A}^{\mathrm{gr}}_{k,l,n} := \bigoplus_{r,s \geqslant 0} \mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}.$$

By proposition 3.6.1 we obtain an isomorphism of $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_l) \otimes U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$-modules

$$\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n} \cong \bigoplus_{\substack{\lambda \in \Lambda_{\underline{k},r} \\ \mu \in \Lambda_{\underline{l},s}}} (V^*_\lambda \otimes V_\mu) \otimes (V^*_\lambda \otimes V_\mu), \tag{66}$$

where $\underline{k} := \min\{k, n\}$ and $\underline{l} := \min\{l, n\}$. Here we consider the left copy of $V^*_\lambda \otimes V_\mu$ as a module for $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_l)$ and the right copy of $V^*_\lambda \otimes V_\mu$ as a module for $U_R \otimes U_R \cong U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$. Of course by restriction along the coproduct map $U_R \to U_R \otimes U_R$ we can also consider the right copy as a U_R-module.

Proposition 3.13.3. *For $a\otimes a'\in\mathscr{P}^{(r)}\otimes_{\tilde{\mathcal{R}}}\mathscr{D}^{(r')}$ and $b\otimes b'\in\mathscr{P}^{(s)}\otimes_{\tilde{\mathcal{R}}}\mathscr{D}^{(s')}$ we have*

$$\mathsf{P}(a\otimes a')\,\mathsf{P}(b\otimes b')-\mathsf{P}((a\otimes a')(b\otimes b'))\in\bigoplus_{i=1}^{\min\{u,u'\}}\mathscr{P}\mathscr{D}^{(u-i,u'-i)},\qquad(67)$$

where $u:=r+s$ and $u':=r'+s'$.

Proof. We have $\mathsf{P}(a\otimes a')\mathsf{P}(b\otimes b')=aa'bb'$. Using the explicit relations of $\mathscr{P}\mathscr{D}$ (see remark 3.7.4) we can move the $\partial_{i,j}$ past the $t_{i,j}$ to express $a'b$ as $a'b=\sum a''b''$ where $a''\in\mathscr{P}$ and $b''\in\mathscr{D}$. Thus

$$\mathsf{P}(a\otimes a')\,\mathsf{P}(b\otimes b')=\sum aa''b''b'.$$

Similarly, using the relations of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ we can move the $1\otimes\partial_{i,j}$ past the $t_{i,j}\otimes1$ and as a result we obtain $(1\otimes a')(b\otimes1)=\sum\underline{a}''\otimes\underline{b}''$ where $\underline{a}''\in\mathscr{P}$ and $\underline{b}''\in\mathscr{D}$. It follows that

$$\mathsf{P}((a\otimes a')(b\otimes b'))=\sum a\underline{a}''\underline{b}''b'.$$

The only difference between the relations of $\mathscr{P}\mathscr{D}$ and $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ is (R6) vs. (R6′). Since (R6′) is the homogenized form of (R6′), it follows that

$$\sum a''b''-\sum\underline{a}''\underline{b}''\in\bigoplus_{i=1}^{\min\{r',s\}}\mathscr{P}\mathscr{D}^{(s-i,r'-i)}.$$

From the latter inclusion (67) follows immediately. □

Recall that $\mathrm{gr}(\mathscr{A}_{k,l,n})$ denotes the associated graded algebra corresponding to the degree filtration on $\mathscr{A}_{k,l,n}$, i.e. the filtration obtained by setting $\deg(t_{i,j})=\deg(\partial_{i,j})=1$. Note that we have a canonical isomorphism $\mathscr{A}^{\mathrm{gr}}_{k,l,n}\cong\mathrm{gr}(\mathscr{A}^{\mathrm{gr}}_{k,l,n})$ since $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ is graded.

The maps $\mathsf{P}_{k,l,n}:\mathscr{A}^{\mathrm{gr}}_{k,l,n}\to\mathscr{A}_{k,l,n}$ do *not* induce isomorphisms of associative algebras. However, the following statement holds.

Corollary 3.13.4. *The associated graded map $\mathrm{gr}(\mathsf{P}_{k,l,n})$ induces a U_R-equivariant isomorphism between $\mathscr{A}^{\mathrm{gr}}_{k,l,n}\cong\mathrm{gr}(\mathscr{A}^{\mathrm{gr}}_{k,l,n})$ and $\mathrm{gr}(\mathscr{A}_{k,l,n})$. When $k=l=m$, the latter map is U_{LR}-equivariant.*

Proof. From proposition 3.13.3 it follows that the associated graded map $\mathrm{gr}(\mathsf{P}_{k,l,n})$ is an isomorphism of associative algebras from $\mathrm{gr}(\mathscr{A}^{\mathrm{gr}}_{k,l,n})\cong\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ onto $\mathrm{gr}(\mathscr{A}_{k,l,n})$. The equivariance statements follow from proposition 3.13.1. □

3.14. The algebras $\mathring{U}_L$, $\mathring{U}_R$ and $\mathring{U}_{LR}$

We set

$$\mathring{U}_{LR}:=\phi_U^{-1}(\mathscr{P}\mathscr{D}):=\{x\in U_{LR}:\phi_U(x)\in\mathscr{P}\mathscr{D}\}.$$

Also, we set

$$\mathring{U}_L:=\left\{x\in U_L:x\otimes1\in\mathring{U}_{LR}\right\}\quad\text{and}\quad\mathring{U}_R:=\left\{x\in U_R:1\otimes x\in\mathring{U}_{LR}\right\}.$$

Recall that the adjoint action of U_{LR} is $\mathrm{ad}_y(x) := \sum y_1 x S(y_2)$ for $x, y \in U_{LR}$. Recall that we equip $\mathrm{End}_{\Bbbk}(\mathscr{P})$ with the U_{LR}-module structure of lemma 2.3.1 and we denote the latter action by $x \cdot T$ for $x \in U_{LR}$ and $T \in \mathrm{End}_{\Bbbk}(\mathscr{P})$. Then it is straightforward from the definition of ϕ_U that

$$\phi_U(\mathrm{ad}_y(x)) = y \cdot \phi_U(x) \quad \text{for } x, y \in U_{LR}. \tag{68}$$

Lemma 3.14.1. $\mathrm{ad}_x(\mathring{U}_{LR}) \subseteq \mathring{U}_{LR}$ *for* $x \in U_{LR}$.

Proof. Follows immediately from (68) and proposition 3.12.1. □

Remark 3.14.2. When $m \leqslant n$, proposition 3.6.1 implies that the map $\mathcal{L}_{\mathscr{P}} : U_L \to \mathrm{End}_{\Bbbk}(\mathscr{P})$ is an injection. The argument is similar to the proof of [KS97, theorem 7.1.5.13].

Let $\mathscr{K}_n$ denote the kernel of $\mathcal{R}_{\mathscr{P}} : U_R \to \mathrm{End}_{\Bbbk}(\mathscr{P})$. For the next proposition recall the notation $\mathscr{F}(H, I)$ defined in (11).

Proposition 3.14.3. *Assume that* $m \leqslant n$. *Then* $\mathring{U}_L \subseteq \mathscr{F}(U_L)$ *and* $\mathring{U}_R \subseteq \mathscr{F}(U_R, \mathscr{K}_n)$.

Proof. Since the actions of U_{LR} on $\mathscr{P}$ and $\mathscr{D}$ are degree preserving, it follows that $\mathscr{P}\mathscr{D}$ is a locally finite U_{LR}-module. The assertions of the proposition follow from the fact that the maps $\mathring{U}_L \xrightarrow{\phi_U} \mathscr{P}\mathscr{D}$ and $\mathring{U}_R/(\mathring{U}_R \cap \mathscr{K}_n) \xrightarrow{\phi_U} \mathscr{P}\mathscr{D}$ are injective and U_{LR}-equivariant (this is equivalent to (68)). □

The following proposition is a 'no-go theorem' that provides evidence that the commutative diagram (1) cannot be fully quantized.

Proposition 3.14.4. *There does not exist a* $\Bbbk$*-algebra* $\widetilde{\mathscr{P}\mathscr{D}}$ *with the following properties:*

(i) $\widetilde{\mathscr{P}\mathscr{D}}$ *is a locally finite* U_{LR}*-module.*
(ii) $\mathscr{P}$ *is a* $\widetilde{\mathscr{P}\mathscr{D}}$*-module and the action map* $\widetilde{\mathscr{P}\mathscr{D}} \otimes \mathscr{P} \to \mathscr{P}$ *is a homomorphism of* U_{LR}*-module.*
(iii) There exists a homomorphism of algebras $\widetilde{\phi} : U_{LR} \to \widetilde{\mathscr{P}\mathscr{D}}$ *such that the diagram*

$$\begin{tikzcd}
U_{LR} \otimes \mathscr{P} \arrow[rr, "x\otimes f \mapsto x\cdot f"] \arrow[dr, "x\otimes f \mapsto \widetilde{\phi}(x)\otimes f"'] & & \mathscr{P} \\
& \widetilde{\mathscr{P}\mathscr{D}} \otimes \mathscr{P} \arrow[ur, "D\otimes f \mapsto D\cdot f"'] &
\end{tikzcd}$$

is commutative.

Proof. Let us denote the U_{LR}-action of (i) by $x \cdot D$ for $x \in U_{LR}$ and $D \in \widetilde{\mathscr{P}\mathscr{D}}$. Let $\mathscr{K} \subseteq \widetilde{\mathscr{P}\mathscr{D}}$ denote the kernel of the map $\widetilde{\mathscr{P}\mathscr{D}} \to \mathrm{End}_{\Bbbk}(\mathscr{P})$ that is induced by the $\widetilde{\mathscr{P}\mathscr{D}}$-module structure on $\mathscr{P}$. We define a new U_{LR}-action on $\widetilde{\mathscr{P}\mathscr{D}}$ by setting

$$x \bullet D := \sum \widetilde{\phi}(x_1) D \widetilde{\phi}(S(x_2)) \quad \text{for } x \in U_{LR} \text{and } D \in \widetilde{\mathscr{P}\mathscr{D}}.$$

By the proof of proposition 3.12.1 we obtain $(x \bullet D) - (x \cdot D) \in \mathscr{K}$ for $x \in U_{LR}$ and $D \in \widetilde{\mathscr{P}\mathscr{D}}$. In particular, for $x, y \in U_{LR}$ if we set $D := \widetilde{\phi}(y)$ then we have

$$\widetilde{\phi}(\mathrm{ad}_x(y)) + \mathscr{K} = x \bullet D + \mathscr{K} = x \cdot D + \mathscr{K}. \tag{69}$$

Now assume that $m \leqslant n$, so that the restriction of $\widetilde{\phi}$ to a map $U_L \otimes 1 \to \mathrm{End}_{\Bbbk}(\mathscr{P})$ is faithful. By (69) it follows that if $y \in U_L \otimes 1$ then the image of $\widetilde{\phi}(\mathrm{ad}_{U_L\otimes 1}(y))$ in $\widetilde{\mathscr{P}\mathscr{D}/\mathscr{K}}$ is finite dimensional. But $\widetilde{\phi}(\mathrm{ad}_{U_L\otimes 1}(y)) \cap \mathscr{K} = \{0\}$, hence $\mathrm{ad}_{U_L\otimes 1}(y)$ is also finite dimensional. Consequently, we have shown that $U_L = \mathscr{F}(U_L)$, which is a contradiction. □

3.15. Relation between theorem A(i) and theorem A(ii)

Our goal in this subsection is to prove lemma 3.15.1 below, which implies that theorem A(ii) follows by symmetry from theorem A(i).

From the symmetry of the defining relations of $\mathscr{P}\mathscr{D}_{m\times n}$ with respect to the two indices of the generators $t_{i,j}$ and $\partial_{i,j}$ it follows that there exists an isomorphism of algebras

$$\eta_{m,n} : \mathscr{P}\mathscr{D}_{m\times n} \to \mathscr{P}\mathscr{D}_{n\times m},$$

such that $\eta_{m,n}(t_{i,j}) = t_{j,i}$ and $\eta_{m,n}(\partial_{i,j}) = \partial_{j,i}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Note that $\eta_{m,n}$ restricts to an isomorphism $\mathscr{P}_{m\times n} \cong \mathscr{P}_{n\times m}$. This naturally results in an isomorphism of algebras $\mathrm{End}_{\Bbbk}(\mathscr{P}_{m\times n}) \cong \mathrm{End}_{\Bbbk}(\mathscr{P}_{n\times m})$.

Lemma 3.15.1. *The following assertions hold.*

(i) For $x \otimes y \in U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$ and $D \in \mathscr{P}\mathscr{D}_{m\times n}$ we have

$$\eta_{m,n}\left((x\otimes y)\cdot D\right) = (y \otimes x)\cdot \eta_{m,n}(D).$$

(ii) The induced isomorphism $\mathrm{End}_{\Bbbk}(\mathscr{P}_{m\times n}) \cong \mathrm{End}_{\Bbbk}(\mathscr{P}_{n\times m})$ maps the images of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$ in $\mathrm{End}_{\Bbbk}(\mathscr{P}_{m\times n})$ onto the images of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$ in $\mathrm{End}_{\Bbbk}(\mathscr{P}_{n\times m})$.

Proof. (i) It suffices to prove the assertion when x and y are selected from the standard generators of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{gl}_n)$, respectively. If $D = t_{i,j}$ or $D = \partial_{i,j}$, then the assertion follows from symmetry of the effect of the generators on the indices of the $t_{i,j}$ and the $\partial_{i,j}$ (see remark 3.6.2). For general $D \in \mathscr{P}\mathscr{D}_{m\times n}$ the assertion follows from the fact that $\mathscr{P}\mathscr{D}_{m\times n}$ and $\mathscr{P}\mathscr{D}_{n\times m}$ are module algebras over $U_q(\mathfrak{gl}_m)\otimes U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_n)\otimes U_q(\mathfrak{gl}_m)$, respectively.

(ii) Follows immediately from (i). □

3.16. Relation to $\mathbb{C}[\mathrm{Mat}_{m,n}]_q$ and $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$

We can now relate our algebras $\mathscr{P}$ and $\mathscr{P}\mathscr{D}$ to the $\mathbb{C}$-algebras $\mathbb{C}[\mathrm{Mat}_{m,n}]_q$ and $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ (where $0 < q < 1$) that are introduced in [SSV04, BKV06]. We remark that in [BKV06] only the special case $m = n$ is considered, and the latter algebras are denoted by $\mathbb{C}[\mathrm{Mat}_n]_q$ and $\mathrm{Pol}(\mathrm{Mat}_n)_q$, respectively. The algebra $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ is defined in [SSV04, section 2] in terms of the generators z_j^i and $(z_j^i)^*$ where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. These generators satisfy the relations (2.1)–(2.7) of [SSV04]. For the reader's convenience we describe the relations of $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$. The relations among the z_j^i (which are (2.1)–(2.3) of [SSV04]) are identical to the relations among the $t_{i,j}$, the only difference being that q becomes a complex-valued parameter. In a similar way, the relations among the $(z_j^i)^*$ (which are (2.4)–(2.6) of [SSV04]) are identical to the relations among the $\partial_{i,j}$. The mixed relations (which correspond to (2.7) of [SSV04]) are

$$(z_j^i)^* z_l^k = q^2 \sum_{1\leqslant a,b\leqslant n} \sum_{1\leqslant c,d\leqslant m} \left(r_{j,l}^{b,a}\right)\left(r_{i,k}^{d,c}\right) z_a^c \left(z_b^d\right)^* + [\![j,l]\!]\,[\![i,k]\!]\left(1-q^2\right),$$

where $r_{i,i}^{i,i}=1$, $r_{i,j}^{i,j}=q^{-1}$ for $i\neq j$, $r_{i,i}^{j,j}=1-q^{-2}$ for $j>i$, and $r_{k,l}^{i,j}=0$ otherwise. The algebra $\mathbb{C}[\mathrm{Mat}_{m,n}]_q$ is the subalgebra of $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ that is generated by the z_j^i.

Throughout this subsection we set $A:=\mathbb{Z}[q,q^{-1}]$. The relations of $\mathscr{P}\mathscr{D}$ in definition 3.7.3 are defined over A. Thus we obtain an integral form $\mathscr{P}\mathscr{D}^A$ of $\mathscr{P}\mathscr{D}$ by considering the free A-submodule of $\mathscr{P}\mathscr{D}$ that is generated by the monomials (56). Evaluation at $q_\circ$ for $0<q_\circ<1$ results in a ring homomorphism $A\to\mathbb{C}$. Set $\mathscr{P}\mathscr{D}_{\langle q_\circ\rangle}:=\mathscr{P}\mathscr{D}^A\otimes_A\mathbb{C}$.

Corollary 3.16.1. *The algebras $\mathscr{P}\mathscr{D}_{\langle q_\circ\rangle}$ and $\mathrm{Pol}(\mathrm{Mat}_{m,n})_{q_\circ}$ are isomorphic by the assignments $t_{i,j}\mapsto\frac{z_j^i}{\sqrt{1-q_\circ^2}}$ and $\partial_{i,j}\mapsto\frac{(z_j^i)^*}{\sqrt{1-q_\circ^2}}$.*

Proof. It is straightforward to check that these assignments intertwine the relations (R1)–(R6) with the relations (2.1)–(2.7) of [SSV04]. □

In the rest of this subsection we relate the actions of quantized enveloping algebras on $\mathscr{P}$ and on $\mathbb{C}[\mathrm{Mat}_{m,n}]_q$. Set $\mathscr{P}^A:=\mathscr{P}\cap\mathscr{P}\mathscr{D}^A$. Then $\mathscr{P}^A$ is a free A-module, with an A-basis that consists of the monomials

$$t_{1,1}^{a_{1,1}}\cdots t_{1,n}^{a_{1,n}}\cdots t_{m,1}^{a_{m,1}}\cdots t_{m,n}^{a_{m,n}}.$$

Let $U_q^A(\mathfrak{gl}_n)$ denote the restricted integral A-form of $U_q(\mathfrak{gl}_n)$. For integral forms of quantized enveloping algebras see for example [CP94, section 9.3]. The explicit description of $U_q^A(\mathfrak{gl}_n)$ is given for example in [RT10]. We denote the analogous integral form of the algebra $U_{LR}\cong U_q(\mathfrak{gl}_m)\otimes U_q(\mathfrak{gl}_n)$ by U_{LR}^A. Thus

$$U_{LR}^A\cong U_q^A(\mathfrak{gl}_m)\otimes_A U_q^A(\mathfrak{gl}_n).$$

For $0<q_\circ<1$ set $U_{q_\circ}(\mathfrak{gl}_n):=U_q^A(\mathfrak{gl}_n)\otimes_A\mathbb{C}$. By remark 3.6.2 the map $U_{LR}\otimes\mathscr{P}\to\mathscr{P}$ that describes the U_{LR}-module structure on $\mathscr{P}$ restricts to a map $U_{LR}^A\otimes\mathscr{P}^A\to\mathscr{P}^A$. After the scalar extension $(-)\otimes_A\mathbb{C}$ and using the isomorphism $\mathscr{P}^A\otimes_A\mathbb{C}\cong\mathbb{C}[\mathrm{Mat}_{m,n}]_{q_\circ}$ we obtain a structure of a $U_{q_\circ}(\mathfrak{gl}_m)\otimes U_{q_\circ}(\mathfrak{gl}_n)$-module on $\mathbb{C}[\mathrm{Mat}_{m,n}]_{q_\circ}$ that corresponds to a map

$$\left(U_{q_\circ}(\mathfrak{gl}_m)\otimes U_{q_\circ}(\mathfrak{gl}_n)\right)\otimes\mathbb{C}[\mathrm{Mat}_{m,n}]_{q_\circ}\to\mathbb{C}[\mathrm{Mat}_{m,n}]_{q_\circ},\tag{70}$$

or equivalently a homomorphism of algebras

$$\phi_{U,q_\circ}:U_{q_\circ}(\mathfrak{gl}_m)\otimes U_{q_\circ}(\mathfrak{gl}_n)\to\mathrm{End}_{\mathbb{C}}\left(\mathbb{C}[\mathrm{Mat}_{m,n}]_{q_\circ}\right).\tag{71}$$

The next proposition relates the latter module structure to the one given in [SSV04, section 9–10] and [BKV06, section 3]. We remark that in [SSV04, BKV06], the module algebra structure on $\mathbb{C}[\mathrm{Mat}_{m,n}]_{q_\circ}$ is with respect to $U_{q_\circ}(\mathfrak{gl}_m)^{\mathrm{cop}}\otimes U_{q_\circ}(\mathfrak{gl}_n)^{\mathrm{cop}}$. We denote the latter module structure by the map

$$\phi_{\mathrm{SSV}}:U_{q_\circ}(\mathfrak{gl}_m)^{\mathrm{cop}}\otimes U_{q_\circ}(\mathfrak{gl}_n)^{\mathrm{cop}}\to\mathrm{End}_{\mathbb{C}}\left(\mathbb{C}[\mathrm{Mat}_{m,n}]_{q_\circ}\right).$$

Let $x\mapsto x^\natural$ for $n\geqslant 1$ be the $\mathbb{C}$-linear isomorphism of Hopf algebras $U_{q_\circ}(\mathfrak{gl}_n)\to U_{q_\circ}(\mathfrak{gl}_n)^{\mathrm{op}}$ that is given by the same relations as (45) but for $q:=q_\circ$.

Proposition 3.16.2. *With $\phi_{U,\langle q_\circ\rangle}$ and ϕ_{SSV} as above, we have*

$$\phi_{U,q_\circ}(x\otimes y)=\phi_{\mathrm{SSV}}\left(\vartheta_L\left(S\left(x^\natural\right)\right)\otimes\vartheta_R\left(S\left(y^\natural\right)\right)\right)\quad \textit{for } x\otimes y\in U_{q_\circ}(\mathfrak{gl}_m)\otimes U_{q_\circ}(\mathfrak{gl}_n),$$

where ϑ_L and ϑ_R are the automorphisms of the Hopf algebra $U_{q_\circ}(\mathfrak{gl}_n)$ that are uniquely defined by setting

$$\vartheta_L(E_i):=q_\circ^{-\frac{1}{2}}F_{m-i}\quad,\quad \vartheta_L(F_i):=q_\circ^{\frac{1}{2}}E_{m-i}\quad,\quad \vartheta_L(K_{\varepsilon_i}):=K^{-1}_{\varepsilon_{m+1-i}},$$

and

$$\vartheta_R(E_i):=q_\circ^{-\frac{1}{2}}F_i\quad,\quad \vartheta_R(F_i):=q_\circ^{\frac{1}{2}}E_i\quad,\quad \vartheta_L(K_{\varepsilon_i}):=K_{\varepsilon_i}.$$

Proof. Follows from comparing remark 3.6.2 with [SSV04, section 8] or [BKV06, equations (14), (15)]. Note that the coproduct of $U_{q_\circ}(\mathfrak{gl}_n)$ in [BKV06] is co-opposite to the coproduct considered in the present paper, but this is corrected by composing with $x\mapsto S(x^\natural)$. □

3.17. Some technical statements about the action of $\mathscr{PD}$ on $\mathscr{P}$

In this subsection we prove several technical statements about the interaction between the $\partial_{i,j}$ on $\mathscr{P}$. We will need these statements in the upcoming sections of this paper. In order to make our exposition more organized we have collected all of them in one subsection. The reader may find it easier to skip this subsection and return to it whenever there is a reference.

Recall that the action of $D\in\mathscr{PD}$ on $f\in\mathscr{P}$ is denoted by $D\cdot f$ (see section 3.10).

Lemma 3.17.1. *Assume that either $i\notin\{a_1,\dots,a_r\}$ or $j\notin\{b_1,\dots,b_r\}$, then $\partial_{i,j}t_{a_1,b_1}\cdots t_{a_r,b_r}$ belongs to the left ideal of $\mathscr{PD}$ that is generated by the $\partial_{i',j'}$ satisfying $i'\geqslant i$ and $j'\geqslant j$. In particular, $\partial_{i,j}\cdot(t_{a_1,b_1}\cdots t_{a_r,b_r})=0$.*

Proof. We use induction on r. For $r=1$ the assertion follows from relations (R3)–(R6). Next suppose $r>1$. If $i\neq a_1$ and $j\neq b_1$ then $\partial_{i,j}t_{a_1,b_1}=t_{a_1,b_1}\partial_{i,j}$ and we can use the induction hypothesis. If $i=a_1$ then $j\notin\{b_1,\dots,b_r\}$ and we can write

$$\partial_{i,j}t_{a_1,b_1}\cdots t_{a_r,b_r}=qt_{i,b_1}\partial_{i,j}t_{a_2,b_2}\cdots t_{a_r,b_r}+\left(q-q^{-1}\right)\sum_{i'>i}t_{i',b_1}\partial_{i',j}t_{a_2,b_2}\cdots t_{a_r,b_r},$$

and again the induction hypothesis is applicable to each summand on the right hand side. The argument for the case $j=b_1$ is similar. □

Lemma 3.17.2. *Assume that either $\{i_1,\dots,i_s\}\nsubseteq\{a_1,\dots,a_r\}$ and $\{j_1,\dots,j_s\}\nsubseteq\{b_1,\dots,b_r\}$. Then*

$$\partial_{i_1,j_1}\cdots\partial_{i_s,j_s}\cdot(t_{a_1,b_1}\cdots t_{a_r,b_r})=0.$$

Proof. Without loss of generality assume that $i_1\notin\{a_1,\dots,a_r\}$. Relations (R1′) and (R2′) imply that we can replace $\partial_{i_1,a}\partial_{b,c}$ by either $\partial_{b,c}\partial_{i_1,a}$ or $q^{\pm1}\partial_{b,c}\partial_{i_1,a}$ or $\partial_{b,c}\partial_{i_1,a}\pm(q-q^{-1})\partial_{b,a}\partial_{i_1,c}$. Using the latter replacements we can express $\partial_{i_1,j_1}\cdots\partial_{i_s,j_s}$ as a linear combination of monomials that belong to the left ideal $\breve{\mathscr{I}}:=\sum_{j=1}^n\mathscr{D}\partial_{i_1,j}$ of $\mathscr{D}$. From lemma 3.17.1 it follows that elements of $\breve{\mathscr{I}}$ annihilate $t_{a_1,b_1}\cdots t_{a_r,b_r}$. □

For the following corollary recall that $\overrightarrow{M}^{\mathbf{i}}_{\mathbf{j}}$ is the quantum minor defined in (7).

Corollary 3.17.3. *Let* $\mathbf{i} := (i_1, \ldots, i_r)$ *and* $\mathbf{j} := (j_1, \ldots, j_r)$ *be r-tuples of integers that satisfy*

$$1 \leqslant i_1 < \cdots < i_r \leqslant m \quad \text{and} \quad 1 \leqslant j_1 < \cdots < j_r \leqslant n.$$

Then $\overrightarrow{M}_{\mathbf{j}}^{\mathbf{i}} \cdot (t_{a_1,b_1} \cdots t_{a_s,b_s}) = 0$ *when* $a_i \geqslant i_1 + 1$ *for all* $1 \leqslant i \leqslant s$.

Proof. This follows from lemma 3.17.2 since $\overrightarrow{M}_{\mathbf{j}}^{\mathbf{i}}$ is a linear combination of monomials of the form $\partial_{i_{\sigma(1)},j_1} \cdots \partial_{i_{\sigma(r)},j_r}$. □

Lemma 3.17.4. *Suppose that* $f, g \in \mathscr{P}$ *satisfy* $\partial_{i_1,j_1} \cdots \partial_{i_r,j_r} \cdot f = g$ *for some* $1 \leqslant i_1, \ldots, i_r \leqslant m$ *and* $1 \leqslant j_1, \ldots, j_r \leqslant n$. *Then for any* $1 \leqslant i'_1 \leqslant \ldots \leqslant i'_s \leqslant m$ *and* $1 \leqslant j'_1 \leqslant \ldots \leqslant j'_s \leqslant n$ *that satisfy either* $\min\{j_u\}_{u=1}^r > \max\{j'_u\}_{u=1}^s$ *or* $\min\{i_u\}_{u=1}^r > \max\{i'_u\}_{u=1}^s$ *we have*

$$\partial_{i_1,j_1} \cdots \partial_{i_r,j_r} \cdot \left(f t_{i'_1,j'_1} \cdots t_{i'_s,j'_s}\right) = g t_{i'_1,j'_1} \cdots t_{i'_s,j'_s}.$$

Proof. We assume $\min\{j_u\}_{u=1}^r > \max\{j'_u\}_{u=1}^s$ (the other case follows by symmetry). Recall that $\mathscr{I}$ is the left ideal of $\mathscr{P}\mathscr{D}$ generated by $\mathscr{D}^{(1)}$ (see section 3.10). Set $f' := \partial_{i_r,j_r} \cdot f$. Then $\partial_{i_r,j_r} f = f' + \sum_{(i',j')} b_{i',j'} \partial_{i',j'}$, where the $b_{i',j'} \in \mathscr{P}\mathscr{D}$ and the sum is over all pairs (i',j') that satisfy $i_r \leqslant i' \leqslant m$ and $j_r \leqslant j' \leqslant n$. In particular $j' \notin \{j'_1, \ldots, j'_s\}$, hence by lemma 3.17.1 we obtain

$$\partial_{i_r,j_r} f t_{i'_1,j'_1} \cdots t_{i'_s,j'_s} = f' t_{i'_1,j'_1} \cdots t_{i'_s,j'_s} + \sum_{(i',j')} b_{i',j'} \partial_{i',j'} t_{i'_1,j'_1} \cdots t_{i'_s,j'_s} \in f' t_{i'_1,j'_1} \cdots t_{i'_s,j'_s} + \mathscr{I}.$$

This means $\partial_{i_r,j_r} \cdot (f t_{i'_1,j'_1} \cdots t_{i'_s,j'_s}) = f' t_{i'_1,j'_1} \cdots t_{i'_s,j'_s}$. The proof is completed by induction on r. □

Recall the operators $\mathbf{D}'_{k,r}$ from section 1. We have

$$\mathbf{D}'_{1,0} + \left(q^2 - 1\right) \mathbf{D}'_{1,1} = 1 + \left(q^2 - 1\right) \sum_{i=1}^{n} t_{m,i} \partial_{m,i}.$$

The next lemma is a consequence of [BKV06, theorem 1] but we give an elementary, independent proof. This also makes the proofs of theorems A and B independent of [BKV06, theorem 1].

Lemma 3.17.5. *Set* $D := \mathbf{D}'_{1,0} + (q^2 - 1)\mathbf{D}'_{1,1}$. *Then* $D \cdot t_{a_1,b_1} \cdots t_{a_r,b_r} = q^{2\sum_{i=1}^r [\![m,a_i]\!]} t_{a_1,b_1} \cdots t_{a_r,b_r}$.

Proof. By remark 3.8.6 the monomials

$$f = f(a_{1,1}, \ldots, a_{m,n}) := t_{m,n}^{a_{m,n}} \cdots t_{m,1}^{a_{m,1}} \cdots t_{1,n}^{a_{1,1}} \cdots t_{1,1}^{a_{1,1}}$$

form a basis of $\mathscr{P}$. From relations (R1) and (R2) it follows that it suffices to prove the assertion for such monomials. By lemma 3.17.4 the assertion is reduced to the case where $a_{i,j} = 0$ for $i < m$. For $j > i$ we have $t_{m,i} \partial_{m,i} t_{m,j} = q^2 t_{m,j} t_{m,i} \partial_{m,i}$. By successive application of the latter relation, followed by lemmas 3.11.3 and 3.17.4, we obtain

$$\left(q^2 - 1\right) t_{m,i} \partial_{m,i} \cdot f = q^{2\sum_{j=i+1}^n a_{m,j}} \left(q^{2a_{m,i}} - 1\right) f.$$

After summing up over $1 \leqslant i \leqslant n$, the assertion of the lemma is reduced to the algebraic identity

$$1+\sum_{i=1}^{n} q^{2\sum_{j=i+1}^{n} a_{m,j}}\left(q^{2a_{m,i}}-1\right)=q^{2\sum_{i=1}^{n} a_{m,i}},$$

which can be verified by a straightforward computation. □

Definition 3.17.6. Given any two ordered pairs of integers (i,j) and (i',j'), we write $(i,j) \lhd (i',j')$ if $i \leqslant i'$ and $j \leqslant j'$ and at least one of the latter inequalities is strict.

Let $\mathscr{I}_{a,b}$ denote the left ideal of $\mathscr{P}\mathscr{D}$ that is generated by the $\partial_{i,j}$ where $i \geqslant a$ and $j \geqslant b$.

Lemma 3.17.7. *Let $a \geqslant 0$ and let $1 \leqslant k \leqslant n$. Then $\partial_{1,k} t_{1,k}^{a+1} = \mathbf{c}(a) t_{1,k}^{a} + D$ where $\mathbf{c}(a)$ is as in (62) and $D \in \mathscr{I}_{1,k}$.*

Proof. Follows by induction on a. For $a=0$ the assertion follows from the relation

$$\partial_{1,k} t_{1,k} = 1 + q^2 t_{1,k}\partial_{1,k} + D_1 \quad \text{where } D_1 \in \sum_{(1,k)\lhd(i,\ell)} \mathscr{P}\mathscr{D}\partial_{i,\ell}. \tag{72}$$

Suppose that for a given $a \geqslant 0$ we have $\partial_{1,k} t_{1,k}^{a} = \mathbf{c}(a-1) t_{1,k}^{a-1} + D_2$ with $D_2 \in \mathscr{I}_{1,k}$. Using (72) we obtain

$$\partial_{1,k} t_{1,k}^{a+1} = \left(1+q^2 t_{1,k}\partial_{1,k}+D_1\right) t_{1,k}^{a} = \left(1+q^2\mathbf{c}\left(a-1\right)\right) t_{1,k}^{a} + q^2 t_{1,k} D_2 + D_1 t_{1,k}^{a}.$$

From lemma 3.17.1 it follows that $D_1 t_{1,k}^{a} \in \mathscr{I}_{1,k}$. Finally note that $\mathbf{c}(a) = 1 + q^2\mathbf{c}(a-1)$. □

Lemma 3.17.8. *Let $a,b \geqslant 0$ and let $1 \leqslant k \leqslant n$.*

(i) If $b > a$ then $\partial_{1,k}^{b+1} t_{1,k}^{a+1} \in \mathscr{I}_{1,k}$.
(ii) If $b \leqslant a$ then $\partial_{1,k}^{b+1} t_{1,k}^{a+1} = \mathbf{c}(a,b) t_{1,k}^{a-b} + D$ for some $\mathbf{c}(a,b) \in \Bbbk$, where $D \in \mathscr{I}_{1,k}$. Furthermore $\mathbf{c}(a,0) = \mathbf{c}(a)$ and $\mathbf{c}(a,b+1) = \mathbf{c}(a,b)\mathbf{c}(a-b-1)$ for $b < a$.

Proof. (i) Follows from the equality $\partial_{1,k}^{b+1} t_{1,k}^{a+1} = \partial_{1,k}^{b-a}\partial_{1,k}^{a+1} t_{1,k}^{a+1}$ and lemma 3.17.7.
(ii) We use induction on b. For $b=0$ this is lemma 3.17.7. If $b+1 \leqslant a$ then

$$\begin{aligned}\partial_{1,k}^{b+2} t_{1,k}^{a+1} &= \partial_{1,k}\partial_{1,k}^{b+1} t_{1,k}^{a+1}\\ &= \mathbf{c}\left(a,b\right)\partial_{1,k} t_{1,k}^{a-b} + \partial_{1,k} D = \mathbf{c}\left(a,b\right)\mathbf{c}\left(a-b-1\right) t_{1,k}^{a-b-1} + \mathbf{c}\left(a,b\right) D_1 + \partial_{1,k} D,\end{aligned}$$

where $D_1, D \in \mathscr{I}_{1,k}$. Part (ii) follows immediately. □

Lemma 3.17.9. *Let $a,b \geqslant 0$ and let $1 \leqslant k \leqslant n$. Assume that $f \in \mathscr{P}$ is a product of the $t_{1,j}$ for $j \leqslant k-1$. Then the following hold:*

(i) If $b > a$ then $\partial_{1,k}^{b} t_{1,k}^{a} f \in \mathscr{I}_{1,k}$.
(ii) If $b \leqslant a$ then $\partial_{1,k}^{b+1} t_{1,k}^{a+1} f = \mathbf{c}(a,b) t_{1,k}^{a-b} f + D$ where $D \in \mathscr{I}_{1,k}$ and $\mathbf{c}(a,b)$ is as in lemma 3.17.8.

Proof. (i) Follows from lemmas 3.17.8(i) and 3.17.1.
(ii) From lemma 3.17.8(ii) we have $\partial_{1,k}^{b+1} t_{1,k}^{a+1} f = \mathbf{c}(a,b) t_{1,k}^{a-b} f + Df$, where $D \in \mathscr{I}_{1,k}$. The assumption on f and lemma 3.17.1 imply that $Df \in \mathscr{I}_{1,k}$. □

Remark 3.17.10. It is easy to verify that $\mathbf{c}(a,b) = \mathbf{c}(a)\mathbf{c}(a-1)\cdots\mathbf{c}(a-b)$ for $a \geqslant b \geqslant 0$. We extend the domain of $\mathbf{c}(a,b)$ to pairs (a,b) satisfying $a,b \geqslant -1$ by setting $\mathbf{c}(a,b) = 0$ for $-1 \leqslant a < b$ and $\mathbf{c}(a,b) = 1$ for $a \geqslant b = -1$. Note that $\mathbf{c}(a,b)$ is always a polynomial in q^2 with integer coefficients. Furthermore, when $a \geqslant b$ the degree of $\mathbf{c}(a,b)$ as a polynomial in q is $(b+1)(2a-b)$.

For a k-tuple of non-negative integers $\mathsf{a} := (a_1,\ldots,a_k)$, where $k \leqslant n$, we set $t^{\mathsf{a}} := t_{1,k}^{a_k}\cdots t_{1,1}^{a_1}$ and $\partial^{\mathsf{a}} := \partial_{1,1}^{a_1}\cdots\partial_{1,k}^{a_k}$.

Lemma 3.17.11. *Let $1 \leqslant k' < k_r < \ldots < k_1 \leqslant n$. Also, let $a_1,\ldots,a_{k'} \geqslant 0$ and $b_1,\ldots,b_r \geqslant 0$. Set $\mathsf{a} := (a_1,\ldots,a_{k'})$ and $f := t^{\mathsf{a}} := t_{1,k'}^{a_{k'}}\cdots t_{1,1}^{a_1}$. Then*

$$\partial_{1,k'}^{b} t_{1,k_1}^{b_1}\cdots t_{1,k_r}^{b_r} f = f_1 + D,$$

where $f_1 \in \mathscr{P}$ and $D \in \mathscr{I}_{1,k'}$. If $a_{k'} < b$ then $f_1 = 0$. If $a_{k'} \geqslant b$ then

$$f_1 = q^{b(b_1+\cdots+b_r)}\mathbf{c}(a_{k'}-1,b-1)\, t_{1,k_1}^{b_1}\cdots t_{1,k_r}^{b_r} t^{\mathsf{a}'} \quad \text{where} \quad \mathsf{a}' := (a_1,\ldots,a_{k'-1},a_{k'}-b)\,.$$

Proof. The assertion is trivial for $b = 0$. If $b_1 = \cdots = b_r = 0$ then the assertion follows from lemma 3.17.9(ii) and remark 3.17.10. Next assume without loss of generality that $b_1 \geqslant 1$. First suppose that $b = 1$. Using lemma 3.17.1 we obtain

$$\begin{aligned}\partial_{1,k'} t_{1,k_1}^{b_1}\cdots t_{1,k_r}^{b_r} f &= q t_{1,k_1}\partial_{1,k'} t_{1,k_1}^{b_1-1} t_{1,k_2}^{b_2}\cdots t_{1,k_r}^{b_r} f + \left(q-q^{-1}\right)\sum_{1<i\leqslant m} t_{i,k_1}\partial_{i,k'} t_{1,k_1}^{b_1-1} t_{1,k_2}^{b_2}\cdots t_{1,k_r}^{b_r} f\\ &= q t_{1,k_1}\partial_{1,k'} t_{1,k_1}^{b_1-1} t_{1,k_2}^{b_2}\cdots t_{1,k_r}^{b_r} f + D_1,\end{aligned}$$

where $D_1 \in \mathscr{I}_{2,k'}$. By repeating the above calculation and then using lemma 3.17.7 we obtain

$$\begin{aligned}\partial_{1,k'} t_{1,k_1}^{b_1}\cdots t_{1,k_r}^{b_r} f &= q^{b_1+\cdots+b_r} t_{1,k_1}^{b_1}\cdots t_{1,k_r}^{b_r}\partial_{1,k'} t^{\mathsf{a}} + D_2,\\ &= q^{b_1+\cdots+b_r}\mathbf{c}\left(a_{k'}-1\right) t_{1,k_1}^{b_1}\cdots t_{1,k_r}^{b_r} t^{\mathsf{a}-\mathsf{e}_{k'}} + D_2,\end{aligned}$$

where $D_2 \in \mathscr{I}_{1,k'}$, $\mathsf{a}-\mathsf{e}_{k'} := (a_1,\ldots,a_{k'-1},a_{k'}-1)$ and we define $\mathbf{c}(-1) := 0$. This completes the proof for $b = 1$. For $b > 1$ we just repeat the above argument. $\square$

Lemma 3.17.12. *Let $\mathsf{a} := (a_1,\ldots,a_n)$ and $\mathsf{b} := (b_1,\ldots,b_n)$ be n-tuples of non-negative integers. Then the following statements hold.*

(i) $\partial^{\mathsf{b}}\cdot t^{\mathsf{a}} = 0$ if $b_i > a_i$ for at least one $1 \leqslant i \leqslant n$.
(ii) Assume that $a_i \geqslant b_i$ for all $1 \leqslant i \leqslant n$. Then

$$\partial^{\mathsf{b}}\cdot t^{\mathsf{a}} = \left(q^{\sum_{i=2}^{n}(a_i-b_i)(b_1+\cdots+b_{i-1})}\prod_{i=1}^{n}\mathbf{c}\left(a_i-1,b_i-1\right)\right) t^{\mathsf{a}-\mathsf{b}}, \tag{73}$$

and

$$t^{\mathsf{b}}\partial^{\mathsf{b}}\cdot t^{\mathsf{a}} = \left(q^{\sum_{i=2}^{n}(2a_i-2b_i)(b_1+\cdots+b_{i-1})}\prod_{i=1}^{n}\mathbf{c}\left(a_i-1,b_i-1\right)\right) t^{\mathsf{a}}. \tag{74}$$

Proof. (i) Follows from lemmas 3.17.9 and 3.17.11.
(ii) By lemma 3.17.11, $\partial^{\mathsf{b}} t^{\mathsf{a}} = \mathbf{c}(a_n - 1, b_n - 1)\partial^{\mathsf{b}'} t_{1,n}^{a_n - b_n} t^{\mathsf{a}'} + D_1$ where $\mathsf{a}' := (a_1, \ldots, a_{n-1})$, $\mathsf{b}' := (b_1, \ldots, b_{n-1})$ and $D_1 \in \mathscr{I}_{1,n}$. Again by lemma 3.17.11,

$$\partial^{\mathsf{b}'} t_{1,n}^{a_n - b_n} t^{\mathsf{a}'} = q^{(a_n - b_n) b_{n-1}} \mathbf{c}\left(a_{n-1} - 1, b_{n-1} - 1\right) \partial^{\mathsf{b}''} t_{1,n}^{a_n - b_n} t_{1,n-1}^{a_{n-1} - b_{n-1}} t^{\mathsf{a}''} + D_2,$$

where $\mathsf{a}'' := (a_1, \ldots, a_{n-2})$, $\mathsf{b}'' := (b_1, \ldots, b_{n-2})$ and $D_2 \in \mathscr{I}_{1,n-1}$. Continuing in this fashion we finally obtain (73). For (74) we should compute the scalar relating $t^{\mathsf{b}} t^{\mathsf{a}-\mathsf{b}}$ and t^{a}. This is straightforward using the relations $t_{1,i} t_{1,j} = q t_{1,j} t_{1,i}$ for $i < j$. $\square$

4. Differential operators associated to the K_λ

Let $\mathring{U}_L$, $\mathring{U}_R$ and $\mathring{U}_{LR}$ be defined as in section 3.14. The main goal of this section is to prove that certain elements of the Cartan subalgebras of U_L and U_R belong to $\mathring{U}_L$ and $\mathring{U}_R$. This is established in proposition 4.1.1.

4.1. Cartan elements in $\mathring{U}_L$, $\mathring{U}_R$ and $\mathring{U}_{LR}$

For $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant n$ we set

$$\lambda_{L,a} := -\sum_{i=a}^{m} 2\varepsilon_i \quad \text{and} \quad \lambda_{R,b} := -\sum_{i=b}^{n} 2\varepsilon_i. \tag{75}$$

As in (37) these weights correspond to $K_{\lambda_{L,a}} \in U_{\mathfrak{h},L}$ and $K_{\lambda_{R,b}} \in U_{\mathfrak{h},R}$, respectively.

Proposition 4.1.1. *Let $K_{\lambda_{L,a}} \in U_L$ and $K_{\lambda_{R,b}} \in U_R$ be as in (75), where $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant n$. Then $K_{\lambda_{L,a}} \in \mathring{U}_L$ and $K_{\lambda_{R,b}} \in \mathring{U}_R$.*

Proof. We only prove the assertion for $K_{\lambda_{L,a}}$ (for $K_{\lambda_{R,b}}$ the argument is similar). First we verify the case $a = m$. By a straightforward computation based on remark 3.6.2 we have

$$K_{\lambda_{L,m}} \cdot t_{a_1,b_1} \cdots t_{a_r,b_r} = q^{2\sum_{i=1}^{r} [\![m,a_i]\!]} t_{a_1,b_1} \cdots t_{a_r,b_r}.$$

By lemma 3.17.5 the action of $\mathbf{D}'_{1,0} + (q^2 - 1)\mathbf{D}'_{1,1}$ on $\mathscr{P}$ is the same as the action of $K_{\lambda_{L,m}}$. Thus, by proposition 3.11.4 we obtain

$$\phi_U\left(K_{\lambda_{L,m}} \otimes 1\right) = \mathbf{D}'_{1,0} + \left(q^2 - 1\right)\mathbf{D}'_{1,1}. \tag{76}$$

To complete the proof, by lemma 3.14.1 it suffices to verify that for any $a < m$, the $\mathrm{ad}(U_L)$-invariant subalgebra of U_L that is generated by $K_{\lambda_{L,a+1}}$ and $K_{\lambda_{L,m}}$ also contains $K_{\lambda_{L,a}}$. Denoting the standard generators of U_L by E_i, F_i, $K_i^{\pm 1}$, we set

$$E'_{\varepsilon_i - \varepsilon_j} := \left[E_i, [\ldots, E_{j-1}]_q\right]_q \quad \text{and } F'_{\varepsilon_i - \varepsilon_j} := \left[F_{j-1}, [\ldots, F_i]_{q^{-1}}\right]_{q^{-1}} \quad \text{for } 1 \leqslant \mathrm{i} < \mathrm{j} \leqslant \mathrm{m}.$$

Let $u := E'_{\varepsilon_a - \varepsilon_m} K_{-\varepsilon_a - \varepsilon_m}$ and $v := F'_{\varepsilon_a - \varepsilon_m} K_{\lambda_{L,a+1}}$. By a simple induction we can verify that

$$u = \left(1 - q^2\right)^{-1} \mathrm{ad}_{E_a} \cdots \mathrm{ad}_{E_{m-1}} \left(K_{\lambda_{L,m}}\right) \text{ and } v = \left(1 - q^{-2}\right)^{-1} \mathrm{ad}_{F_{m-1}} \cdots \mathrm{ad}_{F_a} \left(K_{\lambda_{L,a+1}}\right),$$

so that $u, v \in \mathring{U}_L$ by lemma 3.14.1. For $x, y \in U_L$ set $[x, y] := xy - yx$. Since $\mathring{U}_L$ is an algebra,

$$\left[E'_{\varepsilon_a - \varepsilon_m}, F'_{\varepsilon_a - \varepsilon_m}\right] K_{-\varepsilon_a - \varepsilon_m} K_{\lambda_{L,a+1}} = uv - q^{-2} vu \in \mathring{U}_L. \tag{77}$$

But the left hand side of (77) is equal to

$$\left(q-q^{-1}\right)^{-1}\left(K_{\varepsilon_a-\varepsilon_m}-K^{-1}_{\varepsilon_a-\varepsilon_m}\right)K_{-\varepsilon_a-\varepsilon_m}K_{\lambda_{L,a+1}}=\left(q-q^{-1}\right)^{-1}\left(K_{-2\varepsilon_m}K_{\lambda_{L,a+1}}-K_{\lambda_{L,a}}\right).$$

It follows that

$$K_{\lambda_{L,a}}=-\left(q-q^{-1}\right)\left(uv-q^{-2}vu\right)+K_{\lambda_{L,m}}K_{\lambda_{L,a+1}}\in\mathring{U}_L.$$

□

Proposition 4.1.2. *U_L is generated as an algebra by $\mathring{U}_L$ and $\{K_{\varepsilon_i}\}_{i=1}^m$. Similarly, U_R is generated as an algebra by $\mathring{U}_R$ and $\{K_{\varepsilon_i}\}_{i=1}^n$.*

Proof. We give the proof for U_L (for U_R the proof is similar). Let $\mathcal{A}$ denote the subalgebra of U_L generated by $\mathring{U}_L$ and $\{K_{\varepsilon_i}\}_{i=1}^m$. Set $\rho:=\sum_{i=1}^m i\varepsilon_i$. Then $K_{-2\rho}=K_{\lambda_{L,1}}\cdots K_{\lambda_{L,m}}$, hence by proposition 4.1.1 we have $K_{-2\rho}\in\mathring{U}_L$. Lemma 3.14.1 implies that $E_iK_{-2\rho}=(1-q^2)^{-1}\mathrm{ad}_{E_i}(K_{-2\rho})\in\mathring{U}_L$, so that $E_i\in\mathcal{A}$. By a similar argument we can prove that $F_i\in\mathcal{A}$ as well. Also by our assumption $K_{\varepsilon_i}\in\mathcal{A}$ for $1\leqslant i\leqslant m$, hence $\mathcal{A}=U_L$. □

5. Explicit formulas for $\phi_U(K_{\lambda_{L,a}}\otimes 1)$ and $\phi_U(1\otimes K_{\lambda_{R,b}})$

In this section we compute explicit formulas for $\phi_U(K_{\lambda_{L,a}}\otimes 1)$ and $\phi_U(1\otimes K_{\lambda_{R,b}})$, where $K_{\lambda_{L,a}}$ and $K_{\lambda_{R,b}}$ are defined in (75). These explicit formulas are used in the proof of theorem C.

5.1. Eigenvalues of $\mathbf{D}_{n,r}$ and q-factorial Schur polynomials

For any integer partition ν such that $\ell(\nu)\leqslant n$, let s_ν denote the q-factorial Schur polynomial in n variables associated to ν, defined by

$$s_\nu\left(x_1,\ldots,x_n;q\right):=\frac{\det\left(\prod_{k=0}^{\nu_j+n-j-1}\left(x_i-q^k\right)\right)_{1\leqslant i,j\leqslant n}}{\prod_{1\leqslant i<j\leqslant n}\left(x_i-x_j\right)}.$$

Recall that $\mathbf{D}(r,a,b)\in\mathscr{PD}_{a\times b}$ is the q-differential operator defined in (8). We will need the following statement, which is a variation of [BKV06, theorem 1].

Proposition 5.1.1. *Let λ be an integer partition satisfying $\ell(\lambda)\leqslant n$. Then the restriction of $\mathbf{D}(r,n,n)\in\mathscr{PD}_{n\times n}$ to the irreducible U_{LR}-submodule $V^*_\lambda\otimes V^*_\lambda$ of $\mathscr{P}_{n\times n}$ is a scalar multiple of identity, the scalar being*

$$\varphi_{\lambda,r,n}(q):=\frac{(-1)^r q^{r-r^2-2r(n-r)}}{\left(1-q^2\right)^r}s_{(1^r)}\left(q^{2(\lambda_1+n-1)},\ldots,q^{2(\lambda_{n-1}+1)},q^{2\lambda_n};q^2\right).\tag{78}$$

Proof. We show that the assertion follows from an analogous result in the setting of operators in $\mathrm{Pol}(\mathrm{Mat}_n)_q$ acting on $\mathbb{C}[\mathrm{Mat}_n]_q$ (see section 3.16) that is proved in [BKV06, theorem 1]. In the following proof we use the notation introduced in section 3.16. In particular we set $A:=\mathbb{Z}[q,q^{-1}]$.

Step 1. For each irreducible component $V^*_\lambda\otimes V^*_\lambda$ of $\mathscr{P}_{n\times n}$ (see proposition 3.6.1) we choose a lowest weight vector $v_\lambda\in\mathscr{P}^A_{n\times n}$ for the U_{LR}-action and set $W^A_\lambda:=U^A_{LR}\cdot v_\lambda$. From the explicit formulas of the action of U_{LR} (see remark 3.6.2) it follows that $W^A_\lambda\subseteq\mathscr{P}^A_{n\times n}$. Furthermore, the canonical map $W^A_\lambda\otimes_A\Bbbk\to V^*_\lambda\otimes V^*_\lambda$ is an isomorphism.

Step 2. By corollary 3.16.1 we obtain a map

$$\mathscr{P}\mathscr{D}^A_{n\times n} \xrightarrow{D\mapsto D\otimes 1} \mathscr{P}\mathscr{D}^A_{n\times n}\otimes_A \mathbb{C} \xrightarrow{\cong} \mathrm{Pol}\left(\mathrm{Mat}_n\right)_{q_\circ}, \tag{79}$$

that restricts to a map

$$\mathscr{P}^A_{n\times n} \xrightarrow{D\mapsto D\otimes 1} \mathscr{P}^A_{n\times n}\otimes_A \mathbb{C} \xrightarrow{\cong} \mathbb{C}\left[\mathrm{Mat}_n\right]_{q_\circ}. \tag{80}$$

We also have a commutative diagram

$$\begin{tikzcd}
U^A_{LR}\otimes_A \mathscr{P}^A_{n\times n} \arrow[r] \arrow[d, "(-)\otimes_A\mathbb{C}"'] & \mathscr{P}^A_{n\times n} \arrow[d, "(-)\otimes_A\mathbb{C}"] \\
U_{q_\circ}(\mathfrak{gl}_n)\otimes U_{q_\circ}(\mathfrak{gl}_n)\otimes \mathbb{C}[\mathrm{Mat}_n]_{q_\circ} \arrow[r] & \mathbb{C}[\mathrm{Mat}_n]_{q_\circ}
\end{tikzcd}$$

where the top horizontal map is the restriction of the U_{LR}-module structure on $\mathscr{P}_{n\times n}$ and the bottom horizontal map is (70) in the special case $m=n$. Let us denote both of the maps (79) and (80) by $\beta_{q_\circ}$. Then $\beta_{q_\circ}(t_{i,j}) = (1-q_\circ^2)^{-\frac{1}{2}}z^i_j$ and $\beta_{q_\circ}(\partial_{i,j}) = (1-q_\circ^2)^{-\frac{1}{2}}(z^i_j)^*$. From the definition of $\mathbf{D}(r,n,n)$ it is clear that $\mathbf{D}(r,n,n)\in\mathscr{P}\mathscr{D}^A_{n\times n}$. In addition $\beta_{q_\circ}\big((1-q^2)^r\mathbf{D}(r,n,n)\big) = y_r$, where y_r is the operator defined in [BKV06, equation (11)].

Step 3. From proposition 3.16.2 it follows that $\beta_{q_\circ}(v_\lambda)$ is the joint highest weight vector for the irreducible submodule $\mathbb{C}[\mathrm{Mat}_n]_{q_\circ,\lambda}$ of $\mathbb{C}[\mathrm{Mat}_n]_{q_\circ}$ that is defined in [BKV06, section 2]. Thus Step 1 and the commutative diagram of Step 2 imply that $\beta_{q_\circ}(W^A_\lambda) = \beta_{q_\circ}(U^A_{LR}\cdot v_\lambda)\subseteq \mathbb{C}[\mathrm{Mat}_n]_{q_\circ,\lambda}$.

Step 4. Fix λ such that $\ell(\lambda)\leqslant n$, choose any vector $w\in W^A_\lambda$, and set

$$w' := \left(1-q^2\right)^r\left(\mathbf{D}(r,n,n)-\varphi_{\lambda,r,n}\right)\cdot w.$$

From Steps 1–2 above it follows that $\beta_{q_\circ}(w') = y_r\cdot\beta_{q_\circ}(w) - (1-q_\circ^2)\varphi_{\lambda,r,n}(q_\circ)\beta_{q_\circ}(w)$. Since $\beta_{q_\circ}(w)\in\mathbb{C}[\mathrm{Mat}]_{q_\circ,\lambda}$, by [BKV06, theorem 1] we obtain $\beta_{q_\circ}(w')=0$. Since evaluations at $q_\circ$ for infinitely many $q_\circ$ separate the points of $\mathscr{P}^A_{n\times n}$, it follows that $w'=0$.

Step 5. By Step 4 we have $\mathbf{D}(r,n,n)\cdot w = \varphi_{\lambda,r,n}(q)w$ for $w\in W^A_\lambda$. Since W^A_λ spans $V^*_\lambda\otimes V^*_\lambda$ over $\mathbb{k}$, the same assertion holds for all $w\in V^*_\lambda\otimes V^*_\lambda$.

□

Remark 5.1.2. In our forthcoming work [LSS22b], we prove a broad extension of proposition 5.1.1 for Capelli operators on quantum symmetric spaces.

The polynomials s_ν are specializations of the *interpolation Macdonald polynomials* R_λ defined in [Sah96] (see also [Kn97] and [Ok97]). In the rest of this section we follow the notation of [Sah11, section 0.3]. Let $R_\lambda(x_1,\dots,x_n;q,t)$ denote the unique symmetric polynomial with coefficients in $\mathbb{Q}(q,t)$ that satisfies the following conditions:

(i) $\deg R_\lambda = |\lambda|$.
(ii) $R_\lambda(q^{\mu_1},\dots,q^{\mu_i}t^{1-i},\dots,q^{\mu_n}t^{1-n};q,t)=0$ for all partitions $\mu\neq\lambda$ that satisfy $|\mu|\leqslant|\lambda|$.
(iii) R_λ can be expressed as $R_\lambda = m_\lambda + \sum_{\mu\neq\lambda}c_{\mu,\lambda}m_\mu$, where the m_μ denote the monomial symmetric polynomials.

It is known [Kn97, proposition 2.8] that

$$s_\lambda(x_1,\dots,x_n;q) = q^{(n-1)|\lambda|} R_\lambda\left(q^{1-n}x_1,\dots,q^{1-n}x_n;q,q\right).$$

The proof of lemma 5.1.4 below uses Okounkov's binomial theorem for interpolation Macdonald polynomials [Ok97]. We remark that in [Ok97] the interpolation Macdonald polynomials are defined slightly differently, and are denoted by the P^*_λ, but one can show that

$$P^*_\lambda(x_1,\dots,x_n;q,t) = R_\lambda\left(x_1,x_2t^{-1},x_nt^{-n+1};q,t\right). \tag{81}$$

For two integer partitions λ,μ such that $\ell(\lambda),\ell(\mu)\leqslant n$, let $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t}$ denote the (q,t)-binomial coefficient defined in [Ok97]. Thus

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t} := \frac{P^*_\mu\left(q^{\lambda_1},\dots,q^{\lambda_n};q,t\right)}{P^*_\lambda\left(q^{\lambda_1},\dots,q^{\lambda_n};q,t\right)}. \tag{82}$$

Lemma 5.1.3. *For $0\leqslant r\leqslant n$ we have $\begin{bmatrix}1^n\\ 1^r\end{bmatrix}_{q,q} = q^{-r(n-r)}\frac{(q^n-1)\cdots(q^{n-r+1}-1)}{(q^r-1)\cdots(q-1)}$.*

Proof. The proof is a straightforward but somewhat tedious calculation based on a general combinatorial formula in [Sah11, theorem 0.8] for the (q,t)-binomial coefficients. We give a brief outline of this calculation. In the notation of [Sah11], the value of (82) can be expressed as a sum of the form $\sum_T wt(T)$, where T is a standard tableau of shape $\lambda\backslash\mu$. For $\lambda := (1^n)$ and $\mu := (1^r)$, there is only one such tableau. By direct calculation one obtains

$$\lambda^i = \left(1^{n-i}\right)\ ,\ a_{\lambda^i,\lambda^{i+1}} = \frac{t^{-n+i+1}\left(1-t^{n-i}\right)}{1-t}\ ,\ \frac{|\overline{\lambda^i}|-|\overline{\lambda^{i+1}}|}{|\overline{\lambda}|-|\overline{\lambda^{i+1}}|} = \frac{t^i(1-t)}{1-t^{i+1}}.$$

From these, the assertion of the lemma follows immediately. □

Lemma 5.1.4. *Set $\nu_r := (1^r)$ for $0\leqslant r\leqslant n$. Then*

$$\sum_{r=0}^n q^{-\binom{r}{2}-r(n-r)} s_{\nu_r}\left(q^{n-1}x_1,\dots,q^{n-i}x_i,\dots,x_n;q\right) = x_1\cdots x_n.$$

Proof. This is stated in [BKV06, proposition 10] without a proof. We show that it is a special case of Okounkov's binomial theorem [Ok97, equation (1.11)]. More specifically, from (81) it follows that

$$P^*_{\nu_r}(x_1,\dots,x_n;q,q) = q^{(1-n)r} s_{\nu_r}\left(q^{n-1}x_1,\dots,x_n;q\right).$$

We now consider the identity [Ok97, equation (1.11)] for $t := q$ and $\lambda := (1^n)$. Then the left hand side of [Ok97, equation (1.11)] is equal to $x_1\cdots x_n$, whereas its right hand side is equal to

$$\sum_{r=0}^n \begin{bmatrix}1^n\\ 1^r\end{bmatrix}_{q,q} q^{-\binom{r}{2}} \frac{(q^r-1)\cdots(q-1)}{(q^n-1)\cdots(q^{n-r+1}-1)} s_{\nu_r}\left(q^{n-1}x_1,\dots,x_n;q\right).$$

To complete the proof, we use lemma 5.1.3. □

5.2. The explicit formulas

Let $\mathbf{D}_{n,r}$ and $\mathbf{D}'_{m,r}$ be as in section 1. For $0 \leqslant r \leqslant m$ we set

$$\mathbf{D}_r := \mathbf{D}_{n,r} = \mathbf{D}'_{m,r}. \tag{83}$$

Proposition 5.2.1. $\phi_U(K_{\lambda_{L,1}} \otimes 1) = \phi_U(1 \otimes K_{\lambda_{R,1}}) = \sum_{r=0}^{m}(q^2-1)^r \mathbf{D}_r$.

Proof. Both $K_{\lambda_{L,1}} \otimes 1$ and $1 \otimes K_{\lambda_{R,1}}$ act on $\mathscr{P}^{(d)}$ by the scalar q^{2d} (this is easy to verify using remark 3.6.2). Since $\mathscr{P}$ is a faithful $\mathscr{P}\mathscr{D}$-module, by proposition 3.6.1 it suffices to verify that for every partition λ that satisfies $\ell(\lambda) \leqslant \min\{m,n\}$ and $|\lambda| = d$, the restriction of $\sum_{r=0}^{m}(q^2-1)^r \mathbf{D}_r$ to the irreducible U_{LR}-submodule $V_\lambda^* \otimes V_\lambda^*$ of $\mathscr{P}$ is multiplication by the scalar q^{2d}.

Step 1. First we prove the assertion in the case $m = n$. In this case $\mathbf{D}_r = \mathbf{D}(r,n,n)$, hence by proposition 5.1.1 it is enough to verify that

$$\sum_{r=0}^{n} q^{r-r^2-2r(n-r)} s_{\nu_r}\left(q^{2(\lambda_1+n-1)},\ldots,q^{2(\lambda_{n-1}+1)},q^{2\lambda_n};q^2\right) = q^{2(\lambda_1+\cdots+\lambda_n)}. \tag{84}$$

Equality (84) follows from lemma 5.1.4 after substituting q by $q^{\frac{1}{2}}$.

Step 2. Henceforth assume $m < n$ (by lemma 3.15.1 the proof when $m > n$ is similar). Let

$$\boldsymbol{e} = \boldsymbol{e}_{m\times n}^{n\times n} : \mathscr{P}\mathscr{D} \to \mathscr{P}\mathscr{D}_{n\times n}$$

be the embedding of algebras defined in (4), so that $\boldsymbol{e}(t_{i,j}) = t_{i+n-m,j}$ and $\boldsymbol{e}(\partial_{i,j}) = \partial_{i+n-m,j}$. Set $\tilde{\mathbf{D}}_r := \mathbf{D}(r,n,n)$. By corollary 3.17.3, $\overline{M}_{\mathbf{j}}^{\mathbf{i}} \cdot (\boldsymbol{e}(\mathscr{P})) = 0$ unless $\mathbf{i} = (u_1,\ldots,u_r)$ satifies $u_1 \geqslant n-m+1$. Thus, for every $f \in \mathscr{P}$ we have $\tilde{\mathbf{D}}_r \cdot \boldsymbol{e}(f) = \boldsymbol{e}(\mathbf{D}_r) \cdot \boldsymbol{e}(f)$ when $0 \leqslant r \leqslant m$, and $\tilde{\mathbf{D}}_r \cdot \boldsymbol{e}(f) = 0$ when $m < r \leqslant n$.

Step 3. Recall that $\mathscr{I}$ denotes the left ideal of $\mathscr{P}\mathscr{D}$ that is generated by $\mathscr{D}^{(1)}$. Let $\mathscr{I}'$ denote the left ideal of $\mathscr{P}\mathscr{D}_{n\times n}$ that is generated by $\mathscr{D}_{n\times n}^{(1)}$. Let $\boldsymbol{e} : \mathscr{P}\mathscr{D} \to \mathscr{P}\mathscr{D}_{n\times n}$ be as in Step 2. For $D \in \mathscr{P}\mathscr{D}$ and $f \in \mathscr{P}$ we have $(D \cdot f - Df) \in \mathscr{I}$, hence

$$\boldsymbol{e}(D \cdot f) - \boldsymbol{e}(D)\boldsymbol{e}(f) = \boldsymbol{e}(D \cdot f - Df) \in \mathscr{I}'.$$

But also $\boldsymbol{e}(D) \cdot \boldsymbol{e}(f) - \boldsymbol{e}(D)\boldsymbol{e}(f) \in \mathscr{I}'$. From the last two relations we obtain $\boldsymbol{e}(D) \cdot \boldsymbol{e}(f) - \boldsymbol{e}(D \cdot f) \in \mathscr{I}'$. But in addition $\boldsymbol{e}(D) \cdot \boldsymbol{e}(f) - \boldsymbol{e}(D \cdot f) \in \mathscr{P}_{n\times n}$, hence $\boldsymbol{e}(D) \cdot \boldsymbol{e}(f) = \boldsymbol{e}(D \cdot f)$.

Step 4. Let $f \in \mathscr{P}^{(d)}$. From Step 3 and Step 2 it follows that

$$\boldsymbol{e}\left(\sum_{r=0}^{m}\left(q^2-1\right)^r \mathbf{D}_r \cdot f\right) = \boldsymbol{e}\left(\sum_{r=0}^{m}\left(q^2-1\right)^r \mathbf{D}_r\right) \cdot \boldsymbol{e}(f) = \sum_{r=0}^{n}\left(q^2-1\right)^r \tilde{\mathbf{D}}_r \cdot \boldsymbol{e}(f). \tag{85}$$

From Step 1 it follows that $\sum_{r=0}^{n}(q^2-1)^r \tilde{\mathbf{D}}_r \cdot \boldsymbol{e}(f) = q^{2d}\boldsymbol{e}(f)$. Since $\boldsymbol{e}$ is an injection, from (85) we obtain $\sum_{r=0}^{m}(q^2-1)^r \mathbf{D}_r \cdot f = q^{2d} f$.

□

Proposition 5.2.2. *For* $1 \leqslant a \leqslant m$ *and* $1 \leqslant b \leqslant n$ *we have*

$$\phi_U\left(K_{\lambda_{L,a}} \otimes 1\right) = \sum_{r=0}^{m-a+1}\left(q^2-1\right)^r \mathbf{D}'_{m-a+1,r} \quad \text{and} \quad \phi_U\left(1 \otimes K_{\lambda_{R,b}}\right) = \sum_{r=0}^{n-b+1}\left(q^2-1\right)^r \mathbf{D}_{n-b+1,r}. \tag{86}$$

Proof. We give the proof for $K_{\lambda_{L,a}} \otimes 1$ only (the proof for $1 \otimes K_{\lambda_{R,b}}$ is similar). Every element of $\mathscr{P}$ is expressible as a linear combination of monomials of the form $t_{i_1,j_1} \cdots t_{i_k,j_k}$ where $i_1 \geqslant \cdots \geqslant i_k$. Choose $k' \leqslant k$ such that $i_{k'} \geqslant a$ and $i_{k'+1} < a$. Then

$$\left(K_{\lambda_{L,a}} \otimes 1\right) \cdot t_{i_1,j_1} \cdots t_{i_k,j_k} = q^{2k'} t_{i_1,j_1} \cdots t_{i_k,j_k}.$$

Set $D := \sum_{r=0}^{m-a+1} (q^2-1) \mathbf{D}'_{m-a+1,r}$. From lemma 3.17.4 it follows that

$$D \cdot \left(t_{i_1,j_1} \cdots t_{i_k,j_k}\right) = \left(D \cdot t_{i_1,j_1} \cdots t_{i_{k'},j_{k'}}\right) t_{i_{k'+1},j_{k'+1}} \cdots t_{i_k,j_k}.$$

Hence it suffices to prove that

$$D \cdot t_{i_1,j_1} \cdots t_{i_{k'},j_{k'}} = q^{2k'} t_{i_1,j_1} \cdots t_{i_{k'},j_{k'}}. \tag{87}$$

Set $\tilde{m} := m-a+1$. Let $\boldsymbol{e} = \boldsymbol{e}^{m\times n}_{\tilde{m}\times n} : \mathscr{P}\mathscr{D}_{\tilde{m}\times n} \to \mathscr{P}\mathscr{D}$ be the embedding of algebras defined in (4), so that $\boldsymbol{e}(t_{i,j}) = t_{i+a-1,j}$ and $\boldsymbol{e}(\partial_{i,j}) = \partial_{i+a-1,j}$. Set $\tilde{\mathbf{D}}_r := \mathbf{D}(r,\tilde{m},n) \in \mathscr{P}\mathscr{D}_{\tilde{m}\times n}$. Similar to the proof of proposition 5.2.1 we have $\boldsymbol{e}(\tilde{\mathbf{D}}_r) = \mathbf{D}'_{m-a+1,r}$ and $\boldsymbol{e}(\tilde{\mathbf{D}}_r \cdot f) = \boldsymbol{e}(\tilde{\mathbf{D}}_r) \cdot \boldsymbol{e}(f)$ for $f \in \mathscr{P}_{\tilde{m}\times n}$. proposition 5.2.1 for $\mathscr{P}\mathscr{D}_{\tilde{m}\times n}$ yields $\sum_{r=0}^{m-a+1} \tilde{\mathbf{D}}_r \cdot f = q^{2k'} f$ for $f := t_{i_1-a+1,j_1} \cdots t_{i_{k'}-a+1,j_{k'}} \in \mathscr{P}_{\tilde{m}\times n}$. By applying $\boldsymbol{e}$ to both sides we obtain (87). $\square$

6. Some properties of polarization operators

In this section we investigate invariance and generation properties of the $\mathsf{L}_{i,j}$, the $\mathsf{R}_{i,j}$, and their variants.

6.1. Invariants and the operators $\mathsf{L}_{i,j}$, $\mathsf{R}_{i,j}$

Recall from (3) that $\mathcal{Y}^{\mathcal{Z}}$ denotes the centralizer of $\mathcal{Z}$ in $\mathcal{Y}$.

Lemma 6.1.1. $\mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}_\bullet} = \mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}}$ *and* $\mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{R}_\bullet} = \mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{R}}$.

Proof. We only give the proofs of the two assertions for $\mathscr{L}$. The inclusion $\mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}_\bullet} \supseteq \mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}}$ is trivial because $\mathscr{L}_\bullet \subseteq \mathscr{L}$. To prove $\mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}_\bullet} \subseteq \mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}}$, choose any $T \in \mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}_\bullet}$. From proposition 5.2.2 it follows that T commutes with

$$\phi_U\left(K_{2\varepsilon_a} \otimes 1\right) = \phi_U\left(\left(K^{-1}_{\lambda_{L,a}} \otimes 1\right)\left(K_{\lambda_{L,a+1}} \otimes 1\right)\right) \quad \text{for } 1 \leqslant a \leqslant m,$$

where we assume $K_{\lambda_{L,m+1}} := 1$. From proposition 3.6.1 (and also from remark 3.6.2) it follows that $\phi_U(K_{\varepsilon_i} \otimes 1)$ is a diagonalizable operator whose eigenvalues are powers of q. In particular, the eigenspaces of $\phi_U(K_{2\varepsilon_i} \otimes 1)$ and $\phi_U(K_{\varepsilon_i} \otimes 1)$ are the same. Thus T also commutes with $\phi_U(K_{\varepsilon_i} \otimes 1)$. Finally, proposition 4.1.2 implies that $T \in \mathrm{End}_{\Bbbk}(\mathscr{P})^{\mathscr{L}}$. $\square$

As in section 2.3 set

$$\mathscr{P}\mathscr{D}_{(\epsilon_L)} := \left\{D \in \mathscr{P}\mathscr{D} \,:\, x \cdot D := \epsilon_L(x) D \text{for } x \in U_L\right\},$$

where ϵ_L denotes the counit of U_L. We define $\mathscr{P}\mathscr{D}_{(\epsilon_R)}$, $(\mathscr{A}_{k,l,n})_{(\epsilon_R)}$, $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$ and $\left(\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}\right)_{(\epsilon_R)}$ similarly (where ϵ_R denotes the counit of U_R).

Lemma 6.1.2. *$\mathscr{PD}^{\mathscr{L}\bullet} = \mathscr{PD}^{\mathscr{L}} = \mathscr{PD}_{(\epsilon_L)}$ and $\mathscr{PD}^{\mathscr{R}\bullet} = \mathscr{PD}^{\mathscr{R}} = \mathscr{PD}_{(\epsilon_R)}$.*

Proof. From lemma 6.1.1 it follows that $\mathscr{PD}^{\mathscr{L}\bullet} = \mathscr{PD}^{\mathscr{L}}$ and $\mathscr{PD}^{\mathscr{R}\bullet} = \mathscr{PD}^{\mathscr{R}}$. By proposition 3.12.1, the action of U_{LR} on $\mathscr{PD}$ is the restriction of the action of U_{LR} on $\mathrm{End}_{\Bbbk}(\mathscr{P})$ that is defined in lemma 2.3.1. Thus lemma 2.3.1 implies that $\mathscr{PD}^{\mathscr{L}} = \mathscr{PD}_{(\epsilon_L)}$ and $\mathscr{PD}^{\mathscr{R}} = \mathscr{PD}_{(\epsilon_R)}$. □

Lemma 6.1.3. *$(\mathscr{A}_{k,l,n})_{(\epsilon_R)}$ is a subalgebra of $\mathscr{A}_{k,l,n}$ and $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$ is a subalgebra of $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$.*

Proof. Follows immediately from the fact that both $\mathscr{A}_{k,l,n}$ and $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ are U_R-module algebras. □

Recall that by definition, $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ is a subalgebra of $\mathscr{PD}^{\mathrm{gr}} := \mathscr{PD}^{\mathrm{gr}}_{m\times n}$ where $m := \max\{k,l\}$. For $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$ define $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j} \in \mathscr{A}^{\mathrm{gr}}_{k,l,n}$ by

$$\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j} := \sum_{r=1}^{n} \tilde{t}_{i,r}\tilde{\partial}_{j,r} = \sum_{r=1}^{n} t_{m-i+1,r}\partial_{m-j+1,r}. \tag{88}$$

Under the isomorphism of corollary 3.13.4 the $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$ correspond to the $\mathrm{gr}(\tilde{\mathsf{L}}_{i,j}) \in \mathrm{gr}(\mathscr{A}_{k,l,n})$.

Lemma 6.1.4. *$\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j} \in \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$ and $\tilde{\mathsf{L}}_{i,j} \in (\mathscr{A}_{k,l,n})_{(\epsilon_R)}$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$.*

Proof. Recall that $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ is a U_R-module algebra because it is a U_R-stable subalgebra of $\mathscr{PD}^{\mathrm{gr}}$. For $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$ the assertion follows from the formulas of remark 3.6.2. For example

$$\begin{aligned} E_s \cdot \tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j} = E_s \cdot \sum_{r=1}^{n} \tilde{t}_{i,r}\tilde{\partial}_{j,r} &= \sum_{r=1}^{n} (E_s \cdot \tilde{t}_{i,r})\left(K_s \cdot \tilde{\partial}_{j,r}\right) + \sum_{r=1}^{n} \tilde{t}_{i,r}\left(E_s \cdot \tilde{\partial}_{j,r}\right) \\ &= \left(-q^{-1}\tilde{t}_{i,n-s}\right)\left(q\tilde{\partial}_{j,n+1-s}\right) + (\tilde{t}_{i,n-s})\left(\tilde{\partial}_{j,n+1-s}\right) = 0. \end{aligned}$$

Since the map $\mathsf{P}_{k,l,n} : \mathscr{A}^{\mathrm{gr}}_{k,l,n} \to \mathscr{A}_{k,l,n}$ is a U_R-module homomorphism, we have $\tilde{\mathsf{L}}_{i,j} \in (\mathscr{A}_{k,l,n})_{(\epsilon_R)}$. □

Lemma 6.1.5. *The U_L-submodule of $\mathscr{PD}$ that is generated by $\mathsf{L}_{m,m}$ contains $\mathsf{L}_{i,j}$ for $1 \leqslant i,j \leqslant m$. Similarly, the U_R-submodule of $\mathscr{PD}$ that is generated by $\mathsf{R}_{n,n}$ contains $\mathsf{R}_{i,j}$ for $1 \leqslant i,j \leqslant n$.*

Proof. We only give the proof for the assertion about the U_R-submodule generated by $\mathsf{R}_{n,n}$ (the other assertion is proved similarly). Denote this submodule by $\mathscr{M}$. First we prove the following relations for the U_R-action:

$$E_j \cdot \mathsf{R}_{i,j+1} = \mathsf{R}_{i,j} \text{and } F_i \cdot \mathsf{R}_{i+1,j} = -q\mathsf{R}_{i,j} \text{for } j \neq i \quad , \quad F_i \cdot \mathsf{R}_{i+1,i} = -q\mathsf{R}_{i,i} + q^{-1}\mathsf{R}_{i+1,i+1}. \tag{89}$$

The proofs of these relations are similar and based on the explicit formulas given in remark 3.6.2. For example using $\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$ we have

$$\begin{aligned} F_i \cdot \mathsf{R}_{i+1,i} = F_i \cdot \sum_{r=1}^{n} t_{r,i+1}\partial_{r,i} &= \sum \left((F_i \cdot t_{r,i+1})(\partial_{r,i}) + \left(K_i^{-1} \cdot t_{r,i+1}\right)(F_i \cdot \partial_{r,i})\right) \\ &= \sum_{r=1}^{n} \left(-qt_{r,i}\partial_{r,i} + q^{-1}t_{r,i+1}\partial_{r,i+1}\right) = -q\mathsf{R}_{i,i} + q^{-1}\mathsf{R}_{i+1,+1}. \end{aligned}$$

Since $\mathsf{R}_{n,n} \in \mathscr{M}$, from the second relation in (89) for $j = n$ we obtain $\mathsf{R}_{i,n} \in \mathscr{M}$ for $i \leqslant n$. Then using the first relation in (89) successively for $j = n-1, \ldots, i+1$ we obtain $\mathsf{R}_{i,j} \in \mathscr{M}$ for $i < j$. The above argument can be repeated with the roles of E_i and F_i switched. This yields $\mathsf{R}_{i,j} \in \mathscr{M}$ for $i > j$. Finally, from $\mathsf{R}_{n,n}$ and the third relation in (89) we obtain $\mathsf{R}_{i,i} \in \mathscr{M}$ for $1 \leqslant i \leqslant n$. $\square$

Corollary 6.1.6. $\mathsf{L}_{i,j} \in \mathscr{L}_\bullet$ *for all* $1 \leqslant i,j \leqslant m$ *and* $\mathsf{R}_{i,j} \in \mathscr{R}_\bullet$ *for* $1 \leqslant i,j \leqslant n$.

Proof. From (68) it follows that $\mathscr{L}_\bullet$ is U_L-stable. By (76) we have $\phi_U(K_{\lambda_{L,m}} \otimes 1) = 1 + (q^2 - 1)\mathsf{L}_{m,m}$, hence $\mathsf{L}_{m,m} \in \mathscr{L}_\bullet$. Hence by lemma 6.1.5 we have $\mathsf{L}_{i,j} \in \mathscr{L}_\bullet$. The proof of $\mathsf{R}_{i,j} \in \mathscr{R}_\bullet$ is similar. $\square$

6.2. The $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_l)$-module decomposition of $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$

Given two irreducible U_R-modules V_λ and V_μ, the canonical isomorphism $(V^*_\lambda \otimes V_\mu)_{(\epsilon_R)} \cong \mathrm{Hom}_{U_R}(V_\lambda, V_\mu)$ implies that

$$\dim (V^*_\lambda \otimes V_\mu)_{(\epsilon_R)} = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases} \tag{90}$$

Thus, from (66) it follows that as $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_l)$-modules we have

$$\left(\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}\right)_{(\epsilon_R)} = 0 \text{ for } r \neq s \text{ and } \left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}\right)_{(\epsilon_R)} \cong \bigoplus_{\lambda \in \Lambda_{d,r}} V^*_\lambda \otimes V_\lambda, \text{ where } d := \min\{k,l,n\}. \tag{91}$$

7. The map $\Gamma_{k,l,n}$

Let $k,l,n \geqslant 1$ be integers such that $k,l \leqslant n$. In this section we define a map

$$\Gamma_{k,l,n} : \mathscr{P}_{k\times l} \to \mathscr{A}^{\mathrm{gr}}_{k,l,n}$$

that is a bijection onto the subalgebra $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$ of $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$. A similar map was also used in [LZZ11]. The ideas of the proofs of lemmas 7.1.1 and 7.1.5 are taken from [LZZ11]. Recall from definition 3.7.2 that $\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n} \cong \mathscr{P}_{n\times n} \otimes \mathscr{D}_{n\times n}$ as a vector space.

7.1. Construction of $\Gamma_{k,l,n}$

For $n \geqslant 1$ set

$$\Gamma_n : \mathscr{P}_{n\times n} \to \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}\,,\ \Gamma_n := (1 \otimes \iota) \circ \Delta_{\mathscr{P}}, \tag{92}$$

where $\iota : \mathscr{P}_{n\times n} \to \mathscr{D}_{n\times n}$ is the anti-isomorphism of bialgebras defined in (50) and $\Delta_{\mathscr{P}}$ is the coproduct of $\mathscr{P}_{n\times n}$. In particular in Sweedler's notation we have $\Gamma_n(u) = \sum u_1 \otimes \iota(u_2)$ for $u \in \mathscr{P}_{n\times n}$.

Lemma 7.1.1. *Let* $\epsilon_{\mathscr{D}}$ *be the counit of* $\mathscr{D}_{n\times n}$. *Then the map* $1 \otimes \epsilon_{\mathscr{D}} : \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n} \to \mathscr{P}_{n\times n}$ *is a left inverse to* Γ_n. *In particular,* Γ_n *is an injection.*

Proof. This is equivalent to the relation $\sum \epsilon_{\mathscr{D}}(\iota(u_2))u_1 = u$ for $u \in \mathscr{P}_{n\times n}$. Since ι is a linear bijection, it suffices to verify that $\sum \epsilon_{\mathscr{D}}(\iota(u_2))\iota(u_1) = \iota(u)$. Since ι is an anti-isomorphism of coalgebras, the latter relation follows from the defining property of the counit $\epsilon_{\mathscr{D}}$. $\square$

Recall from proposition 3.9.4(i) that the assignments $\tilde{t}_{i,j} \mapsto \tilde{t}_{i,j}$ and $\tilde{\partial}_{i,j} \mapsto \tilde{\partial}_{i,j}$ result in an embedding of algebras

$$\boldsymbol{e}^{n\times n}_{k\times l} : \mathscr{P}\mathscr{D}_{k\times l} \to \mathscr{P}\mathscr{D}_{n\times n}. \tag{93}$$

We denote the restriction of the map (93) to the subalgebra $\mathscr{P}_{k\times l}$ by the same notation, that is

$$\boldsymbol{e}^{n\times n}_{k\times l} : \mathscr{P}_{k\times l} \to \mathscr{P}_{n\times n}. \tag{94}$$

Lemma 7.1.2. *For $\boldsymbol{e} = \boldsymbol{e}^{n\times n}_{k\times l}$ as in (94) we have $\Gamma_n(\boldsymbol{e}(\mathscr{P}_{k\times l})) \subseteq \mathscr{A}^{\mathrm{gr}}_{k,l,n}$.*

Proof. For a monomial $t_{i_1,j_1}\cdots t_{i_r,j_r} \in \mathscr{P}_{n\times n}$ we have

$$\Delta(t_{i_1,j_1}\cdots t_{i_r,j_r}) = \Delta(t_{i_1,j_1})\cdots\Delta(t_{i_r,j_r}) = \sum_{1\leqslant a_1,\ldots,a_r\leqslant n} t_{i_1,a_1}\cdots t_{i_r,a_r} \otimes t_{a_1,j_1}\cdots t_{a_r,j_r}.$$

Set $m' := n-k$ and $n' := n-l$. From the above equality it follows that

$$\Gamma_n(\boldsymbol{e}(t_{i_1,j_1}\cdots t_{i_r,j_r})) = \sum_{1\leqslant a_1,\ldots,a_r\leqslant n} t_{i_1+m',a_1}\cdots t_{i_r+m',a_r} \otimes \partial_{j_r+n',a_r}\cdots\partial_{j_1+n',a_1}.$$

Thus $\Gamma_n(\boldsymbol{e}(t_{i_1,j_1}\cdots t_{i_r,j_r})) \in \mathscr{A}^{\mathrm{gr}}_{k,l,n}$ (see remark 3.8.4). $\square$

Lemma 7.1.2 justifies that the following definition is valid.

Definition 7.1.3. We define $\Gamma_{k,l,n} : \mathscr{P}_{k\times l} \to \mathscr{A}^{\mathrm{gr}}_{k,l,n}$ to be the unique map that makes the diagram

$$\begin{tikzcd}
\mathscr{P}_{k\times l} \arrow[r, "\boldsymbol{e}^{n\times n}_{k\times l}"] \arrow[dr, dotted, "\Gamma_{k,l,n}"'] & \mathscr{P}_{n\times n} \arrow[r, "\Gamma_n"] & \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n} \\
& \mathscr{A}^{\mathrm{gr}}_{k,l,n} \arrow[ur, hook] &
\end{tikzcd}$$

commutative.

Lemma 7.1.4. *The map $\Gamma_{k,l,n}$ is injective.*

Proof. Since $\boldsymbol{e}^{n\times n}_{k\times l}$ is an injective map, this follows from lemma 7.1.1. $\square$

Lemma 7.1.5. $\Gamma_{k,l,n}(\mathscr{P}^{(d)}_{k\times l}) = \left(\mathscr{A}^{\mathrm{gr},(d,d)}_{k,l,n}\right)_{(\epsilon_R)}$ *for $d \geqslant 0$.*

Proof. First we prove that $\Gamma_{k,l,n}(\mathscr{P}_{k\times l}) \subseteq \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$. By definition 7.1.3 it suffices to prove that $\Gamma_n(\mathscr{P}_{n\times n}) \subseteq \left(\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}\right)_{(\epsilon_R)}$. By standard properties of the antipode of U_R,

$$\sum x_2 S^{-1}(x_1) = \epsilon_R(x)1 \quad \text{for } x \in U_R. \tag{95}$$

It follows that for $x \in U_R$ and $u \in \mathscr{P}_{n\times n}$ we have

$$\begin{aligned}
x\cdot\Gamma_n(u) &= \sum (x_1\cdot u_1)\otimes(x_2\cdot\iota(u_2))\\
&= \sum \left(u_{11}\langle\iota(u_{12}),S^{-1}(x_1)\rangle\right)\otimes\left(\iota(u_2)_1\langle\iota(u_2)_2,x_2\rangle\right) && \text{(By (52) and (53))}\\
&= \sum \langle\iota(u_2),S^{-1}(x_1)\rangle\langle\iota(u_3),x_2\rangle u_1\otimes\iota(u_4) && \text{(By coassociativity)}\\
&= \sum \epsilon_R(x)\langle\iota(u_2),1\rangle u_1\otimes\iota(u_3) && \text{(By (13) and (95))}\\
&= \epsilon_R(x)\sum u_1\otimes\iota(u_2) = \epsilon_R(x)\Gamma_n(u). && \text{(By counit relation of } \mathscr{D}_{n\times n})
\end{aligned}$$

Thus we have proved $x\cdot\Gamma_n(u) = \epsilon_R(x)\Gamma_n(u)$, that is, $\Gamma_n(u) \in \left(\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}\right)_{(\epsilon_R)}$. From (92) it follows that $\Gamma_{k,l,n}\left(\mathscr{P}^{(r)}_{k\times l}\right) \subseteq \mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}$. Consequently,

$$\Gamma_{k,l,n}\left(\mathscr{P}^{(r)}_{k\times l}\right) \subseteq \mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n} \cap \left(\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}\right)_{(\epsilon_R)} = \left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}\right)_{(\epsilon_R)}. \tag{96}$$

By lemma 7.1.4 , to complete the proof it suffices to verify that the two sides of (96) have equal dimensions. Since $k,l \leqslant n$, from (91) and proposition 3.6.1 it follows that both of these vector spaces have dimension equal to $\sum_{\lambda\in\Lambda_{d,r}} \mathsf{d}(\lambda,k)\mathsf{d}(\lambda,l)$ where $\mathsf{d}(\lambda,k)$ (respectively, $\mathsf{d}(\lambda,l)$) denotes the dimension of the $U_q(\mathfrak{gl}_k)$-module (respectively, $U_q(\mathfrak{gl}_l)$-module) associated to λ. $\square$

8. The product $\star_{k,l,n}$ on $\mathscr{P}_{k\times l}$

Throughout this section we assume that $m=n$ (so that $U_L \cong U_R \cong U_q(\mathfrak{gl}_n)$) and $1 \leqslant k,l \leqslant n$.

8.1. An explicit formula for the product of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}$

Recall that $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$ are subalgebras of $U_q(\mathfrak{gl}_n)^\bullet$ (see definition 3.2.3 and definition 3.4.1). Also recall that given $f,g \in U_q(\mathfrak{gl}_n)^\bullet$, we define $\left\langle f\otimes g,\mathcal{R}^{(n)}\right\rangle$ and $\left\langle f\otimes g,\underline{\mathcal{R}}^{(n)}\right\rangle$ as in (33).

Proposition 8.1.1. *Let $a,a' \in \mathscr{P}_{n\times n}$ and $b,b' \in \mathscr{D}_{n\times n}$. Then the product of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}$ satisfies*

$$(a\otimes b)(a'\otimes b') = \sum\langle\iota(a'_1)^\natural\otimes(b_1)^\natural,\mathcal{R}^{(n)}\rangle\langle\iota(a'_3)\otimes b_3,\mathcal{R}^{(n)}\rangle aa'_2\otimes b_2b',$$

where $(\Delta_{\mathscr{P}}\otimes 1)\circ\Delta_{\mathscr{P}}(a') = \sum a'_1\otimes a'_2\otimes a'_3$ and $(\Delta_{\mathscr{P}}\otimes 1)\circ\Delta_{\mathscr{P}}(b) = \sum b_1\otimes b_2\otimes b_3$, with $\Delta_{\mathscr{P}}$ and $\Delta_{\mathscr{D}}$ denoting the coproducts of $\mathscr{P}_{n\times n}$ and $\mathscr{D}_{n\times n}$ respectively.

Proof. We need to compute $\left(\check{\underline{\mathcal{R}}}_{LR}\right)_{B,A}(b\otimes a')$ where $B := \mathscr{D}_{n\times n}$ and $A := \mathscr{P}_{n\times n}$. Recall from (24) that $\check{\underline{\mathcal{R}}}_{LR} = (\check{\underline{\mathcal{R}}}_L)_{13}(\check{\underline{\mathcal{R}}}_R)_{24}$. The map

$$\mathscr{P}_{n\times n} \to U_q(\mathfrak{gl}_n)^\circ \;,\; a \mapsto a\circ\xi_{-1/q}$$

intertwines between the U_R-module structures $\mathcal{R}_{\mathscr{P}}$ and right translation on $U_q(\mathfrak{gl}_n)^\circ$, in the sense of remark 2.2.1. Thus lemma 2.7.2 implies that

$$(\underline{\mathcal{R}}_R)_{24}(b\otimes a') = \sum b_1\otimes a'_1\langle b_2\otimes\left(a'_2\circ\xi_{-1/q}\right),\underline{\mathcal{R}}^{(n)}\rangle. \tag{97}$$

Similarly, the maps

$$\mathscr{D}_{n\times n} \to U_q(\mathfrak{gl}_n)^\circ\ ,\ b \mapsto b^\natural \quad \text{and} \quad \mathscr{P}_{n\times n} \to U_q(\mathfrak{gl}_n)^\circ\ ,\ a \mapsto (a\circ\xi_{-q})^\natural$$

intertwine the actions $\mathcal{L}_{\mathscr{D}}$ and $\mathcal{L}_{\mathscr{P}}$ (on $\mathscr{D}_{n\times n}$ and $\mathscr{P}_{n\times n}$ respectively) with right translation. This is because the map $u\mapsto u^\natural$ on $U_q(\mathfrak{gl}_n)^\circ$ that is defined in (46) is an anti-automorphism of the coalgebra structure of $U_q(\mathfrak{gl}_n)^\circ$. Thus

$$(\underline{\mathcal{R}}_L)_{13}(b\otimes a') = \sum b_2\otimes a'_2\langle (b_1)^\natural\otimes(a'_1\circ\xi_{-q})^\natural, \underline{\mathcal{R}}^{(n)}\rangle. \tag{98}$$

From (97) and (98) it follows that

$$(a\otimes b)(a'\otimes b') = \sum \langle (b_1)^\natural\otimes(a'_1\circ\xi_{-q})^\natural, \underline{\mathcal{R}}^{(n)}\rangle\langle b_3\otimes(a'_3\circ\xi_{-1/q}), \underline{\mathcal{R}}^{(n)}\rangle aa'_2\otimes b_2b'. \tag{99}$$

Since $S^2(x)=\xi_{q^2}(x)$ for $x\in U_q(\mathfrak{gl}_n)$, we have $\iota(v)\circ S = v\circ\xi_{-1/q}\circ S^2 = v\circ\xi_{-q}$ for $v\in U_q(\mathfrak{gl}_n)^\circ$, and thus lemma 3.3.1 implies that $(v\circ\xi_{-q})^\natural = (\iota(v)\circ S)^\natural = \iota(v)^\natural\circ S^{-1}$. By a similar argument $v\circ\xi_{-1/q} = \iota(v)\circ S^{-1}$. Thus in (99) we can substitute $(a'_1\circ\xi_{-q})^\natural$ by $\iota(a'_1)^\natural\circ S^{-1}$ and $a'_3\circ\xi_{-1/q}$ by $\iota(a'_3)\circ S^{-1}$. After these substitutions, the assertion of the proposition follows from (43). $\square$

8.2. *The product $\star_{k,l,n}$ on $\mathscr{P}_{k\times l}$ and the map $\Gamma_{k,l,n}$*

We start by defining a binary product

$$\mathscr{P}_{n\times n}\otimes\mathscr{P}_{n\times n}\to\mathscr{P}_{n\times n}\ ,\ u\otimes v\mapsto u\star_n v.$$

Definition 8.2.1. For $u,v\in\mathscr{P}_{n\times n}$ we set

$$u\star_n v := \sum \left\langle \iota(v_1)^\natural\otimes\iota(u_3)^\natural, \mathcal{R}^{(n)}\right\rangle\left\langle \iota(v_3)\otimes\iota(u_2), \mathcal{R}^{(n)}\right\rangle u_1v_2, \tag{100}$$

where the sum ranges over summands of $(\Delta_{\mathscr{P}}\otimes 1)\circ\Delta_{\mathscr{P}}(u) = \sum u_1\otimes u_2\otimes u_3$ and $(\Delta_{\mathscr{P}}\otimes 1)\circ\Delta_{\mathscr{P}}(v) = \sum v_1\otimes v_2\otimes v_3$.

For the next proposition recall that $\Gamma_n:\mathscr{P}_{n\times n}\to\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}$ is the map defined in (92).

Proposition 8.2.2. $\Gamma_n(u\star_n v) = \Gamma_n(u)\Gamma_n(v)$ *for* $u,v\in\mathscr{P}_{n\times n}$.

Proof. By proposition 8.1.1 for $a:=u_1$, $b:=\iota(u_2)$, $a':=v_1$ and $b':=\iota(v_2)$ we obtain

$$\begin{aligned}\Gamma_n(u)\Gamma_n(v) &= \sum (u_1\otimes\iota(u_2))(v_1\otimes\iota(v_2))\\ &= \sum\langle\iota(v_1)^\natural\otimes\iota(u_4)^\natural, \mathcal{R}^{(n)}\rangle\langle\iota(v_3)\otimes\iota(u_2), \mathcal{R}^{(n)}\rangle u_1v_2\otimes\iota(u_3)\iota(v_4).\end{aligned}$$

Since $u\mapsto\iota(u)$ is an anti-automorphism of algebras, by (100) we also have

$$\Gamma_n(u\star_n v) = \sum\langle\iota(v_1)^\natural\otimes\iota(u_4)^\natural, \mathcal{R}^{(n)}\rangle\langle\iota(v_4)\otimes\iota(u_3), \mathcal{R}^{(n)}\rangle u_1v_2\otimes\iota(v_3)\iota(u_2).$$

After changing the indices as $(u_1,u_2,u_3,u_4) = (u_1,u_{21},u_{22},u_3)$ and $(v_1,v_2,v_3,v_4) = (v_1,v_2,v_{31},v_{32})$ by coassociativity, the equality $\Gamma_n(u\star_n v)=\Gamma_n(u)\Gamma_n(v)$ reduces to

$$\sum\langle\iota(v_{31})\otimes\iota(u_{21}), \mathcal{R}^{(n)}\rangle\iota(u_{22})\iota(v_{32}) = \sum\langle\iota(v_{32})\otimes\iota(u_{22}), \mathcal{R}^{(n)}\rangle\iota(v_{31})\iota(u_{21}). \tag{101}$$

Set $f := \iota(v_3)$ and $g := \iota(u_2)$. Since $u \mapsto \iota(u)$ is an anti-automorphism of coalgebras, from (43) it follows that (101) is equivalent to

$$\sum \langle f_2 \otimes g_2, \mathcal{R}^{(n)} \rangle g_1 f_1 = \sum \langle f_1 \otimes g_1, \mathcal{R}^{(n)} \rangle f_2 g_2,$$

which is a consequence of (35). □

Recall that throughout this section $1 \leqslant k, l \leqslant n$.

Lemma 8.2.3. *Let $\boldsymbol{e} := \boldsymbol{e}^{n\times n}_{k\times l}$ be as in (94). Then $\boldsymbol{e}(u) \star_n \boldsymbol{e}(v) \in \boldsymbol{e}(\mathscr{P}_{k\times l})$ for $u, v \in \mathscr{P}_{k\times l}$.*

Proof. Set $D := \Gamma_n(\boldsymbol{e}(u) \star_n \boldsymbol{e}(v))$. By lemma 7.1.5 we have

$$\Gamma_n(\boldsymbol{e}(u)) = \Gamma_{k,l,n}(u) \in \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)} \quad \text{and} \quad \Gamma_n(\boldsymbol{e}(v)) = \Gamma_{k,l,n}(v) \in \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}.$$

Since $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$ is a subalgeba of $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$, from proposition 8.2.2 it follows that $D \in \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$. Again by lemma 7.1.5 there exists $w \in \mathscr{P}_{k\times l}$ such that $D = \Gamma_{k,l,n}(w) = \Gamma_n(\boldsymbol{e}(w))$. From injectivity of Γ_n (see lemma 7.1.1) it follows that $\boldsymbol{e}(u) \star_n \boldsymbol{e}(v) = \boldsymbol{e}(w)$. Consequently we obtain $\boldsymbol{e}(u) \star_n \boldsymbol{e}(v) \in \boldsymbol{e}(\mathscr{P}_{k\times l})$. □

Lemma 8.2.3 validates the following definition.

Definition 8.2.4. For $u, v \in \mathscr{P}_{k\times l}$ we define a binary product

$$\mathscr{P}_{k\times l} \otimes \mathscr{P}_{k\times l} \to \mathscr{P}_{k\times l}, \ u \otimes v \mapsto u \star_{k,l,n} v,$$

by setting $u \star_{k,l,n} v := \boldsymbol{e}^{-1}(\boldsymbol{e}(u) \star_n \boldsymbol{e}(v))$ where $\boldsymbol{e} := \boldsymbol{e}^{n\times n}_{k\times l}$ is the map (94).

Proposition 8.2.5. *Let $u, v \in \mathscr{P}_{k\times l}$. Then the following statements hold:*

(i) $\Gamma_{k,l,n}(u \star_{k,l,n} v) = \Gamma_{k,l,n}(u)\Gamma_{k,l,n}(v)$.

(ii) *If* $u \in \mathscr{P}^{(r)}_{k\times l}$ *and* $v \in \mathscr{P}^{(s)}_{k\times l}$ *then* $u \star_{k,l,n} v \in \mathscr{P}^{(r+s)}_{k\times l}$ *and* $\Gamma_{k,l,n}(u \star_{k,l,n} v) \in \left(\mathscr{A}^{\mathrm{gr},(r+s,r+s)}_{k,l,n}\right)_{(\epsilon_R)}$.

Proof. (i) By proposition 8.2.2 we have

$$\begin{aligned}\Gamma_{k,l,n}\left(u \star_{k,l,n} v\right) &= \Gamma_{k,l,n}\left(\boldsymbol{e}^{-1}\left(\boldsymbol{e}(u) \star_n \boldsymbol{e}(v)\right)\right) \\ &= \Gamma_n\left(\boldsymbol{e}(u) \star_n \boldsymbol{e}(n)\right) = \Gamma_n(\boldsymbol{e}(u))\Gamma_n(\boldsymbol{e}(v)) = \Gamma_{k,l,n}(u)\Gamma_{k,l,n}(v).\end{aligned}$$

(ii) By definitions of $\star_{k,l,n}$ and $\Gamma_{k,l,n}$ the assertions reduce to proving that for $u \in \mathscr{P}^{(r)}_{n\times n}$ and $v \in \mathscr{P}^{(s)}_{n\times n}$ we have $u \star_n v \in \mathscr{P}^{(r+s)}_{n\times n}$ and $\Gamma_n(u \star_n v) \in \mathscr{P}\mathscr{D}^{\mathrm{gr},(r+s,r+s)}_{n\times n}$. The latter assertions follow from (100) and the fact that for $d \geqslant 0$ we have $\Delta_{\mathscr{P}}(\mathscr{P}^{(d)}_{n\times n}) \subseteq \mathscr{P}^{(d)}_{n\times n} \otimes \mathscr{P}^{(d)}_{n\times n}$. □

8.3. The map Υ

Let $\Upsilon : \mathscr{P}_{n\times n}\otimes\mathscr{P}_{n\times n}\to\mathscr{P}_{n\times n}\otimes\mathscr{P}_{n\times n}$ be the map defined by

$$\Upsilon(u\otimes v) := \sum\langle\iota(v_1)^\natural\otimes\iota(u_3)^\natural,\mathcal{R}^{(n)}\rangle\langle\iota(v_3)\otimes\iota(u_2),\mathcal{R}^{(n)}\rangle u_1\otimes v_2. \qquad (102)$$

Lemma 8.3.2. *Let* $\mathsf{m}_{n\times n}:\mathscr{P}_{n\times n}\otimes\mathscr{P}_{n\times n}\to\mathscr{P}_{n\times n}$ *denote the usual product of the algebra* $\mathscr{P}_{n\times n}$*. Then the following statements hold.*

(i) $u\star_n v = \mathsf{m}_{n\times n}\circ\Upsilon(u\otimes v)$ *for* $u,v\in\mathscr{P}_{n\times n}$.
(ii) $u\star_{k,l,n} v = \boldsymbol{e}^{-1}\big(\mathsf{m}_{n\times n}(\Upsilon(\boldsymbol{e}(u)\otimes\boldsymbol{e}(v)))\big)$ *for* $u,v\in\mathscr{P}_{k\times l}$ *where* $\boldsymbol{e}:=\boldsymbol{e}^{n\times n}_{k\times l}$ *is as in (94).*

Proof. Straightforward from definition 8.2.1. □

Lemma 8.3.3. $\langle\partial_{i_1,j_1}\cdots\partial_{i_r,j_r},1\rangle = 0$ *unless when* $i_k = j_k$ *for* $1\leqslant k\leqslant r$.

Proof. Follows immediately from the definition of the $\partial_{i,j}$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between $U_q(\mathfrak{gl}_n)^\circ$ and $U_q(\mathfrak{gl}_n)$. □

Proposition 8.3.3. *Let* $\boldsymbol{e}:=\boldsymbol{e}^{n\times n}_{k\times l}$ *be as in (94). Set* $\mathscr{W}_{d_1,d_2}:=\mathscr{P}^{(d_1)}_{k\times l}\otimes\mathscr{P}^{(d_2)}_{k\times l}$ *for* $d_1,d_2\geqslant 0$ *and* $\mathscr{W}'_{d_1,d_2}:=(\boldsymbol{e}\otimes\boldsymbol{e})(\mathscr{W}_{d_1,d_2})$*. Then* $\Upsilon(\mathscr{W}'_{d_1,d_2})\subseteq\mathscr{W}'_{d_1,d_2}$.

Proof. From the defining formula of Υ and the fact that the coproduct of $\mathscr{P}_{n\times n}$ maps $\mathscr{P}^{(a)}_{n\times n}$ into $\mathscr{P}^{(a)}_{n\times n}\otimes\mathscr{P}^{(a)}_{n\times n}$ we obtain $\Upsilon\left(\mathscr{P}^{(a)}_{n,n}\otimes\mathscr{P}^{(b)}_{n\times n}\right)\subseteq\mathscr{P}^{(a)}_{n,n}\otimes\mathscr{P}^{(b)}_{n\times n}$. The claim follows if we prove that

$$\Upsilon(\boldsymbol{e}(u)\otimes\boldsymbol{e}(v))\in\boldsymbol{e}\otimes\boldsymbol{e}(\mathscr{P}_{k\times l}\otimes\mathscr{P}_{k\times l})\quad\text{for } u,v\in\mathscr{P}_{k\times l}. \qquad (103)$$

It suffices to prove this assertion for monomials $u=t_{i_1,j_1}\cdots t_{i_r,j_r}$ and $v=t_{p_1,q_1}\cdots t_{p_s,q_s}$ in $\mathscr{P}_{k\times l}$. Set $m':=n-k$ and $n':=n-l$. Then

$$((\Delta\otimes 1)\circ\Delta)(\boldsymbol{e}(u)) = \sum\boldsymbol{e}(u)_1\otimes\boldsymbol{e}(u)_2\otimes\boldsymbol{e}(u)_3,$$

where for indices $1\leqslant a_1,b_1,\dots,a_r,b_r\leqslant n$ we have

$$\boldsymbol{e}(u)_1 = t_{m'+i_1,a_1}\cdots t_{m'+i_r,a_r},\quad \boldsymbol{e}(u)_2 = t_{a_1,b_1}\cdots t_{a_r,b_r},\quad \boldsymbol{e}(u)_3 = t_{b_1,n'+j_1}\cdots t_{b_r,n'+j_r}.$$

Similarly,

$$((\Delta\otimes 1)\circ\Delta)(\boldsymbol{e}(v)) = \sum\boldsymbol{e}(v)_1\otimes\boldsymbol{e}(v)_2\otimes\boldsymbol{e}(v)_3,$$

where for indices $1\leqslant c_1,d_1,\dots,c_s,d_s\leqslant n$ we have

$$\boldsymbol{e}(v)_1 = t_{m'+p_1,c_1}\cdots t_{m'+p_s,c_s}\quad,\quad \boldsymbol{e}(v)_2 = t_{c_1,d_1}\cdots t_{c_s,d_s}\quad,\quad \boldsymbol{e}(v)_3 = t_{d_1,n'+q_1}\cdots t_{d_s,n'+q_s}.$$

In the rest of this proof we set

$$\mathsf{a}:=(a_1,\dots,a_r)\ ,\ \mathsf{b}:=(b_1,\dots,b_r)\ ,\ \mathsf{c}:=(c_1,\dots,c_s)\ ,\ \mathsf{d}:=(d_1,\dots,d_s).$$

Step 1. From (102) it follows that

$$\Upsilon(\boldsymbol{e}(u)\otimes\boldsymbol{e}(v)) = \sum_{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}} M(\mathsf{b},\mathsf{c})M'(\mathsf{a},\mathsf{b},\mathsf{d})\,t_{m'+i_1,a_1}\cdots t_{m'+i_r,a_r}\otimes t_{c_1,d_1}\cdots t_{c_s,d_s}, \qquad (104)$$

where using the fact that the map $u \mapsto u^\natural$ of (46) induces an isomorphism between $U(\mathfrak{gl}_n)^\circ$ and $(U(\mathfrak{gl}_n)^\circ)^{\mathrm{cop}}$ we have

$$M(\mathsf{b},\mathsf{c}) := \langle \partial_{m'+p_s,c_s} \cdots \partial_{m'+p_1,c_1} \otimes \partial_{b_r,n'+j_r} \cdots \partial_{b_1,n'+j_1} \, , \mathcal{R}^{(n)} \rangle$$

and

$$M'(\mathsf{a},\mathsf{b},\mathsf{d}) := \langle \partial_{n'+q_s,d_s} \cdots \partial_{n'+q_1,d_1} \otimes \partial_{b_r,a_r} \cdots \partial_{b_1,a_1} \, , \mathcal{R}^{(n)} \rangle .$$

Note that by definition, both of the R-matrix pairings $M(\mathsf{b},\mathsf{c})$ and $M'(\mathsf{a},\mathsf{b},\mathsf{d})$ correspond to the action of $U_q(\mathfrak{gl}_n)$ on $\mathscr{D}_{n\times n} \subseteq U_q(\mathfrak{gl}_n)^\circ$ by *right* translation $\mathcal{R}_{\mathscr{D}}$ (see section 2.7).

Step 2. We prove that if $c_t \leqslant m'$ for some $1 \leqslant t \leqslant s$ then $M(\mathsf{b},\mathsf{c}) = 0$. To this end, we investigate the effect of the action of $\mathcal{R}^{(n)}$ on the first component in $M(\mathsf{b},\mathsf{c})$, i.e. on $\partial_{m'+p_s,c_s} \cdots \partial_{m'+p_1,c_1}$. Recall that $\mathcal{R}^{(n)}$ acts by a product of 2-tensors of the form $E_\beta \otimes F_\beta$, where $\beta = \varepsilon_{\ell_1} - \varepsilon_{\ell_2}$ for $1 \leqslant \ell_1 < \ell_2 \leqslant n$, followed by $e^{h\sum_{i=1}^n H_i \otimes H_i}$ (which acts by scalars on tensor product of monomials in the ∂'s and we can ignore it in the argument that follows). The s-fold coproduct of E_β is a sum of s-tensors of the form $X := X_s \otimes \cdots \otimes X_1$ with components in $\{E_\beta, K_\beta, 1\}$. From remark 3.6.2 it follows that the action of X on any monomial $\partial_{m'+p_s,\bar{c}_s} \cdots \partial_{m'+p_1,\bar{c}_1}$ does not increase the indices $\bar{c}_1, \ldots, \bar{c}_s$ and leaves the indices $n'+p_1, \ldots, n'+p_s$ unchanged. Thus lemma 8.3.3 implies that $M(\mathsf{b},\mathsf{c}) = 0$.

Step 3. We prove that if $d_t \leqslant n'$ for some $1 \leqslant t \leqslant s$ then $M'(\mathsf{a},\mathsf{b},\mathsf{d}) = 0$. The argument is similar to Step 2, by investigating the action of root vectors E_β on the first component of $M'(\mathsf{a},\mathsf{b},\mathsf{d})$, that is on $\partial_{n'+q_s,d_s} \cdots \partial_{n'+q_1,d_1}$.

Step 4. We prove that if $b_t \leqslant n'$ for some $1 \leqslant t \leqslant r$ then $M(\mathsf{b},\mathsf{c}) = 0$. Again the argument is similar to Step 2. This time use the fact that the action of the root vectors F_β does not decrease the indices $n'+j_1, \ldots, n'+j_r$.

Step 5. We prove that if $a_t \leqslant n'$ for some $1 \leqslant t \leqslant r$ then $M(\mathsf{b},\mathsf{c})M'(\mathsf{a},\mathsf{b},\mathsf{d}) = 0$. The proof is slightly more complicated than Steps 2–4. By Step 4 we can assume that $\min\{b_1, \ldots, b_r\} \geqslant n'+1$. As in Steps 2–4 we can express $M'(\mathsf{a},\mathsf{b},\mathsf{d})$ as a sum over the values

$$C_{\partial_E,\partial_F,\beta_1,\ldots,\beta_N} \left(\langle (E_{\beta_1} \cdots E_{\beta_N}) \cdot \partial_E \, , 1 \rangle\right) \left(\langle (F_{\beta_1} \cdots F_{\beta_N}) \cdot \partial_F \, , 1 \rangle\right), \tag{105}$$

where $\partial_E := \partial_{n'+q_s,d_s} \cdots \partial_{n'+q_1,d_1}$, $\partial_F := \partial_{b_r,a_r} \cdots \partial_{b_1,a_1}$ and $C_{\partial_E,\partial_F,\beta_1,\ldots,\beta_N}$ is a scalar in $\Bbbk$ that results from the action of $e^{h\sum_{i=1}^n H_i \otimes H_i}$ (again, this scalar does not play a role in the argument that follows). For $\beta = \varepsilon_{\ell_1} - \varepsilon_{\ell_2}$ with $1 \leqslant \ell_1 < \ell_2 \leqslant n$ we have

$$E_\beta \cdot \partial_{\ell',\ell''} = \begin{cases} 0 & \text{if } \ell_2 \neq \ell'', \\ (-1)^{\ell_2-\ell_1-1} \partial_{\ell',\ell_1} & \text{if } \ell_2 = \ell'', \end{cases} \tag{106}$$

and

$$F_\beta \cdot \partial_{\ell',\ell''} = \begin{cases} 0 & \text{if } \ell_1 \neq \ell'', \\ (-1)^{\ell_2-\ell_1-1} \partial_{\ell',\ell_2} & \text{if } \ell_1 = \ell''. \end{cases} \tag{107}$$

First suppose that there exists $1 \leqslant N' \leqslant N$ such that $\beta_{N'} = \varepsilon_{\ell_1} - \varepsilon_{\ell_2}$ with $\ell_1 \leqslant n'$. Then by (106) and an argument similar to Step 2 we obtain $\langle (E_{\beta_1} \cdots E_{\beta_N}) \cdot \partial_E \, , 1 \rangle = 0$, hence the corresponding value (105) vanishes. Next suppose that for all $1 \leqslant N' \leqslant N$ we have $\beta_{N'} = \varepsilon_{\ell_1} - \varepsilon_{\ell_2}$ where $\ell_1 \geqslant n'+1$. The r-fold coproduct of $F_{\beta_1} \cdots F_{\beta_N}$ is a sum of r-tensors $X = X_r \otimes \cdots \otimes X_1$

whose components are products of the $F_{\beta_{N'}}$ and the $K^{-1}_{\beta_{N'}}$. If no $F_{\beta_{N'}}$ occurs in X_t then lemma 8.3.3 implies that $\langle X\cdot\partial_F,1\rangle=0$ unless $a_t=b_t\geqslant n'+1$. If at least one $F_{\beta_{N'}}$ occurs in X_t then from (107) it follows that $\langle X\cdot\partial_F,1\rangle=0$ unless $a_t\geqslant n'+1$ (because we must have $a_t=\ell_1$). Since we have assumed that $a_t\leqslant n'$ we obtain $\langle (F_{\beta_1}\cdots F_{\beta_N})\cdot\partial_F,1\rangle=0$. Thus all the values (105) vanish and we have $M'(\mathsf{a},\mathsf{b},\mathsf{d})=0$.

Step 6. From Steps 2–5 it follows that the only two-tensors on the right hand side of (104) that have a nonzero coefficient are those that belong to $(\boldsymbol{e}\otimes\boldsymbol{e})\,(\mathscr{P}_{k\times l}\otimes\mathscr{P}_{k\times l})$. This completes the proof of (103).

□

Proposition 8.3.4. Υ *induces linear bijections* $\mathscr{P}^{(r)}_{n,n}\otimes\mathscr{P}^{(s)}_{n\times n}\to\mathscr{P}^{(r)}_{n,n}\otimes\mathscr{P}^{(s)}_{n\times n}$ *for all* $r,s\geqslant 0$.

Proof. Since $\mathscr{P}^{(r)}_{n,n}\otimes\mathscr{P}^{(s)}_{n\times n}$ is finite dimensional, it suffices to prove that Υ is an injection. Set $\Upsilon^\iota:=(\iota^{-1}\otimes\iota^{-1})\circ\Upsilon\circ(\iota\otimes\iota)$ with ι as in (49). Since ι is an antiautomorphism of bialgebras and preseves the grading of $\mathscr{P}_{n\times n}$, we have

$$\Upsilon^\iota:\mathscr{D}^{(r)}_{n\times n}\otimes\mathscr{D}^{(s)}_{n\times n}\to\mathscr{D}^{(r)}_{n\times n}\otimes\mathscr{D}^{(s)}_{n\times n},$$
$$\Upsilon^\iota(u\otimes v)=\sum\langle (v_3)^\natural\otimes(u_1)^\natural,\mathcal{R}^{(n)}\rangle\langle v_1\otimes u_2,\mathcal{R}^{(n)}\rangle u_3\otimes v_2.$$

It suffices to prove injectivity of Υ^ι.

Step 1. Define $\Upsilon^{(1)}:\mathscr{D}^{(r)}_{n\times n}\otimes\mathscr{D}^{(s)}_{n\times n}\to\mathscr{D}^{(r)}_{n\times n}\otimes\mathscr{D}^{(s)}_{n\times n}$ by

$$\Upsilon^{(1)}(u\otimes v):=\sum\langle u_1\otimes v_1,\underline{\mathcal{R}}^{(n)}\rangle u_2\otimes v_2.$$

From lemma 2.7.2(i) we have

$$\begin{aligned}\sum f_1\otimes g_1\langle g_2\otimes f_2,\underline{\mathcal{R}}^{(n)}\rangle&=\sigma\left(\sum g_1\otimes f_1\langle g_2\otimes f_2,\underline{\mathcal{R}}^{(n)}\rangle\right)\\&=\sigma\circ\underline{\mathcal{R}}^{(n)}(g\otimes f)=\left(\mathcal{R}^{(n)}\right)^{-1}(f\otimes g).\end{aligned}$$

From this and lemma 2.7.2(ii) it follows that

$$\begin{aligned}\Upsilon^{(1)}\circ\Upsilon^\iota(u\otimes v)&=\sum\langle (v_4)^\natural\otimes(u_1)^\natural,\mathcal{R}^{(n)}\rangle\langle v_1\otimes u_2,\mathcal{R}^{(n)}\rangle\langle u_3\otimes v_2,\underline{\mathcal{R}}^{(n)}\rangle u_4\otimes v_3\\&=\sum\langle (v_3)^\natural\otimes(u_1)^\natural,\mathcal{R}^{(n)}\rangle\langle\left(\mathcal{R}^{(n)}\right)^{-1}(v_1\otimes u_2),\mathcal{R}^{(n)}\rangle u_3\otimes v_2\\&=\sum\langle (v_2)^\natural\otimes(u_1)^\natural,\mathcal{R}^{(n)}\rangle u_2\otimes v_1.\end{aligned}$$

Step 2. Define $\Upsilon^{(2)}:\mathscr{D}^{(r)}_{n\times n}\otimes\mathscr{D}^{(s)}_{n\times n}\to\mathscr{D}^{(r)}_{n\times n}\otimes\mathscr{D}^{(s)}_{n\times n}$ by $\Upsilon^{(2)}(u\otimes v):=u^\natural\otimes v$. Set

$$\check{\Upsilon}:=\left(\Upsilon^{(2)}\right)^{-1}\circ\left(\Upsilon^{(1)}\circ\Upsilon^\iota\right)\circ\Upsilon^{(2)}.$$

Since $u\mapsto u^\natural$ is an antiautomorphism of coalgebras on $\mathscr{D}_{n\times n}$, we obtain

$$\check{\Upsilon}(u\otimes v)=\sum\langle (v_2)^\natural\otimes u_2,\mathcal{R}^{(n)}\rangle u_1\otimes v_1.$$

Step 3. From lemmas 3.4.2–3.4.4 it follows that the assignment $v \mapsto (v \circ S)^\natural$ induces an isomorphism of coalgebras $\mathscr{P}_{n\times n} \to \mathscr{D}_{n\times n}$ that preserves the grading. Define a map

$$\Upsilon^{(3)} : \mathscr{D}^{(r)}_{n\times n} \otimes \mathscr{P}^{(s)}_{n\times n} \to \mathscr{D}^{(r)}_{n\times n} \otimes \mathscr{D}^{(s)}_{n\times n} \,,\; \Upsilon^{(3)}(u \otimes v) := u \otimes \left((v \circ S)^\natural\right).$$

Using (43) we obtain

$$\left(\Upsilon^{(3)}\right)^{-1} \circ \breve{\Upsilon} \circ \Upsilon^{(3)}(u \otimes v) = \sum \langle (v_2 \circ S) \otimes u_2, \mathcal{R}^{(n)} \rangle u_1 \otimes v_1 = \sum \langle u_2 \otimes v_2, \underline{\mathcal{R}}^{(n)} \rangle u_1 \otimes v_1.$$

Lemma 2.7.2(i) implies that $\left(\Upsilon^{(3)}\right)^{-1} \circ \breve{\Upsilon} \circ \Upsilon^{(3)}(u \otimes v) = \underline{\mathcal{R}}^{(n)}(u \otimes v)$. Since the map $u \otimes v \mapsto \underline{\mathcal{R}}^{(n)}(u \otimes v)$ is an injection, Υ^ι is also an injection. □

Corollary 8.3.5. *Let $\boldsymbol{e} := \boldsymbol{e}^{n\times n}_{k\times l}$ be as in (94) and let $\mathsf{m}_{k\times l} : \mathscr{P}_{k\times l} \otimes \mathscr{P}_{k\times l} \to \mathscr{P}_{k\times l}$ be the usual product of $\mathscr{P}_{k\times l}$. Then the following statements hold.*

(i) $u \star_{k,l,n} v = \mathsf{m}_{k\times l} \circ (\boldsymbol{e}^{-1} \otimes \boldsymbol{e}^{-1}) \circ \Upsilon(\boldsymbol{e}(u) \otimes \boldsymbol{e}(v))$ for $u, v \in \mathscr{P}_{k\times l}$.
(ii) For $r, s \geqslant 0$ the map $\mathscr{P}^{(r)}_{k\times l} \otimes \mathscr{P}^{(s)}_{k\times l} \to \mathscr{P}^{(r+s)}_{k\times l}$ given by $u \otimes v \mapsto u \star_{k,l,n} v$ is surjective.

Proof. (i) By proposition 8.3.3 we have $\Upsilon(\boldsymbol{e}(u) \otimes \boldsymbol{e}(v)) \in (\boldsymbol{e} \otimes \boldsymbol{e})(\mathscr{P}_{k\times l} \otimes \mathscr{P}_{k\times l})$. The assertion follows from the relation $\mathsf{m}_{k\times l} = \boldsymbol{e}^{-1} \circ \mathsf{m}_{n\times n} \circ (\boldsymbol{e} \otimes \boldsymbol{e})$ and lemma 8.3.2(ii).
(ii) By propositions 8.3.4 and 8.3.3 the map

$$\Upsilon_{k,l,n} := \left(\boldsymbol{e}^{-1} \otimes \boldsymbol{e}^{-1}\right) \circ \Upsilon \circ (\boldsymbol{e} \otimes \boldsymbol{e})$$

is a linear bijection on $\mathscr{P}^{(r)}_{k\times l} \otimes \mathscr{P}^{(s)}_{k\times l}$. From (i) it follows that $u \star_{k,l,n} v = \mathsf{m}_{k\times l}(\Upsilon_{k,l,n}(u \otimes v))$. The latter equality reduces the assertion to surjectivity of $\mathsf{m}_{k\times l} : \mathscr{P}^{(r)}_{k\times l} \otimes \mathscr{P}^{(s)}_{k\times l} \to \mathscr{P}^{(r+s)}_{k\times l}$, which is a trivial statement. □

9. Proofs of theorems A and B

We begin by describing our strategy for proving theorems A and B. Lemma 3.15.1 implies that theorem A(ii) follows by symmetry from theorem A(i). Furthermore, lemma 6.1.2 implies that theorem A(i) is the special case of theorem B for $k = l = m$. Thus, it suffices to prove theorem B.

We now give an outline of the proof of theorem B. By corollary 3.13.4 we have a U_R-equivariant isomorphism of $\Bbbk$-algebras $\mathrm{gr}(\mathscr{A}_{k,l,n}) \cong \mathscr{A}^{\mathrm{gr}}_{k,l,n}$. Recall that by definition, $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ is a subalgebra of $\mathscr{P}\mathscr{D}^{\mathrm{gr}} := \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m\times n}$ where $m := \max\{k, l\}$. theorem B for $\mathrm{gr}(\mathscr{A}_{k,l,n})$ is equivalent to the following assertion for $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$.

Theorem B′. *The algebra $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$ is generated by the $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$, where $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$ is defined as in (88).*

Let $\mathcal{B} \subseteq \mathscr{A}^{\mathrm{gr}}_{k,l,n}$ denote the subalgebra generated by the $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$. Lemma 6.1.4 implies that $\mathcal{B} \subseteq \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$. Let $\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}$ be defined as in (65). Since by (91) we have $\left(\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}\right)_{(\epsilon_R)} = 0$ for $r \neq s$, to prove theorem B′ it suffices to verify that

$$\left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}\right)_{(\epsilon_R)} \subseteq \mathcal{B} \quad \text{for } r \geqslant 0. \tag{108}$$

We will verify (108) in the case $n \geqslant \max\{k,l\}$ in section 9.1 and in the case $n < \max\{k,l\}$ in section 9.2. This completes the proof of theorem B′ in both cases. Then in section 9.3 we reduce theorem B for $\mathscr{A}_{k,l,n}$ to theorem B′.

9.1. *Proof of theorem B′ when* $n \geqslant \max\{k,l\}$

We prove by induction on r that

$$\left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}\right)_{(\epsilon_R)} \subseteq \mathcal{B}_r \quad \text{for } r \geqslant 0,$$

where

$$\mathcal{B}_r := \mathrm{Span}_{\Bbbk}\left\{\tilde{\mathsf{L}}^{\mathrm{gr}}_{i_1,j_1}\cdots\tilde{\mathsf{L}}^{\mathrm{gr}}_{i_r,j_r} \,:\, 1 \leqslant i_1,\ldots,i_r \leqslant k\,,\ 1 \leqslant j_1,\ldots,j_r \leqslant l\right\}.$$

For $r=0$ the assertion is trivial. For $r=1$, from lemma 7.1.5 it follows that $\left(\mathscr{A}^{\mathrm{gr},(1,1)}_{k,l,n}\right)_{(\epsilon_R)}$ is spanned by the $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j} = \Gamma_{k,l,n}(\tilde{t}_{i,j})$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$, where $\tilde{t}_{i,j}$ is defined as in (5) for $a := k$ and $b := l$. Finally, assume $r > 1$ and choose any $D \in \left(\mathscr{A}^{(r,r)}_{k,l,n}\right)_{(\epsilon_R)}$. By lemma 7.1.5 we have $D = \Gamma_{k,l,n}(u)$ for some $u \in \mathscr{P}^{(r)}_{k\times l}$. By corollary 8.3.5(ii) the linear map

$$\mathscr{P}^{(1)}_{k\times l} \otimes \mathscr{P}^{(r-1)}_{k\times l} \to \mathscr{P}^{(r)}_{k\times l}\,,\ u\otimes v \mapsto u \star_{k,l,n} v$$

is a surjection. Thus, we can express u as a sum of products of the form $u' \star_{k,l,n} u''$ where $u' \in \mathscr{P}^{(1)}_{k\times l}$ and $u'' \in \mathscr{P}^{(r-1)}_{k\times l}$. By proposition 8.2.5(i),

$$\Gamma_{k,l,n}\left(u' \star_{k,l,n} u''\right) = \Gamma_{k,l,n}\left(u'\right)\Gamma_{k,l,n}\left(u''\right).$$

From lemma 7.1.5 and the induction hypothesis it follows that $\Gamma_{k,l,n}(u') \in \mathcal{B}_1$ and $\Gamma_{k,l,n}(u'') \in \mathcal{B}_{r-1}$. Consequently, $D = \Gamma_{k,l,n}(u) = \sum \Gamma_{k,l,n}(u' \star_{k,l,n} u'') = \sum \Gamma_{k,l,n}(u')\Gamma_{k,l,n}(u'') \in \mathcal{B}_r$.

9.2. *Proof of theorem B′ when* $n < \max\{k,l\}$

Set $\underline{k} := \min\{k,n\}$ and $\underline{l} := \min\{l,n\}$. We use a reduction to theorem B′ for the case of $\mathscr{A}^{\mathrm{gr}}_{\underline{k},\underline{l},n}$, which follows from section 9.1. This technique is also used in [LZZ11]. However, the arguments of [LZZ11] do *not* extend routinely to the present setting. The reason is that unlike [LZZ11], the products of the generators $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$ are not weight vectors for the Cartan subalgebras of $U_q(\mathfrak{gl}_k)$ and $U_q(\mathfrak{gl}_l)$. As explained below, in order to circumvent this technical difficulty we use proposition 2.6.2.

Set $\underline{m} := \max\{\underline{k}, \underline{l}\}$ and $m := \max\{k, l\}$. Recall that by definition, $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ is a subalgebra of $\mathscr{P}\mathscr{D}^{\mathrm{gr}} = \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m\times n}$ and $\mathscr{A}^{\mathrm{gr}}_{\underline{k},\underline{l},n}$ is a subalgebra of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{\underline{m}\times n}$. Let

$$\boldsymbol{e}^{\mathrm{gr}} := (\boldsymbol{e}^{\mathrm{gr}})^{m\times n}_{\underline{m}\times n} : \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{\underline{m}\times n} \to \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m\times n}$$

be the map defined in proposition 3.9.4(i). By checking the images of generators of $\mathscr{A}^{\mathrm{gr}}_{\underline{k},\underline{l},n}$ we obtain

$$\boldsymbol{e}^{\mathrm{gr}}\left(\mathscr{A}^{\mathrm{gr}}_{\underline{k},\underline{l},n}\right) \subseteq \mathscr{A}^{\mathrm{gr}}_{k,l,n}.$$

Lemma 9.2.1. $\boldsymbol{e}^{\mathrm{gr}}\left(\left(\mathscr{A}^{\mathrm{gr}}_{\underline{k},\underline{l},n}\right)_{(\epsilon_R)}\right) = \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)} \cap \boldsymbol{e}^{\mathrm{gr}}\left(\mathscr{A}^{\mathrm{gr}}_{\underline{k},\underline{l},n}\right)$.

Proof. This follows from U_R-equivariance of the map $\boldsymbol{e}^{\mathrm{gr}}$ (see proposition 3.9.4). □

Recall that $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ is a module over $U_{LR} \otimes U_{LR} = U_L \otimes U_R \otimes U_L \otimes U_R$. Let $U_L^{(k,l)}$ be the subalgebra of $U_{LR} \otimes U_{LR}$ defined by

$$U_L^{(k,l)} := \boldsymbol{\kappa}_{k,m}\left(U_q\left(\mathfrak{gl}_k\right)\right) \otimes 1 \otimes \boldsymbol{\kappa}_{l,m}\left(U_q\left(\mathfrak{gl}_l\right)\right) \otimes 1.$$

Proposition 9.2.2. *The $U_L^{(k,l)}$-submodule of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ that is generated by $\boldsymbol{e}^{\mathrm{gr}}\left(\left(\mathscr{A}^{\mathrm{gr}}_{\underline{k},\underline{l},n}\right)_{(\epsilon_R)}\right)$ is equal to $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$.*

Proof. Set $d := \min\{\underline{k}, \underline{l}\} = \min\{k, l, n\}$. First note that by (90) and (91) we have isomorphisms of $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_l)$-modules

$$\left(\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}\right)_{(\epsilon_R)} = 0 \quad \text{for } r \neq s \quad \text{and} \quad \left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}\right)_{(\epsilon_R)} \cong \bigoplus_{\lambda\in\Lambda_{d,r}} V_\lambda^* \otimes V_\lambda, \tag{109}$$

where V_λ^* (respectively, V_λ) denotes an irreducible $U_q(\mathfrak{gl}_k)$-module (respectively, $U_q(\mathfrak{gl}_l)$-module). Similarly, using the equivariance of $\boldsymbol{e}^{\mathrm{gr}}$ from proposition 3.9.4(ii) we obtain

$$\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)} \cap \boldsymbol{e}^{\mathrm{gr}}\left(\mathscr{A}^{\mathrm{gr},(r,s)}_{\underline{k},\underline{l},n}\right) = 0 \quad \text{for } r \neq s,$$

and an isomorphism of $U_q\left(\mathfrak{gl}_{\underline{k}}\right) \otimes U_q\left(\mathfrak{gl}_{\underline{l}}\right)$-modules

$$\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)} \cap \boldsymbol{e}^{\mathrm{gr}}\left(\mathscr{A}^{\mathrm{gr},(r,r)}_{\underline{k},\underline{l},n}\right) = \boldsymbol{e}^{\mathrm{gr}}\left(\left(\mathscr{A}^{\mathrm{gr},(r,r)}_{\underline{k},\underline{l},n}\right)_{(\epsilon_R)}\right) \cong \bigoplus_{\lambda\in\Lambda_{d,r}} \bar{V}_\lambda^* \otimes \bar{V}_\lambda, \tag{110}$$

where $\bar{V}_\lambda^*$ (respectively, $\bar{V}_\mu$) is an irreducible $U_q\left(\mathfrak{gl}_{\underline{k}}\right)$-module (respectively, $U_q\left(\mathfrak{gl}_{\underline{l}}\right)$-module). In the latter relation we use the bar on $\bar{V}_\lambda^*$ and $\bar{V}_\lambda$ to distinguish $U_q(\mathfrak{gl}_{\underline{k}})$-modules from $U_q(\mathfrak{gl}_k)$-modules and $U_q(\mathfrak{gl}_{\underline{l}})$-modules from $U_q(\mathfrak{gl}_l)$-modules. To complete the proof, we need to verify that the summand $\bar{V}_\lambda^* \otimes \bar{V}_\lambda$ of (110) generates the summand $V_\lambda^* \otimes V_\lambda$ of (109) as a $U_q(\mathfrak{gl}_k) \otimes U_q(\mathfrak{gl}_l)$-module. In what follows, we prove the latter assertion. Let v_λ^* be a highest weight vector of the $U_q(\mathfrak{gl}_k)$-module V_λ^* and let v_λ be a lowest weight vector of the $U_q(\mathfrak{gl}_l)$-module V_λ. It suffices to prove that

$$v_\lambda^* \otimes v_\lambda \in \bar{V}_\lambda^* \otimes \bar{V}_\lambda. \tag{111}$$

The weight of v_λ with respect to the standard Cartan subalgebra of $U_q(\mathfrak{gl}_l)$ is obtained by applying the longest element of the Weyl group S_l to the coefficients of $q^{\sum_{i=1}^d \lambda_i \varepsilon_i}$ (which is the highest weight of $\bar{V}_\lambda$). hence the weight of v_λ is $q^{\sum_{i=1}^d \varepsilon_{l-i+1}\lambda_i}$. By a similar reasoning, the weight of v_λ^* with respect to the standard Cartan subalgebra of $U_q(\mathfrak{gl}_k)$ is $q^{-\sum_{i=1}^d \varepsilon_{k-i+1}\lambda_i}$. Since $k - \underline{k} \leqslant k - d$, we have $K_{\varepsilon_i} \cdot v_\lambda^* = v_\lambda^*$ for $K_{\varepsilon_i} \in U_q(\mathfrak{gl}_k)$ satisfying $1 \leqslant i \leqslant k - \underline{k}$. Similarly, from $l - \underline{l} \leqslant l - d$ it follows that $K_{\varepsilon_i} \cdot v_\lambda = v_\lambda$ for $K_{\varepsilon_i} \in U_q(\mathfrak{gl}_l)$ satisfying $1 \leqslant i \leqslant l - \underline{l}$. Next we express $v_\lambda^* \otimes v_\lambda$ as a linear combination of the basis of $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$ that consists of the monomials (56) (see proposition 3.8.3). Since $v_\lambda^* \otimes v_\lambda \in \mathscr{A}_{k,l,n}^{\mathrm{gr}}$, the monomials that occur must satisfy

$$a_{i,r} = b_{j,r} = 0 \quad \text{for } 1 \leqslant i \leqslant m-k,\ 1 \leqslant j \leqslant m-l,\ 1 \leqslant r \leqslant n. \tag{112}$$

By remark 3.6.2 each of the occurring monomials is a joint eigenvector for the action of

$$\boldsymbol{\kappa}_{k,m}(K_{\varepsilon_i}) \otimes 1 \otimes \boldsymbol{\kappa}_{l,m}\left(K_{-\varepsilon_j}\right) \otimes 1 \quad \text{where } 1 \leqslant i \leqslant k \text{and } 1 \leqslant j \leqslant l,$$

with eigenvalue

$$q^{-\sum_{r=1}^n (a_{m-k+i,r} + b_{m-l+j,r})}.$$

If this eigenvalue is 1 for $i \leqslant k - \underline{k}$ and $j \leqslant l - \underline{l}$, then we must have

$$a_{m-k+i,r} = b_{m-l+j,r} = 0 \quad \text{for } 1 \leqslant i \leqslant k - \underline{k},\ 1 \leqslant j \leqslant l - \underline{l},\ 1 \leqslant r \leqslant n. \tag{113}$$

From (112) and (113) (and the general fact that joint eigenfunctions with distinct eigenvalues are linearly independent) it follows that all of the occurring monomials belong to $\boldsymbol{e}^{\mathrm{gr}}(\mathscr{A}_{\underline{k},\underline{l},n}^{\mathrm{gr}})$. Consequently, $v_\lambda^* \otimes v_\lambda$ belongs to the left hand side of (110). In addition, $v_\lambda^* \otimes v_\lambda$ is the tensor product of a lowest weight vector for a $U_q(\mathfrak{gl}_{\underline{k}})$-module isomorphic to $\bar{V}_\lambda^*$ and a highest weight vector for a $U_q(\mathfrak{gl}_{\underline{l}})$-module isomorphic to $\bar{V}_\lambda$. From the decomposition of the right hand side of (110) we obtain that $v_\lambda^* \in \bar{V}_\lambda^*$ and $v_\lambda \in \bar{V}_\lambda$. This completes the proof of (111). □

We are now ready to complete the proof of (108). From theorem B′ for $\mathscr{A}_{\underline{k},\underline{l},n}^{\mathrm{gr}}$ (which is established in section 9.1) it follows that

$$\boldsymbol{e}^{\mathrm{gr}}\left(\left(\mathscr{A}_{\underline{k},\underline{l},n}^{\mathrm{gr}}\right)_{(\epsilon_R)}\right) \subseteq \mathcal{B}.$$

Thus, by proposition 9.2.2 it suffices to prove that $\mathcal{B}$ is stable under the action of $U_L^{(k,l)}$. The key idea to prove this is that the span of the generators of $\mathcal{B}$ is stable under an algebra larger than $U_L^{(k,l)}$. This enables us to use proposition 2.6.2. Let $\tilde{U}_L^{(k,l)}$ be the subalgebra of $U_{LR} \otimes U_{LR} = U_L \otimes U_R \otimes U_L \otimes U_R$ defined as follows:

(i) If $k \leqslant l$, then $\tilde{U}_L^{(k,l)} := \overline{U}_{k,l} \otimes 1 \otimes U_L \otimes 1$ where $\overline{U}_{k,l}$ is the subalgebra of $U_L = U_q(\mathfrak{gl}_l)$ that is generated by

$$\{E_i\}_{i=1}^{l-1} \cup \{F_i\}_{i=l-k+1}^{l-1} \cup \{K_{\varepsilon_i}\}_{i=1}^{l}.$$

(ii) If $k > l$, then $\tilde{U}_L^{(k,l)} := U_L \otimes 1 \otimes \overline{U}_{k,l} \otimes 1$ where $\overline{U}_{k,l}$ is the subalgebra of $U_L = U_q(\mathfrak{gl}_k)$ that is generated by

$$\{E_i\}_{i=k-l+1}^{k-1} \cup \{F_i\}_{i=1}^{k-1} \cup \{K_{\varepsilon_i}\}_{i=1}^{k}.$$

Note that in both cases we have $U_L^{(k,l)} \subseteq \tilde{U}_L^{(k,l)}$.

Proposition 9.2.3. *$\mathcal{B}$ is stable under the action of $\tilde{U}_L^{(k,l)}$.*

Proof. This follows from proposition 2.6.2 by setting $H := U_L$, $\mathcal{C} := \mathcal{C}^{(m)}$ where $m := \max\{k,l\}$, $\check{R} := \check{\underline{\mathcal{R}}}_L$, $H' := U_R$, $\mathcal{C}' := \mathcal{C}^{(n)}$, $\check{R}' := \check{\underline{\mathcal{R}}}_R$, $A := \mathscr{P}$, $B := \mathscr{D}$, $\overline{H} := \overline{U}_{k,l}$ and

$$\mathcal{E} := \mathrm{Span}_{\Bbbk}\left\{\tilde{\mathsf{L}}_{i,j}^{\mathrm{gr}} : 1 \leqslant i \leqslant k \text{and } 1 \leqslant j \leqslant l\right\}.$$

Checking that $\mathcal{E}$ is stable under the action of $\tilde{U}_L^{(k,l)}$ is a direct calculation based on remark 3.6.2. Also, according to proposition 3.2.2 we can choose $\omega_{V,W}, \overline{\omega}_{V,W} \in H \otimes H$ satisfying the condition of definition 2.5.1(ii) to be finite linear combinations of 2-tensors of the form

$$KF_\beta F_{\beta'} F_{\beta''} \cdots \otimes K' E_\beta E_{\beta'} E_{\beta''} \cdots,$$

where $\beta, \beta', \beta'', \ldots$ are positive roots (see definition 3.2.1) and K, K' are in the Cartan subalgebra. Verifying that the assumptions of proposition 2.6.2(i) and proposition 2.6.2(ii) on $\omega_{V,W}$ and $\overline{\omega}_{V,W}$ hold is then a direct calculation based on the formulas that express the root vectors as commutators of the E_i and the F_i (see definition 3.2.1). □

9.3. Proof of theorem B for $\mathscr{A}_{k,l,n}$

In this subsection we deduce theorem B for $\mathscr{A}_{k,l,n}$ from theorem B′. Let $\mathbb{K}$ be any field. As usual a $\mathbb{K}$-algebra $\mathcal{A}$ is called *filtered* if it has a filtration

$$\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \mathcal{A}^2 \subseteq \cdots$$

such that $\mathcal{A}^i\mathcal{A}^j \subseteq \mathcal{A}^{i+j}$ for $i,j \geqslant 0$. We assume that filtered algebras always satisfy $\mathcal{A}^0 = \mathbb{K}$. As usual $\mathrm{gr}(\mathcal{A}) := \bigoplus_{i=-1}^{\infty} \mathcal{A}^{i+1}/\mathcal{A}^i$ denotes the associated graded algebra of $\mathcal{A}$, where by convention $\mathcal{A}^{-1} = 0$. The following general lemma is standard and can be proved by induction.

Lemma 9.3.1. *Let $\mathcal{A}$ be a filtered algebra and let $a_1, \ldots, a_r \in \mathcal{A}^1$ be such that their images in $\mathcal{A}^1/\mathcal{A}^0$ generate $\mathrm{gr}(\mathcal{A})$. Then $a_1, \ldots, a_r$ generate $\mathcal{A}$.*

The passage from theorem B′ to theorem B relies on the folowing proposition.

Proposition 9.3.2. *Let $\mathcal{A} := \bigoplus_{i=0}^{\infty} \mathcal{A}^{(i)}$ be a graded $\mathbb{K}$-algebra and let $\mathcal{B}$ be a filtered $\mathbb{K}$-algebra. Set $\mathcal{A}^i := \bigoplus_{j=0}^{i} \mathcal{A}^{(j)}$ for $r \geqslant 0$, so that $\mathbb{K} := \mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \mathcal{A}^2 \subseteq \cdots$ is a filtration of $\mathcal{A}$. Let $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ be a filtration-preserving linear map such that $\mathrm{gr}(\mathsf{F}) : \mathrm{gr}(\mathcal{A}) \to \mathrm{gr}(\mathcal{B})$ is an isomorphism of algebras. Suppose that $a_1, \ldots, a_r \in \mathcal{A}^{(1)}$ generate $\mathcal{A}$. Then $\mathsf{F}(a_1), \ldots, \mathsf{F}(a_r)$ generate $\mathcal{B}$.*

Proof. Set $\overline{\mathsf{F}} := \mathrm{gr}(\mathsf{F})$. Since $\mathcal{A} \cong \mathrm{gr}(\mathcal{A})$, we can consider $\overline{\mathsf{F}}$ as a map $\mathcal{A} \to \mathrm{gr}(\mathcal{B})$. Set $b_i := \mathsf{F}(a_i)$ for $1 \leqslant i \leqslant r$. Then $b_i + \mathcal{B}^0 = \overline{\mathsf{F}}(a_i)$, hence $b_1 + \mathcal{B}^0, \ldots, b_r + \mathcal{B}^0$ generate $\mathrm{gr}(\mathcal{B})$. Thus lemma 9.3.1 implies that $b_1, \ldots, b_r$ generate $\mathcal{B}$. □

We return to the proof of theorem B for $\mathcal{A}_{k,l,n}$. We verify that the assumptions of proposition 9.3.2 hold for $\mathcal{A} := \left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$, $\mathcal{B} := (\mathscr{A}_{k,l,n})_{(\epsilon_R)}$ and $\mathsf{F} := \mathsf{P}_{k,l,n}$. Since $\mathsf{P}_{k,l,n} : \mathscr{A}^{\mathrm{gr}}_{k,l,n} \to \mathscr{A}_{k,l,n}$ is an isomorphism of U_R-modules, we have $\mathsf{F}(\mathcal{A}) = \mathcal{B}$. For $r \geqslant 0$ set

$$\mathcal{A}^{(r)} := \mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n} \cap \mathcal{A} \quad , \mathcal{A}^{r} := \bigoplus_{s=0}^{r} \mathcal{A}^{(s)} \ , \tilde{\mathcal{B}}^{(r)} := \mathsf{P}_{k,l,n}\left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}\right) \text{ and } \tilde{\mathcal{B}}^{r} := \bigoplus_{s=0}^{r} \tilde{\mathcal{B}}^{(s)}. \tag{114}$$

Since the U_R-action on $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$ leaves the subspaces $\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}$ stable, we have $\mathcal{A} = \bigoplus_{r=0}^{\infty} \mathcal{A}^{(r)}$. Define a filtration on $\mathcal{B}$ by setting $\mathcal{B}^r := \mathsf{F}(\mathcal{A}^r)$ for $r \geqslant 0$. Since we also have $\mathcal{B} = \mathsf{F}(\mathcal{A})$, the map $\mathrm{gr}(\mathsf{F}) : \mathrm{gr}(\mathcal{A}) \to \mathrm{gr}(\mathcal{B})$ is an isomorphism of graded vector spaces. Next we prove that the latter map is an isomorphism of algebras. To this end, it suffices to verify that

$$\mathsf{F}(D)\,\mathsf{F}(D') - \mathsf{F}(DD') \in \mathcal{B}^{i+j-1} \quad \text{for } D \in \mathcal{A}^{(i)} \text{and } D' \in \mathcal{A}^{(j)}. \tag{115}$$

By proposition 3.13.3 the left hand side of (115) belongs to $\tilde{\mathcal{B}}^{i+j-1}$. Since $\mathcal{A}_{k,l,n}$ is a U_R-module algebra and F is a U_R-module homomorphism, we have $\mathsf{F}(D), \mathsf{F}(D'), \mathsf{F}(DD') \in \mathcal{B}$. It follows that the left hand side of (115) also belongs to $\mathcal{B}$. But since the map $\mathsf{P}_{k,l,n} : \mathscr{A}^{\mathrm{gr}}_{k,l,n} \to \mathscr{A}_{k,l,n}$ is a bijection,

$$\tilde{\mathcal{B}}^{r} \cap \mathcal{B} = \mathsf{P}_{k,l,n}\left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n}\right) \cap \mathsf{P}_{k,l,n}(\mathcal{A}) = \mathsf{P}_{k,l,n}\left(\mathscr{A}^{\mathrm{gr},(r,r)}_{k,l,n} \cap \mathcal{A}\right) \subseteq \mathcal{B}^{r} \quad \text{for } r \geqslant 0.$$

For $r = i + j - 1$ this implies the inclusion (115). Thus we have proved that the assumptions of proposition 9.3.2 hold for $\mathcal{A}$, $\mathcal{B}$ and $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ chosen as above. By theorem B$'$ the $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$ generate $\mathcal{A}$, hence by proposition 9.3.2 the $\tilde{\mathsf{L}}_{i,j}$ generate $\mathcal{B}$. This completes the proof of theorem B for $\mathscr{A}_{k,l,n}$.

10. Proof of theorem C

In this section we give the proof of theorem C(i). The proof of theorem C(ii) is analogous. As a byproduct, in corollary 10.4.1 we obtain explicit generators for $\phi_U^{-1}(\mathscr{L}_{\mathfrak{h},\bullet})$ and $\phi_U^{-1}(\mathscr{R}_{\mathfrak{h},\bullet})$. Henceforth we use E_i, F_i and the K_λ for $\lambda \in \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ to denote elements of $U_R = U_q(\mathfrak{gl}_n)$.

10.1. Parity condition on the λ

For $\lambda, \mu \in \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ expressed as $\lambda := \sum_{i=1}^{n} \lambda_i \varepsilon_i$ and $\mu := \sum_{i=1}^{n} \mu_i \varepsilon_i$ we define $\langle \lambda, \mu \rangle$ as in (40). We also set $\lambda < \mu$ if there exists $1 \leqslant r < n$ such that $\lambda_i = \mu_i$ for all $i \leqslant r$ and $\lambda_{r+1} < \mu_{r+1}$. This defines a total order on $\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$. The following lemma is trivial.

Lemma 10.1.1. *Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ and let $\lambda_{\max}$ denote the maximum of $\mathcal{S}$ with respect to $<$. Let $\gamma_1, \ldots, \gamma_n \in \mathbb{Z}$ be such that $\gamma_n \geqslant 1$ and $\gamma_i \geqslant 1 + \max_{\lambda,\mu \in \mathcal{S}} \left\{ \sum_{i<j\leqslant n} |\lambda_j - \mu_j| \gamma_j \right\}$ for $i < n$. Set $\gamma := \sum_{i=1}^{n} \gamma_i \varepsilon_i$. Then $\langle \lambda_{\max}, \gamma \rangle > \langle \mu, \gamma \rangle$ for all $\mu \in \mathcal{S}$ such that $\mu \neq \lambda_{\max}$.*

Proposition 10.1.2. *Let $\mathcal{I}$ be a finite subset of $\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$. Let $x := \sum_{\lambda \in \mathcal{I}} c_\lambda K_\lambda \in U_{\mathfrak{h},R}$ where $c_\lambda \in \Bbbk^\times$ for $\lambda \in \mathcal{I}$, and assume that $x \in \mathring{U}_R$. Then for all $\lambda := \sum \lambda_i \varepsilon_i \in \mathcal{I}$ and $1 \leqslant i \leqslant n-1$ we have $\lambda_i - \lambda_{i+1} \in 2\mathbb{Z}^{\geqslant 0}$.*

Proof. Step 1. Set $D := \phi_U(x)$. By (68) we have $\phi_U(\mathrm{ad}_y(x)) = (1\otimes y)\cdot D$ for $y \in U_R$. Since $\mathscr{P}\mathscr{D}$ is a locally finite U_R-module, $\phi_U(\mathrm{ad}_{U_R}(x))$ is a finite dimensional subspace of $\mathscr{P}\mathscr{D}$. Furthermore for every $f \in \mathscr{P}$, if we set $W_f := \mathrm{ad}_{U_R}(x)\cdot f := \{\mathrm{ad}_y(x)\cdot f : y \in U_R\}$, then $\dim W_f \leqslant d_\circ$ where $d_\circ := \dim(\phi_U(\mathrm{ad}_{U_R}(x)))$. Note that the upper bound $d_\circ$ on $\dim W_f$ is independent of f.

Step 2. Fix $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ where $1 \leqslant i \leqslant n-1$. It suffices to prove that $\langle\lambda,\alpha_i\rangle \in 2\mathbb{Z}^{\geqslant 0}$ for $\lambda \in \mathcal{I}$. For $r \geqslant 1$ we have

$$\mathrm{ad}_{E_i^r} K_\lambda = \prod_{j=0}^{r-1}\left(1 - q^{\langle\lambda,\alpha_i\rangle - 2j}\right) E_i^r K_\lambda K_i^{-r}.$$

Now take a nonzero $U_{\mathfrak{h},R}$-weight vector $f \in \mathscr{P}$ of weight $q^{-\gamma}$ for $\gamma := \sum_{i=1}^n \gamma_i\varepsilon_i$, where $(\gamma_1,\ldots,\gamma_n)$ is an n-tuple of non-negative integers. We have

$$\begin{aligned}\mathrm{ad}_{E_i^r}(x)\cdot f &= \left(\sum_{\lambda\in\mathcal{I}} c_\lambda \mathrm{ad}_{E_i^r} K_\lambda\right)\cdot f\\ &= q^{r\langle\gamma,\alpha_i\rangle}\left(\sum_{\lambda\in\mathcal{I}} c_\lambda q^{-\langle\lambda,\gamma\rangle}\prod_{j=0}^{r-1}\left(1-q^{\langle\lambda,\alpha_i\rangle-2j}\right)\right) E_i^r\cdot f. \end{aligned} \tag{116}$$

Step 3. For any $\lambda \in \mathcal{I}$, if $\langle\lambda,\alpha_i\rangle \in 2\mathbb{Z}^{\geqslant 0}$ then $\prod_{j=0}^{r-1}\left(1-q^{\langle\lambda,\alpha_i\rangle-2j}\right) = 0$ for all sufficiently large r. Thus, if we set $\mathcal{I}' := \left\{\lambda \in \mathcal{I} : \langle\lambda,\alpha_i\rangle \notin 2\mathbb{Z}^{\geqslant 0}\right\}$ then there exists $r_\circ = r_\circ(\mathcal{I})$ such that for all $r \geqslant r_\circ$ we have

$$\sum_{\lambda\in\mathcal{I}} c_\lambda q^{-\langle\lambda,\gamma\rangle}\prod_{j=0}^{r-1}\left(1-q^{\langle\lambda,\alpha_i\rangle-2j}\right) = \sum_{\lambda\in\mathcal{I}'} c_\lambda q^{-\langle\lambda,\gamma\rangle}\prod_{j=0}^{r-1}\left(1-q^{\langle\lambda,\alpha_i\rangle-2j}\right). \tag{117}$$

Note that the lower bound $r_\circ$ is independent of γ.

Step 4. Assume that $\mathcal{I}' \neq \varnothing$. Choose $r_\circ \in \mathbb{N}$ according to Step 3. Without loss of generality we can also assume that $r_\circ \geqslant d_\circ$. Next choose $r \geqslant r_\circ$. After possibly scaling x by a nonzero element of the polynomial ring $\mathbb{C}[q]$ we can assume that the c_λ are nonzero elements in $\mathbb{C}[q]$. Let $\lambda_{\max}$ denote the maximum of $\mathcal{I}'$ with respect to $<$. Choose γ as in lemma 10.1.1 (with $\mathcal{S} := \mathcal{I}'$). Since the condition on the coefficient γ_i only depends on γ_j for $j > i$, we can also assume that $\gamma_i - \gamma_{i+1} \geqslant 1$. For $\lambda \in \mathcal{I}'$ let $q^{N(r,\lambda)}$ be the lowest power of q that occurs after expanding and simplifying $c_\lambda q^{-\langle\lambda,\gamma\rangle}\prod_{j=0}^{r-1}\left(1-q^{\langle\lambda,\alpha_i\rangle-2j}\right)$. We have

$$N(r,\lambda_{\max}) \leqslant -\langle\lambda_{\max},\gamma\rangle + \deg c_{\lambda_{\max}}(q),$$

because the lowest power $q^{N(r,\lambda)}$ is obtained as follows: from each factor $\left(1-q^{\langle\lambda,\alpha_i\rangle-2j}\right)$ we can choose 1 if $\langle\lambda,\alpha_i\rangle - 2j > 0$ and $q^{\langle\lambda,\alpha_i\rangle-2j}$ otherwise. For all other $\lambda \in \mathcal{I}'$ we have

$$N(r,\lambda) \geqslant -\deg c_\lambda\left(q^{-1}\right) - \langle\lambda,\gamma\rangle - r|\langle\lambda,\alpha_i\rangle| - r(r-1).$$

By the choice of γ, for $\lambda \in \mathcal{I}'$ such that $\lambda \neq \lambda_{\max}$ we have $\langle\lambda_{\max},\gamma\rangle \geqslant 1 + \langle\lambda,\gamma\rangle$. Thus

$$\langle\lambda_{\max},k\gamma\rangle \geqslant k + \langle\lambda,k\gamma\rangle \quad \text{for } k \in \mathbb{N}.$$

Next choose $k \in \mathbb{N}$ such that $k \geqslant 2r_\circ$ and

$$k \geqslant \max_{\lambda \in \mathcal{I}', \lambda \neq \lambda_{\max}} \left\{ \deg c_{\lambda_{\max}}(q) + \deg c_\lambda \left(q^{-1}\right) + 2r_\circ |\langle \lambda, \alpha_i \rangle| + 2r_\circ (2r_\circ - 1) \right\}. \tag{118}$$

If we substitute γ by $k\gamma$, from (118) we obtain that $N(r, \lambda_{\max}) < N(r, \lambda)$ for all $\lambda \in \mathcal{I}' \backslash \{\lambda_{\max}\}$ and $r_\circ \leqslant r \leqslant 2r_\circ$. Together with Step 3, this proves that for the latter choice of γ we have

$$\sum_{\lambda \in \mathcal{I}} c_\lambda q^{-\langle \lambda, \gamma \rangle} \prod_{j=0}^{r-1} \left(1 - q^{\langle \lambda, \alpha_i \rangle - 2j}\right) \neq 0 \quad \text{for } r_\circ \leqslant r \leqslant 2r_\circ,$$

because the coefficient of $q^{N(r, \lambda_{\max})}$ is nonzero.

Step 5. The γ chosen at the end of Step 4 satisfies $\gamma_i - \gamma_{i+1} \geqslant 2r_\circ$, or equivalently $\langle -\gamma, \alpha_i \rangle \leqslant -2r_\circ$ (because $k \geqslant 2r_\circ$). Choose $f \in \mathscr{P}$ of $U_{\mathfrak{h},R}$-weight $q^{-\gamma}$ (for example $f := t_{1,1}^{\gamma_1} \cdots t_{1,n}^{\gamma_n}$). A standard argument based on representation theory of $U_q(\mathfrak{sl}_2)$ implies $E_i^r \cdot f \neq 0$ for $0 \leqslant r \leqslant 2r_\circ$. Since the vectors $E_i^s \cdot f$ for $0 \leqslant s \leqslant 2r_\circ$ have distinct $U_{\mathfrak{h},R}$-weights, they are linearly independent. From Step 2 and Step 4 it follows that the vectors $\mathrm{ad}_{E_i^s}(x) \cdot f$ for $r_\circ \leqslant s \leqslant 2r_\circ$ are also linearly independent. Consequently, $\dim W_f \geqslant r_\circ + 1 \geqslant d_\circ + 1$. This contradicts Step 1. □

10.2. Proof of $\lambda_1 \leqslant 0$

We begin with the following observation.

Remark 10.2.1. Let $D \in \mathscr{P}\mathscr{D}$ and let $\mathsf{a} := (a_1, \ldots, a_n)$ be an n-tuple of non-negative integers. We use the notation $\partial^{\mathsf{a}} := \partial_{1,1}^{a_1} \cdots \partial_{1,n}^{a_n}$ and $t^{\mathsf{a}} := t_{1,n}^{a_n} \cdots t_{1,1}^{a_1}$ for an n-tuple of integers $(a_1, \ldots, a_n)$. Assume that $D \cdot t^{\mathsf{a}} = ct^{\mathsf{a}}$ for some $c \in \Bbbk$. Recall the basis of $\mathscr{P}\mathscr{D}$ that consists of the monomials (58). We can write D as $D = D_1 + D_2 + D_3$ where

(i) D_1 is a linear combination of basis vectors of the form $t^{\mathsf{b}'} \partial^{\mathsf{b}'}$ where b' is an n-tuple of non-negative integers,
(ii) D_2 is a linear combination of basis vectors of the form $t^{\mathsf{a}'} \partial^{\mathsf{b}'}$ where a' and b' are n-tuples of non-negative integers and $\mathsf{a}' \neq \mathsf{b}'$, and
(iii) D_3 is a linear combination of the remaining basis vectors in (58).

Using lemma 3.17.1 and then lemma 3.17.12 we obtain $D \cdot t^{\mathsf{a}} = (D_1 + D_2) \cdot t^{\mathsf{a}} = D_1 \cdot t^{\mathsf{a}}$.

Example 10.2.2. Set $\lambda := \varepsilon_1 + \cdots + \varepsilon_n$ and $x := K_\lambda \in U_R$. Then $x \cdot t_{1,1}^r = q^{-r} t_{1,1}^r$ for $r \geqslant 1$. From remark 10.2.1, lemma 3.17.12 and remark 3.17.10 it follows that if $\phi_U(1 \otimes x) \in \mathscr{R}_\bullet$ then the eigenvalue of $t_{1,1}^r$ with respect to $\phi_U(1 \otimes x)$ should be a ratio of two polynomials such as $\phi_1(q)/\phi_2(q)$ where $\deg \phi_2$ is bounded above (independently of r). Thus, $\phi_U(1 \otimes x) \notin \mathscr{R}_\bullet$ and in particular $\mathscr{R}_\bullet \subsetneq \mathscr{R}$. Consequently, K_λ is a locally finite element of U_R that does not belong to $\mathring{U}_R$.

Proposition 10.2.3. *Let $\mathcal{I}$ be a finite subset of $\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$. Let $x := \sum_{\lambda \in \mathcal{I}} c_\lambda K_\lambda \in U_{\mathfrak{h},R}$ where $c_\lambda \in \Bbbk^\times$ for $\lambda \in \mathcal{I}$, and assume that $x \in \mathring{U}_R$. Then for all $\lambda := \sum \lambda_i \varepsilon_i \in \mathcal{I}$ we have $\lambda_1 \leqslant 0$.*

Proof. Set $D := \phi_U(x)$, so that $D \in \mathscr{P}\mathscr{D}$. Write $D = D_1 + D_2 + D_3$ as in remark 10.2.1 and suppose that $D_1 = \sum_{\mathsf{a} \in \mathcal{Z}} \mathbf{z}_{\mathsf{a}} t^{\mathsf{a}} \partial^{\mathsf{a}}$ where $\mathcal{Z}$ is a finite set of n-tuples of non-negative integers and the $\mathbf{z}_{\mathsf{a}} \in \Bbbk^\times$. After scaling x by a nonzero element of $\mathbb{C}[q]$ if necessary, we can assume that the c_λ and the $\mathbf{z}_{\mathsf{a}}$ are nonzero polynomials in q. Recall that $t^\gamma := t_{1,n}^{\gamma_n}\cdots t_{1,1}^{\gamma_1}$ for $\gamma := \sum_{i=1}^n \gamma_i\varepsilon_i$ in $\mathbb{Z}\varepsilon_1+\cdots+\mathbb{Z}\varepsilon_n$. Then

$$D\cdot t^\gamma = \sum_{\lambda\in\mathcal{I}} c_\lambda q^{-\langle\lambda,\gamma\rangle} t^\gamma.$$

Also, by lemma 3.17.12 and remark 10.2.1 we obtain $D\cdot t^\gamma = D_1\cdot t^\gamma = \sum_{\mathsf{a}\in\mathcal{Z}} \mathbf{z}_{\mathsf{a}}(q)\phi_{\mathsf{a}}(q^2)t^\gamma$, where the ϕ_{a} are polynomials in q with integer coefficients. Note that the $\mathbf{z}_{\mathsf{a}}$ are independent of γ, but the ϕ_{a} can depend on γ.

Set $\tilde\lambda := \lambda_{\max}$ where $\lambda_{\max}$ is the maximum of $\mathcal{I}$ according to the total order introduced in section 10.1. By lemma 10.1.1 we can choose γ such that we have $\langle\tilde\lambda,\gamma\rangle > \langle\mu,\gamma\rangle$ for all $\mu\in\mathcal{I}\backslash\{\tilde\lambda\}$. If the assertion of the proposition is not true, then $\tilde\lambda_1>0$ and thus by choosing γ_1 sufficiently large we can also assume that $\langle\tilde\lambda,\gamma\rangle\geqslant 1$. Thus, for all sufficiently large $k\in\mathbb{N}$ the lowest power of q that occurs in $\sum_{\lambda\in\mathcal{I}} c_\lambda q^{-\langle\lambda,k\gamma\rangle}$ is from the summand $c_{\tilde\lambda}q^{-\langle\tilde\lambda,k\gamma\rangle}$, and is equal to $d-k\langle\tilde\lambda,\gamma\rangle$, where d is the lowest power of q that occurs in $c_{\tilde\lambda}$. By comparing with $\sum_{\mathsf{a}} \mathbf{z}_{\mathsf{a}}(q)\phi_{\mathsf{a}}(q^2)$ it follows that

$$d-k\langle\tilde\lambda,\gamma\rangle \geqslant \min_{\mathsf{a}\in\mathcal{Z}}\left\{-\deg \mathbf{z}_{\mathsf{a}}\left(q^{-1}\right)\right\}.$$

The right hand side is independent of k and γ. However, this is a contradiction since k can be chosen arbitrarily large and $\langle\tilde\lambda,\gamma\rangle\geqslant 1$. □

10.3. Proof of $\lambda_1\in 2\mathbb{Z}^{\leqslant 0}$

In this subsection we strengthen proposition 10.2.3, as follows.

Proposition 10.3.1. *Let $\mathcal{I}$ be a finite subset of $\mathbb{Z}\varepsilon_1+\cdots+\mathbb{Z}\varepsilon_n$. Let $x:=\sum_{\lambda\in\mathcal{I}} c_\lambda K_\lambda \in U_{\mathfrak{h},R}$ where $c_\lambda\in\mathbb{k}^\times$ for $\lambda\in\mathcal{I}$, and assume that $x\in\mathring{U}_R$. Then for every $\lambda:=\sum\lambda_i\varepsilon_i\in\mathcal{I}$ we have $\lambda_1\in 2\mathbb{Z}^{\leqslant 0}$.*

Proof. We assume that the assertion is false, and arrive at a contradiction.

Step 1. Recall from proposition 4.1.1 that the $K_{\lambda_{R,b}}$ for $1\leqslant b\leqslant n$ are contained in $\mathring{U}_R$. The $K_\lambda\in U_{\mathfrak{h},R}$ satisfying $\lambda_i-\lambda_{i+1}\in 2\mathbb{Z}^{\geqslant 0}$ for $1\leqslant i\leqslant n-1$ and $\lambda_1\in 2\mathbb{Z}^{\leqslant 0}$ can be expressed as products of the $K_{\lambda_{R,b}}$. Thus by propositions 10.1.2 and 10.2.3 we can assume that $\lambda_1\in\{-1,-3,-5,\ldots\}$ for all $\lambda\in\mathcal{I}$.

Step 2. Set $D:=\phi_U(x)$ so that $D\in\mathscr{P}\mathscr{D}$. Write D as $D=D_1+D_2+D_3$ according to remark 10.2.1. Suppose that $D_1=\sum_{\mathsf{b}\in\mathcal{Z}}\mathbf{z}_{\mathsf{b}}t^{\mathsf{b}}\partial^{\mathsf{b}}$, where $\mathcal{Z}$ is a finite set of n-tuples of non-negative integers and the $\mathbf{z}_{\mathsf{b}}\in\mathbb{k}^\times$. After scaling x by a nonzero element of $\mathbb{C}[q]$ we can assume that the c_λ and the $\mathbf{z}_{\mathsf{b}}$ are nonzero elements of $\mathbb{C}[q]$. We keep using the notation t^γ for $\gamma:=\sum_{i=1}^n\gamma_i\varepsilon_i$ from the proof of proposition 10.2.3. Then $D\cdot t^\gamma=\sum_{\lambda\in\mathcal{I}}c_\lambda q^{-\langle\lambda,\gamma\rangle}t^\gamma$. From lemma 3.17.12(ii) and remark 3.17.10 it follows that $t^{\mathsf{b}}\partial^{\mathsf{b}}\cdot t^\gamma=\phi_{\mathsf{b}}(q^2)t^\gamma$ where $\phi_{\mathsf{b}}\in\mathbb{C}[q]$ and

$$\deg\phi_{\mathsf{b}} = \sum_{i=1}^n \gamma_i\left(b_1+\cdots+b_i\right) - \sum_{i=1}^n\left(b_i\left(b_1+\cdots+b_i\right)-\frac{b_i\left(b_i-1\right)}{2}\right). \quad (119)$$

For $\mathsf{b}\in\mathcal{Z}$ define $\lambda_{\mathsf{b}}\in\mathbb{Z}\varepsilon_1+\cdots+\mathbb{Z}\varepsilon_n$ by

$$\lambda_{\mathsf{b}} := b_1\varepsilon_1+\left(b_1+b_2\right)\varepsilon_2+\cdots+\left(b_1+\cdots+b_n\right)\varepsilon_n.$$

By (119) we have $\deg\phi_{\mathsf{b}}=\langle\lambda_{\mathsf{b}},\gamma\rangle+C(\mathsf{b})$, where $C(\mathsf{b})$ is independent of γ.

Step 3. Let $\tilde{\lambda} \in \mathcal{I}$ be such that $-\tilde{\lambda}$ is the maximum of $-\mathcal{I} := \{-\lambda : \lambda \in \mathcal{I}\}$ with respect to the total order $<$ of section 10.1. Using lemma 10.1.1 for $-\mathcal{I}$, we can choose γ such that $-\langle\tilde{\lambda},\gamma\rangle > -\langle\mu,\gamma\rangle$ for all $\mu \in \mathcal{I}\backslash\{\tilde{\lambda}\}$. Since $\tilde{\lambda}_1 \in \{-1,-3,-5,\ldots\}$, by choosing the parity of γ_1 suitably we can also assume that $-\langle\tilde{\lambda},\gamma\rangle$ is an odd integer. Then for $k \in \mathbb{N}$ sufficiently large, the highest power of q that occurs in $\sum_{\lambda\in\mathcal{I}} c_\lambda q^{-\langle\lambda,k\gamma\rangle}$ is from the summand $c_{\tilde{\lambda}} q^{-\langle\tilde{\lambda},k\gamma\rangle}$, and is equal to $d - k\langle\tilde{\lambda},\gamma\rangle$, where $d := \deg c_{\tilde{\lambda}}$.

Step 4. Let $\mathsf{b}_{\max} \in \mathcal{Z}$ be such that $\lambda_{\mathsf{b}_{\max}} = \max\{\lambda_{\mathsf{b}} : \mathsf{b} \in \mathcal{Z}\}$, where the maximum is taken with respect to $<$. Note that the map $\mathsf{b} \mapsto \lambda_{\mathsf{b}}$ is an injection. From (119) and lemma 10.1.1 applied to the set $\{\lambda_{\mathsf{b}} : \mathsf{b} \in \mathcal{Z}\}$ it follows that we can choose γ and k in Step 3 such that the following additional property holds: the highest power of q that occurs in $\sum_{\mathsf{b}\in\mathcal{Z}} \mathbf{z}_{\mathsf{b}}(q)\phi_{\mathsf{b}}(q^2)$ is from the summand $\mathbf{z}_{\mathsf{b}_{\max}}(q)\phi_{\mathsf{b}_{\max}}(q^2)$, and is equal to $d' + 2\deg\phi_{\mathsf{b}_{\max}}$, where $d' := \deg \mathbf{z}_{\mathsf{b}_{\max}}$. Note that the values $\deg\phi_{\mathsf{b}}$ depend on γ and k, but the values $\deg \mathbf{z}_{\mathsf{b}}$ only depend on x and in particular they are independent of the choices of γ and k.

Step 5. Recall that $t^{k\gamma}$ is an eigenvector of D, hence $D\cdot t^{k\gamma} = D_1 \cdot t^{k\gamma} = \sum_{\mathsf{b}\in\mathcal{Z}} \mathbf{z}_{\mathsf{b}}(q)\phi_{\mathsf{b}}(q^2)t^{k\gamma}$ by remark 10.2.1. By comparing the highest power of q in the eigenvalue of $t^{k\gamma}$ from Step 3 and Step 4 it follows that

$$d' + 2\deg\phi_{\mathsf{b}_{\max}} = d - k\langle\tilde{\lambda},\gamma\rangle. \tag{120}$$

Since d' is independent of γ and k, the parity of the left hand side of (120) does not change by varying k and γ. However, recall that $\langle\tilde{\lambda},\gamma\rangle$ is an odd integer and the only constraint on k is that it should be sufficiently large. Thus, we can choose k such that the parities of the two sides of (120) are different. This is a contradiction. □

10.4. Completing the proof of theorem C(i)

Theorem C(i) is an immediate consequence of the following corollary and proposition 5.2.2.

Corollary 10.4.1. *Let $\mathcal{I}$ be a finite subset of $\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$. Let $x := \sum_{\lambda\in\mathcal{I}} c_\lambda K_\lambda \in U_{R,\mathfrak{h}}$ where $c_\lambda \in \Bbbk^\times$ for $\lambda \in \mathcal{I}$, and assume that $x \in \mathring{U}_R$. Then x belongs to the subalgebra of $U_{\mathfrak{h},R}$ that is generated by the $K_{\lambda_{R,b}}$ for $1 \leqslant b \leqslant n$.*

Proof. Follows immediately from propositions 10.1.2 and 10.3.1. □

Corollary 10.4.1 implies that $\{K_{\lambda_{R,b}}\}_{b=1}^n$ is a generating set of the algebra $\mathring{U}_R$. An analogous statement holds for U_L. That is, $\{K_{\lambda_{L,a}}\}_{a=1}^m$ is a generating set of the algebra $\mathring{U}_L$.

Data availability statement

No new data were created or analysed in this study.

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Appendix. Commonly used notation

In this section we list the commonly used symbols and notation along with the subsection in which each item is defined.

Introduction: $\Bbbk$, U_L, U_R, U_{LR}, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{L}_\bullet$, $\mathscr{R}_\bullet$, $\mathsf{L}_{i,j}$, $\mathsf{R}_{i,j}$, $\mathcal{Y}^{\mathcal{Z}}$, $\boldsymbol{e}^{m\times n}_{a\times b}$, $\tilde{L}_{i,j}$, $M^{\mathbf{i}}_{\mathbf{j}}$, $\overline{M}^{\mathbf{i}}_{\mathbf{j}}$, $\mathbf{D}(r,a,b)$, $\mathbf{D}_{k,r}$, $\mathbf{D}'_{k,r}$, $\mathbf{R}_a$, $\mathbf{L}_b$, $\mathscr{L}_{\mathfrak{h}}$, $\mathscr{R}_{\mathfrak{h}}$, $\mathscr{L}_{\mathfrak{h},\bullet}$, $\mathscr{R}_{\mathfrak{h},\bullet}$.
Subsection 2.1: $\mathrm{ad}_y(x)$, $\mathscr{F}(H,I)$, $\mathscr{F}(H)$.
Subsection 2.2: V^*, $\langle v^*, v\rangle$, $\mathsf{m}_{v^*,v}$, Δ°, H°.
Subsection 2.3: $V_{(\epsilon)}$.
Subsection 2.4: $\check{R}_{V,W}$, $A\otimes_{\check{R}} B$, $A\otimes_{\check{R},\psi} B$.
Subsection 2.5: $R_{V,W}$, $\omega_{V,W}$, $\overline{\omega}_{V,W}$.
Subsection 2.7: $H^\circ_{\mathcal{C}}$, $\langle f\otimes g, R\rangle$.
Subsection 3.1: $U_q(\mathfrak{gl}_n)$, ε_i, $[\![a,b]\!]$, K_{ε_i}, K_i, K_λ, E_i, F_i, $U_{\mathfrak{h},L}$, $U_{\mathfrak{h},R}$.
Subsection 3.2: $\mathcal{C}^{(n)}$, $\mathcal{R}^{(n)}$, $\underline{\mathcal{R}}^{(n)}$, Exp_q, $\langle\mu,\nu\rangle$, $\check{\mathcal{R}}^{(n)}$, $\underline{\check{\mathcal{R}}}^{(n)}$.
Subsection 3.3: $x^\natural$.
Subsection 3.4: Δ^+_n, $\mathsf{E}_{i,j}$, $\mathscr{P}_{n\times n}$, $\mathscr{D}_{n\times n}$, $t_{i,j}$, $\partial_{i,j}$, $t^\natural_{i,j}$, $\partial^\natural_{i,j}$, $\mathcal{L}_{\mathscr{P}}$, $\mathcal{R}_{\mathscr{P}}$, $\mathcal{L}_{\mathscr{D}}$, $\mathcal{R}_{\mathscr{D}}$, $\Delta_{\mathscr{P}}$, $\Delta_{\mathscr{D}}$, ξ_c, Ξ, ι, $\underline{\iota}$.
Subsection 3.5: $\mathscr{P}$, $\mathscr{P}_{m\times n}$, $\mathscr{D}$, $\mathscr{D}_{m\times n}$, ϕ_U.
Subsection 3.6: $\ell(\lambda)$, $\mathscr{P}^{(d)}$, $\mathscr{D}^{(d)}$, V_λ, $\Lambda_{d,r}$, $|\lambda|$.
Subsection 3.7: $\mathcal{C}_L$, $\mathcal{C}_R$, $\mathcal{R}_L$, $\underline{\mathcal{R}}_L$, $\mathcal{R}_R$, $\underline{\mathcal{R}}_R$, $\check{\mathcal{R}}_L$, $\underline{\check{\mathcal{R}}}_L$, $\check{\mathcal{R}}_R$, $\underline{\check{\mathcal{R}}}_R$, $\mathcal{R}_{LR}$, $\underline{\mathcal{R}}_{LR}$, $\check{\mathcal{R}}_{LR}$, $\underline{\check{\mathcal{R}}}_{LR}$, $\mathscr{P}\mathscr{D}^{\mathrm{gr}}$, $\mathscr{P}\mathscr{D}$.
Subsection 3.9: $\mathscr{A}_{k,l,n}$, $\mathscr{A}^{\mathrm{gr}}_{k,l,n}$, $\boldsymbol{\kappa}_{r,n}$, $(\boldsymbol{e}^{\mathrm{gr}})^{m\times n}_{m'\times n'}$.
Subsection 3.10: ϕ_{PD}, $X\cdot f$.
Subsection 3.11: $\mathscr{P}^{(\leqslant k)}$, $\prec$, $\mathbf{c}(a)$.
Subsection 3.13: P, $\mathsf{P}_{k,l,n}$, $\mathscr{P}\mathscr{D}^{\mathrm{gr},(r,s)}$, $\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}$,
Subsection 3.14: $\mathring{U}_L$, $\mathring{U}_R$, $\mathring{U}_{LR}$.
Subsection 3.15: $\eta_{m,n}$.
Subsection 3.17: $\lhd$, $\mathbf{c}(a,b)$, t^{a}, ∂^{a}.
Subsection 4.1: $\lambda_{L,a}$, $\lambda_{R,b}$, $K_{\lambda_{L,a}}$, $K_{\lambda_{R,b}}$
Subsection 6.1: $\mathscr{P}\mathscr{D}_{(\epsilon_L)}$, $\mathscr{P}\mathscr{D}_{(\epsilon_R)}$, ϵ_L, ϵ_R, $(\mathscr{A}_{k,l,n})_{(\epsilon_R)}$, $\left(\mathscr{A}^{\mathrm{gr}}_{k,l,n}\right)_{(\epsilon_R)}$, $\left(\mathscr{A}^{\mathrm{gr},(r,s)}_{k,l,n}\right)_{(\epsilon_R)}$, $\mathscr{P}\mathscr{D}^{\mathscr{L}}$, $\mathscr{P}\mathscr{D}^{\mathscr{R}}$, $\mathscr{P}\mathscr{D}^{\mathscr{L}\bullet}$, $\mathscr{P}\mathscr{D}^{\mathscr{R}\bullet}$, $\tilde{\mathsf{L}}^{\mathrm{gr}}_{i,j}$.
Subsection 7.1: Γ_n, $\Gamma_{k,l,n}$.
Subsection 8.2: $u\star_n v$, $u\star_{k,l,n} v$.

ORCID iDs

Gail Letzter https://orcid.org/0009-0007-7809-2776
Siddhartha Sahi https://orcid.org/0000-0002-6129-5282
Hadi Salmasian https://orcid.org/0000-0002-1073-7183

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