Journal of Physics A:
Mathematical and Theoretical

PAPER • **OPEN ACCESS**

Integration theory for kinks and sphalerons in one dimension

To cite this article: N S Manton 2024 *J. Phys. A: Math. Theor.* **57** 025202

OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **57** (2024) 025202 (20pp)
https://doi.org/10.1088/1751-8121/ad14ac

Integration theory for kinks and sphalerons in one dimension

N S Manton

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

E-mail: N.S.Manton@damtp.cam.ac.uk

Received 7 September 2023
Accepted for publication 12 December 2023
Published 28 December 2023

Abstract

The static kink, sphaleron and kink chain solutions for a single scalar field ϕ in one spatial dimension are reconsidered. By integration of the Euler–Lagrange equation, or through the Bogomolny argument, one finds that each of these solutions obeys a first-order field equation, an autonomous ODE that can always be formally integrated. We distinguish the BPS case, where the required integral is along a contour in the ϕ-plane, from the semi-BPS case, where the integral is along a contour in the Riemann surface double-covering the ϕ-plane, and is generally more complicated.

Keywords: kink, sphaleron, integration

1. Introduction

A scalar field theory in one spatial dimension, with a single scalar field ϕ and appropriate vacuum structure, is well-known to have stable kink solutions interpolating between adjacent vacua [1–3]. It may also have unstable, sphaleron (or bounce) solutions, representing a saddle point between false and true vacua [4–7]. Additionally, such a theory generally has spatially periodic, kink chain solutions, composed of alternating kinks and antikinks [8]. All these static solutions obey a first-order ODE, obtained as the first integral of the theory's second-order Euler–Lagrange equation. The different solutions arise by varying the constant of integration, as well as the underlying potential $U(\phi)$. Here, we will be concerned almost entirely with solutions of this first-order ODE, which we refer to as the field equation. We will review the solutions, some examples of which are widely known and others less so, focussing on their analytic and algebraic form. We will also find the energies of kink and sphaleron solutions, and the spatial period and energy per period of kink chains. All the calculations reduce to integrals of differentials on the ϕ-plane or its branched double-cover Σ, many of which simplify to elementary integrals.

We start with the energy functional for time-independent fields,

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + U(\phi)\right\}\mathrm{d}x\,, \tag{1}$$

where U is real for all real ϕ. The second-order Euler–Lagrange equation,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \frac{1}{2}\frac{\mathrm{d}U}{\mathrm{d}\phi} = 0\,, \tag{2}$$

has the first integral

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 = U(\phi) + C\,, \tag{3}$$

where C is an arbitrary constant. It is convenient to absorb C into the potential U (i.e. set $C = 0$), as we are interested in particular solutions. Variations of C are accommodated by adjusting $U(\phi)$. Taking the square root of (3), we obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \sqrt{U(\phi)}\,, \tag{4}$$

the first-order autonomous ODE that we call the field equation. Its formal solution is

$$x = \int \frac{\mathrm{d}\phi}{\sqrt{U(\phi)}}\,. \tag{5}$$

It is assumed that U (with the constant C absorbed) is positive for at least some range of ϕ, so that x is real. The further constant of integration in (5) encodes a shift of x, i.e. a spatial translation. We will not be careful about this, so all solutions appearing below can be spatially translated.

We will assume that $U(\phi)$ is a holomorphic or somtimes meromorphic function of ϕ, real along the real ϕ-axis. Simplest is where U is a polynomial in ϕ, but we will later consider U a power of $\cos\phi$. U can have zeros, at real or complex values of ϕ. At a zero of odd multiplicity, $\sqrt{U}$ has a branch point; we will therefore need to consider the Riemann surface Σ that is the branched double-cover of the ϕ-plane. Σ is defined by the algebraic equation

$$V^2 = U(\phi)\,, \tag{6}$$

and on Σ, the differential

$$\frac{\mathrm{d}\phi}{V(\phi)} = \frac{\mathrm{d}\phi}{\sqrt{U(\phi)}} \tag{7}$$

is well-defined. The field equation's formal solution (5) becomes

$$x = \int \frac{\mathrm{d}\phi}{V(\phi)}\,, \tag{8}$$

where the integral is along a suitable open contour on Σ that keeps x real.

There is still a choice of the overall sign for the square root (equivalently, the choice of starting point for the contour on Σ). A reversal of the sign reverses the sign of the derivative of ϕ with respect to x, which has the effect of a spatial reflection, and turns a kink into an antikink. The sign needs to be chosen in a consistent and smooth way along the contour of integration, especially if the contour encircles a branch point. When we later reconsider the Bogomolny argument [9], we will not need to explicitly allow for the sign choice $\pm\sqrt{U}$ (as is usually done), because it is implicit in the choice of contour.

We can gain considerable insight into solutions by considering the integral (8) locally near a zero of U. Let us assume the zero is at $\phi = 0$. Suppose first that the zero is simple, and $U(\phi) = \phi$ in leading approximation. Then

$$x = \int \frac{\mathrm{d}\phi}{\sqrt{\phi}} = 2\sqrt{\phi}, \tag{9}$$

so $\phi(x) = \frac{1}{4}x^2$. We see that although x is a double-valued function of ϕ, and potentially imaginary, the solution $\phi(x)$ itself is real and has a smooth minimum. A smooth, real maximum occurs if $U(\phi) = -\phi$. More generally, if U has only simple zeros, then the differential (7) has no singularities on Σ (as becomes clear after a change of variable $\psi^2 = \phi - a$ near a simple zero of $U(\phi)$ at a). The differential is therefore holomorphic on Σ—it is an Abelian differential of the first kind—and x is its integral from some fixed initial point to the variable point ϕ. Exceptions are if U is a quadratic or linear polynomial; in these cases the differential has a pole at infinity.

Suppose next that U has a double zero, so locally $U(\phi) = \phi^2$. Then the Riemann surface Σ is not branched at $\phi = 0$, and the differential (7) has a simple pole there. More precisely, Σ has two disjoint pieces, where $V = \sqrt{U} = \phi$ and $V = \sqrt{U} = -\phi$, and the poles of the differential have opposite residues on these pieces. The Riemann surface construction has blown-up the node $V = U = 0$ of the curve $V^2 = U(\phi)$ into two 'places'. The local solution (8) is

$$x = \pm \int \frac{\mathrm{d}\phi}{\phi} = \pm \log \phi\,, \tag{10}$$

or equivalently $\phi(x) = e^{\pm x}$. This is the typical exponential growth of a global solution for large negative x, or exponential decay for large positive x, and is called a short-range tail. Finally, suppose U has a zero of even multiplicity $2k$, with $k > 1$, so locally $U(\phi) = \phi^{2k}$. Then the differential (7) has a pole of order k, and its integral is

$$x = \pm \int \frac{\mathrm{d}\phi}{\phi^k} = \mp \frac{1}{(k-1)\,\phi^{k-1}}\,, \tag{11}$$

so $\phi(x)$ is an inverse (fractional) power of $\pm(k-1)x$. In the simplest case $k = 2$, $\phi(x)$ is proportional to $\pm\frac{1}{x}$. This is the basic, long-range tail behaviour of a solution.

If all the zeros of U are double, or of higher even multiplicity, then Σ is globally two copies of the ϕ-plane, V is a polynomial, and (7) is generally an Abelian differential of the third kind, i.e. a meromorphic differential including simple pole parts, on each of these copies. In certain cases, (7) is an Abelian differential of the second kind, i.e. meromorphic, with vanishing simple pole parts.

Many well-known solutions of field theory have the behaviours emerging from this local analysis. The kink solutions in the standard ϕ^4-theory and ϕ^6-theory connect two double zeros of U (true vacua), and have exponentially decaying tails. A special ϕ^8 kink occurs in a theory where U has one quartic zero ($k = 2$) and two double zeros. Here the kink has one long-range and one short-range tail [10]. It is important to distinguish double zeros of U from zeros of higher, even multiplicity. Lohe [11], and more recently Khare *et al* [12], Bazeia *et al* [13], and Gani and collaborators [14–16] have drawn attention to several examples.

A sphaleron is a solution $\phi(x)$ that, as x increases, runs from a double zero ϕ_0 of U (a false vacuum), via a simple zero of U, back to ϕ_0 on the other sheet of Σ. $\phi(x)$ is a smooth quadratic function of x near the simple zero, and its spatial derivative has a node (a zero) here. The spatial derivative is the translation zero-mode of the sphaleron, its zero-frequency mode of small oscillation, so by the Sturm oscillation theorem, there is inevitably a single mode of negative squared frequency with no node. This is the unstable mode of the sphaleron.

Finally, there are kink chain solutions connecting adjacent, simple zeros of U, with U positive between them. The integral (8), determining $x(\phi)$, is finite over a minimal integration contour connecting these zeros, so the contour returns on the second sheet of Σ, and then continues on the first sheet, etc. The result is a spatially periodic solution $\phi(x)$, whose period is a real period of the Abelian differential (7). Simple zeros of U are generic, so, if we allow the constant of integration C to vary, then for almost all C, it is spatially periodic solutions that occur. Like sphalerons, these kink chain solutions are generally unstable.

It is important to understand the genus g of the Riemann surface Σ, as this significantly affects the global nature of the differential (7), and hence the integral solution (8) of the field equation. The genus depends critically on the multiplicities of the zeros of U. In most of the familiar examples of kinks, U has sufficiently many double zeros, or zeros of higher even multiplicity, that the genus reduces to zero and the differential (7) is meromorphic on the ϕ-plane. Using partial fractions, the differential can then be integrated, giving an implicit solution $x = G(\phi)$. In simple cases, this can be inverted to give an explicit solution $\phi(x)$, but inversion is not possible in general.

Conversely, if U is a polynomial of degree $2n$ (or degree $2n-1$), with only simple zeros, then the Riemann surface Σ defined by $V^2 = U(\phi)$, double-covering the ϕ-plane, is of genus $n-1$ and the solution of the field equation is an integral of an Abelian differential of the first kind. (The case $n = 1$ is special—here the genus of Σ is zero but the differential acquires simple poles.) For all n, the solutions are spatially periodic kink chains. If U has a double zero, and further simple zeros, then a sphaleron can occur. The genus is reduced to $n-2$ because of the double zero. Also, the differential is meromorphic, with at least one pair of simple poles. If U has two double zeros, with the remaining zeros simple, then a kink solution can connect the double zeros. For $n > 2$, the genus is reduced to $n-3$ and the differential has at least four simple poles.

Various other possibilities occur if U has more double zeros, or zeros of higher multiplicity. A general statement depends on the observation that any polynomial $U(\phi)$ has an essentially unique decomposition $U(\phi) = P^2(\phi)Q(\phi)$ where P and Q are polynomials and Q has only simple zeros. The curve $V^2 = U(\phi)$ then has the same genus as the curve $v^2 = Q(\phi)$.

2. BPS and semi-BPS kinks

In this section we first review the Bogomolny argument [9] that leads to a simple understanding of kink solutions and their energies. We then clarify what we mean by a BPS kink, and introduce the new notion of a semi-BPS kink. There is also the notion of a semi-BPS sphaleron (a notion independently studied by Alonso-Izquierdo *et al* [17]) and of a semi-BPS kink chain. These are discussed later.

We assume at first that $U(\phi)$ is a polynomial. For the usual Bogomolny argument to work, U needs to be positive semi-definite, with two or more distinct zeros. These zeros of U are quadratic, or of higher even order (otherwise U would take negative values). Physically, they are vacua of the field theory, as they minimise the energy. Let ϕ_- and ϕ_+ be adjacent zeros, with $\phi_+ > \phi_-$. Then we expect there to be a stable kink solution interpolating between ϕ_- as $x \to -\infty$ and ϕ_+ as $x \to \infty$. The kink is the minimal-energy static solution with these boundary conditions, and to find it we perform the following Bogomolny rearrangement of the energy (1) (known much earlier in this one-dimensional context),

$$\begin{aligned} E &= \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + U(\phi)\right\}\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} - \sqrt{U(\phi)}\right)^2 + 2\sqrt{U(\phi)}\,\frac{\mathrm{d}\phi}{\mathrm{d}x}\right\}\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} - \sqrt{U(\phi)}\right)^2\mathrm{d}x + \int_{\phi_-}^{\phi_+}\sqrt{U(\phi)}\,\mathrm{d}\phi\,. \end{aligned} \tag{12}$$

The final integral depends only on the form of U and the endpoints of the integral, and not on the x-dependence of ϕ. It is therefore 'topological'. The energy is minimised when the remaining integral vanishes, i.e. when the first-order field equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \sqrt{U(\phi)} \tag{13}$$

is satisfied. For the kink solution, the positive square root of U should be selected, so that $\frac{\mathrm{d}\phi}{\mathrm{d}x}$ is positive. The negative square root gives the antikink, the spatial reflection of the kink, with opposite boundary conditions.

The field equation obtained by this Bogomolny argument is the same as that obtained as the first integral of the Euler–Lagrange equation, except that the constant of integration C is automatically zero. The argument also gives the formula for the kink energy

$$E = \int_{\phi_-}^{\phi_+}\sqrt{U(\phi)}\,\mathrm{d}\phi = \int_{\phi_-}^{\phi_+} V(\phi)\,\mathrm{d}\phi\,. \tag{14}$$

Here it is important that $U(\phi_\pm) = 0$, otherwise the energy would diverge.

The Bogomolny argument guarantees kink stability, since the kink is the minimal-energy field configuration connecting ϕ_- to ϕ_+. There is also a local argument. The spatial derivative of the kink solution is everywhere positive, as $\sqrt{U(\phi)}$ is positive between ϕ_- and ϕ_+. But this spatial derivative is the translation zero-mode of the kink, a zero-frequency eigenfunction of the second-variation of the energy. As this mode has no node (zero), the Sturm oscillation theorem implies that for the kink, there are no modes of negative squared frequency, i.e. modes of instability.

We now clarify the distinction between a BPS kink and a semi-BPS kink. We call the kink BPS if the polynomial $U(\phi)$ has a square root $V(\phi)$ that is itself a polynomial. The integral of $V(\phi)$ is another polynomial $W(\phi)$, and

$$U(\phi) = \left(\frac{\mathrm{d}W(\phi)}{\mathrm{d}\phi}\right)^2. \tag{15}$$

W is known as the superpotential of U, and it occurs as a fundamental ingredient in supersymmetric theories [18]. The Riemann surface Σ separates into two copies of the ϕ-plane in the BPS case. The differential (7) is meromorphic and can be integrated using partial fractions [19]. For a BPS kink, the energy (14) simplifies to

$$E = \int_{\phi_-}^{\phi_+} V(\phi)\,\mathrm{d}\phi = W(\phi_+) - W(\phi_-)\,. \tag{16}$$

Note that an additive constant of integration in W has no effect, either on the field equation, or on the kink energy.

A kink still exists if the polynomial U does not have a polynomial square root, but has double (or higher even order) zeros at ϕ_- and ϕ_+ and is positive between. Such a polynomial can have real simple zeros outside this range, or complex conjugate pairs of simple zeros, so Σ is a connected double-cover of the ϕ-plane, branched at the simple zeros (and also at any zeros of higher odd order). We call this type of kink semi-BPS. The Bogomolny argument still works, and there is a kink connecting ϕ_- and ϕ_+ whose energy is given by the formula (16). However, the field equation is more difficult to solve, because the differential (7) is only properly defined on Σ. Also, the function $W(\phi)$ is more difficult to determine. The semi-BPS case becomes complicated if U has more than two simple zeros, because Σ then has positive genus, and the kink and its energy depend on elliptic or hyperelliptic integrals. Note that if U is negative for some range of real ϕ, then a semi-BPS kink is only metastable.

3. Examples of BPS kinks

The canonical example of a potential U admitting a BPS kink is the double-well potential of ϕ^4-theory (figure 1) [20, 21]

$$U(\phi) = \left(1-\phi^2\right)^2\,. \tag{17}$$

Here, $V(\phi) = 1-\phi^2$ so the superpotential is $W(\phi) = \phi - \frac{1}{3}\phi^3$. U has double (quadratic) zeros at $\phi = \pm 1$.

The solution of the field equation is

$$x = \int \frac{\mathrm{d}\phi}{1-\phi^2}\,. \tag{18}$$

The integrand is a meromorphic differential on the ϕ-plane, with simple poles at ± 1. The method of partial fractions gives

$$x = \frac{1}{2}\int \left\{ \frac{1}{1-\phi} + \frac{1}{1+\phi} \right\} \mathrm{d}\phi = \frac{1}{2}\log\left(\frac{1+\phi}{1-\phi}\right)\,, \tag{19}$$

which can be inverted to give the explicit solution (figure 2)

$$\phi(x) = \frac{e^{2x}-1}{e^{2x}+1} = \tanh x\,. \tag{20}$$

The kink interpolates between $\phi_- = -1$ and $\phi_+ = 1$, and has short-range (exponentially decaying) tails in both directions because the zeros of U are quadratic. The kink energy is $E = W(1) - W(-1) = \frac{4}{3}$.

The ϕ^6-theory potential [11]

$$U(\phi) = \phi^2\left(1-\phi^2\right)^2 \tag{21}$$

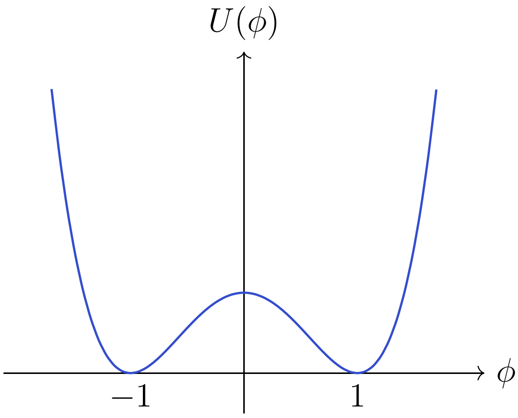

Figure 1. Potential of ϕ^4-theory.

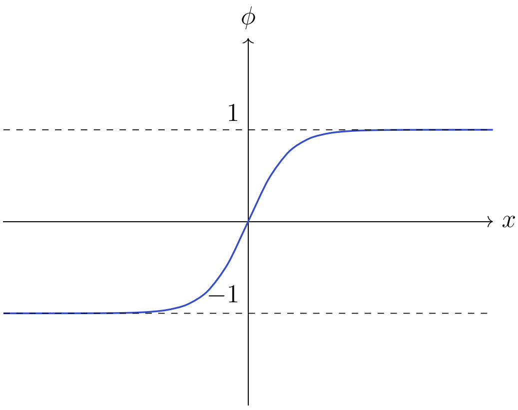

Figure 2. Kink of ϕ^4-theory.

has double zeros at -1, 0 and 1. For the kink connecting 0 to 1, the partial fraction method gives

$$x = \int \frac{d\phi}{\phi(1-\phi^2)} = \frac{1}{2}\int \left\{ \frac{1}{1-\phi} + \frac{2}{\phi} - \frac{1}{1+\phi} \right\} d\phi = \log \frac{\phi}{\sqrt{1-\phi^2}}\,. \tag{22}$$

Because the coefficients of the partial fractions are commensurate, inversion of this expression is straightforward, and the explicit solution is

$$\phi(x) = \frac{1}{\sqrt{1+e^{-2x}}}\,. \tag{23}$$

Here, $V(\phi) = \phi - \phi^3$, so $W = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4$ and the kink energy is $\frac{1}{4}$. For the kink connecting -1 to 0, V needs to be $-\phi + \phi^3$ to be positive, and W also has reversed sign. The kink energy is the same as before.

The potential [11, 15]

$$U(\phi) = \left(a^2 - \phi^2\right)^2 \left(b^2 - \phi^2\right)^2\,, \tag{24}$$

with $b > a > 0$, is a symmetric ϕ^8-theory potential with four double zeros. The field equation can be solved in a similar way, both for the kink connecting $-a$ to a, and for the kink connecting a to b (with the kink from $-b$ to $-a$ similar). The kink from $-a$ to a is implicitly

$$x = \frac{1}{2a\,(b^2 - a^2)} \log\left(\left(\frac{a+\phi}{a-\phi}\right)\left(\frac{b-\phi}{b+\phi}\right)^{a/b}\right). \tag{25}$$

For general a/b the power in this expression is not rational, and inversion is not possible. Even if a/b is rational, inversion generally requires the solution of an algebraic equation of high order, which is often not possible, but some cases are tractable. $W(\phi)$ is easy to calculate for any a and b, and the energy of the kink is $\frac{4}{15}a^3(5b^2 - a^2)$.

In the examples so far, U has had only double zeros, so the kinks have short-range tails. In the next example, U is the symmetric ϕ^8-theory potential with one quartic zero [11, 14],

$$U(\phi) = \phi^4 \left(1 - \phi^2\right)^2. \tag{26}$$

Using partial fractions, the kink solution connecting 0 to 1 is

$$x = \int \frac{\mathrm{d}\phi}{\phi^2\,(1-\phi^2)} = \frac{1}{2}\int \left\{\frac{1}{1-\phi} + \frac{2}{\phi^2} + \frac{1}{1+\phi}\right\} \mathrm{d}\phi = \frac{1}{2}\log\left(\frac{1+\phi}{1-\phi}\right) - \frac{1}{\phi}. \tag{27}$$

This illustrates the generic result that the integral of a rational function is a combination of a rational part and a logarithmic part [19]. These cannot usefully be combined, and there is no explicit formula for $\phi(x)$. Instead, (27) should be regarded as the complete solution. The kink connects $\phi_- = 0$, where the tail is long-range, to $\phi_+ = 1$, where the tail is short-range. In this example, $V(\phi) = \phi^2 - \phi^4$, so $W(\phi) = \frac{1}{3}\phi^3 - \frac{1}{5}\phi^5$ and the kink energy is $E = W(1) - W(0) = \frac{2}{15}$.

It is of interest to find a kink with two long-range tails, interpolating between two quartic zeros of U. The simplest suitable polynomial potential is [10]

$$U(\phi) = \left(1 - \phi^2\right)^4, \tag{28}$$

which has quartic zeros at $\phi = \pm 1$. Here, $V(\phi) = (1-\phi^2)^2$ so $W(\phi) = \phi - \frac{2}{3}\phi^3 + \frac{1}{5}\phi^5$. The kink connecting -1 to 1 therefore has energy $E = W(1) - W(-1) = \frac{16}{15}$. The field equation has the solution

$$x = \int \frac{\mathrm{d}\phi}{\left(1-\phi^2\right)^2}, \tag{29}$$

which, in terms of partial fractions, becomes

$$\begin{aligned} x &= \frac{1}{4}\int \left\{\frac{1}{(1-\phi)^2} + \frac{1}{1-\phi} + \frac{1}{(1+\phi)^2} + \frac{1}{1+\phi}\right\} \mathrm{d}\phi \\ &= \frac{1}{2}\frac{\phi}{1-\phi^2} + \frac{1}{4}\log\left(\frac{1+\phi}{1-\phi}\right). \end{aligned} \tag{30}$$

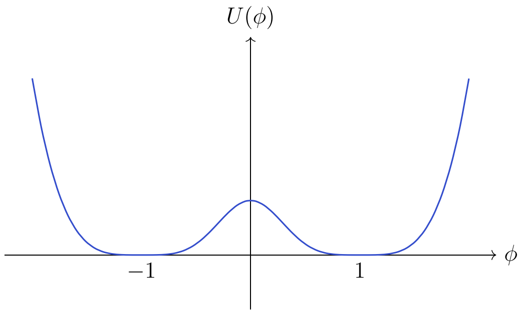

Figure 3. Rational potential $U(\phi) = (1-\phi^2)^4/(1+\phi^2)^2$.

There is no explicit kink solution $\phi(x)$. However one can verify that $\phi(x)$ approaches its asymptotic values ± 1 with long-range $\frac{1}{4x}$ tails modified by logarithmic corrections. More general polynomial potentials with kinks of this type have recently been comprehensively studied [16].

There is also a purely algebraic BPS kink with long-range tails. For this, U needs to be rational rather than polynomial. A suitable potential is (figure 3)

$$U(\phi) = \frac{\left(1-\phi^2\right)^4}{\left(1+\phi^2\right)^2}\,. \tag{31}$$

This again has quartic zeros at $\phi = \pm 1$ and is elsewhere positive. The kink solution connecting -1 and 1 is

$$x = \int \frac{1+\phi^2}{\left(1-\phi^2\right)^2}\,\mathrm{d}\phi = \frac{1}{2}\int \left\{ \frac{1}{\left(1-\phi\right)^2} + \frac{1}{\left(1+\phi\right)^2} \right\} \mathrm{d}\phi\,. \tag{32}$$

The integrand here is an Abelian differential of the second kind, with no simple poles. The integral is therefore rational, and can be inverted. Explicitly,

$$x = \frac{1}{2}\left\{ \frac{1}{1-\phi} - \frac{1}{1+\phi} \right\} = \frac{\phi}{1-\phi^2} \tag{33}$$

so the kink solution is (figure 4)

$$\phi(x) = \frac{\sqrt{1+4x^2}-1}{2x}\,. \tag{34}$$

$\phi(x)$ approaches ± 1 asymptotically with long-range $\frac{1}{2x}$ tails. The superpotential W is more complicated than a polynomial. As

$$V(\phi) = \frac{\left(1-\phi^2\right)^2}{1+\phi^2} = -3+\phi^2+\frac{4}{1+\phi^2}\,, \tag{35}$$

it follows that $W(\phi) = -3\phi + \frac{1}{3}\phi^3 + 4\tan^{-1}\phi$, and the kink energy is $E = -\frac{16}{3} + 2\pi$.

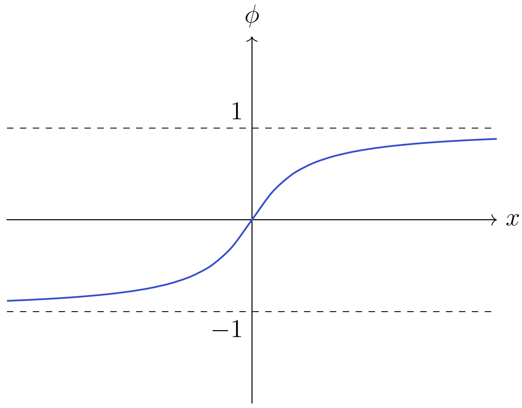

Figure 4. Algebraic kink for rational potential.

4. Example of semi-BPS kink

The only example we consider occurs for the Christ–Lee potential [22, 23]

$$U(\phi) = \frac{1}{2}\left(1-\phi^2\right)^2\left(1+\phi^2\right), \tag{36}$$

which has double zeros at ± 1 and simple zeros at $\pm i$. U has a non-negative square root for all real ϕ, and there is a kink connecting -1 and 1. Because $\sqrt{U}$ has just two branch points, Σ has genus zero, and the kink and its energy can be determined using elementary integration techniques (e.g. the hyperbolic substitution $\phi = \sinh y$), although the integrals are relatively complicated. The kink solution is

$$x = \sqrt{2}\int \frac{\mathrm{d}\phi}{(1-\phi^2)\sqrt{1+\phi^2}} = \frac{1}{2}\log\left(\frac{\sqrt{2(1+\phi^2)}+2\phi}{\sqrt{2(1+\phi^2)}-2\phi}\right), \tag{37}$$

which can be inverted to give the explicit solution

$$\phi(x) = \frac{\sinh x}{\sqrt{\sinh^2 x + 2}}. \tag{38}$$

By a similar integration one finds that

$$\begin{aligned} W(\phi) &= \frac{1}{\sqrt{2}}\int\left(1-\phi^2\right)\sqrt{1+\phi^2}\,\mathrm{d}\phi \\ &= \frac{1}{\sqrt{2}}\left(\frac{3}{8}\phi - \frac{1}{4}\phi^3\right)\sqrt{1+\phi^2} + \frac{5}{8\sqrt{2}}\sinh^{-1}\phi\,. \end{aligned} \tag{39}$$

The kink energy is therefore

$$E = W(1) - W(-1) = \frac{1}{4} + \frac{5}{4\sqrt{2}}\sinh^{-1}1\,. \tag{40}$$

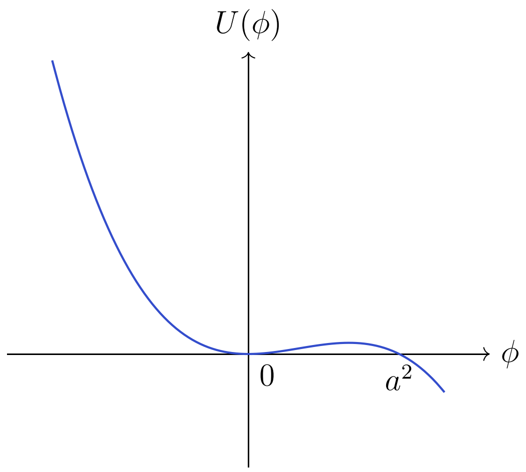

Figure 5. Cubic potential with double zero and simple zero.

5. Examples of sphalerons

A sphaleron occurs if U has a double zero (or a zero of higher even order) ϕ_0 and an adjacent real simple zero ϕ_1, with U positive between them. The sphaleron solution $\phi(x)$ runs from ϕ_0 to ϕ_1 and back to ϕ_0 as x runs from $-\infty$ to ∞. ϕ_1 is attained at some finite x. $V=\sqrt{U}$ is not a polynomial, and is not real on the unattained side of the simple zero, so a sphaleron is semi-BPS. With care, the Bogomolny argument can still be used to obtain the field equation and the sphaleron energy. If U has only one more simple zero, possibly at infinity, then the Riemann surface Σ is of genus zero, and the field equation can be solved by elementary integration.

The simplest example, a scaled version of what was called the 'simplest sphaleron' in [7], has the cubic potential (figure 5) [6]

$$U(\phi)=\frac{4}{a^2}\phi^2\left(a^2-\phi\right)\,, \tag{41}$$

with a double zero at $\phi_0=0$ and a simple zero at $\phi_1=a^2$. As U is negative in the range $\phi>a^2$, and unbounded below, the local minimum of U is not the global minimum, so the constant solution $\phi(x)=0$ is a false vacuum with zero energy. The sphaleron, which extends between 0 and a^2 and back, represents the energy barrier to be crossed, between the false vacuum and configurations of negative energy.

The algebraic curve $V^2=U(\phi)=\frac{4}{a^2}\phi^2(a^2-\phi)$ is a nodal cubic curve of genus zero, and has the rational parametrisation

$$\phi=a^2-t^2\,,\quad V=-\frac{2}{a}t\left(a^2-t^2\right)\,. \tag{42}$$

Σ, the Riemann surface of the curve, is a double-cover of the ϕ-plane. where the single node point $V=\phi=0$ on the curve becomes two distinct places on Σ, with parameter values $t=\pm a$. The sphaleron solution runs from the node $\phi_0=0$ back to the node, encircling the branch point at $\phi_1=a^2$, so it runs along Σ from $t=-a$ to $t=a$.

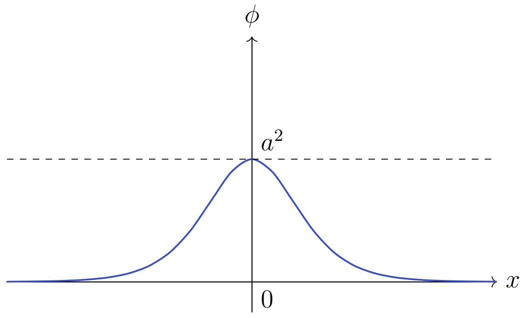

Figure 6. Cubic sphaleron.

The sphaleron solution is

$$x = \int \frac{\mathrm{d}\phi}{V(\phi)} = \frac{a}{2} \int \frac{\mathrm{d}\phi}{\phi\sqrt{a^2 - \phi}} \,. \tag{43}$$

Using the rational parametrisation, this becomes

$$x = a \int \frac{\mathrm{d}t}{a^2 - t^2} = \tanh^{-1} \frac{t}{a} \,. \tag{44}$$

Therefore $t = a \tanh x$, and the explicit solution is (figure 6)

$$\phi(x) = \frac{a^2}{\cosh^2 x} \,. \tag{45}$$

Because the potential is cubic, we call this the cubic sphaleron. Its energy is

$$E = \int V(\phi)\, \mathrm{d}\phi = \frac{2}{a} \int \phi \sqrt{a^2 - \phi}\, \mathrm{d}\phi \,, \tag{46}$$

where the integral runs from node to node around the branch point. In the rational parametrisation, this becomes

$$E = \frac{4}{a} \int_{-a}^{a} \left(a^2 t^2 - t^4\right) \mathrm{d}t = \frac{16}{15} a^4 \,. \tag{47}$$

Notice that E is positive, i.e. larger than the zero energy of the false vacuum.

The sphaleron solution (45) is symmetric in x, and has a maximum at $x = 0$, so the translation zero mode of the sphaleron, its derivative with respect to x, has a zero at $x = 0$, i.e. has one node. By the Sturm oscillation theorem, the second-variation operator must have precisely one nodeless eigenfunction with negative squared frequency–the sphaleron's unstable mode. This argument verifies that the solution $\phi(x)$ is a sphaleron–an unstable static solution of the Euler–Lagrange equation with one mode of instability.

Although the Bogomolny argument correctly gives the field equation for the sphaleron, and also the sphaleron energy (provided the contours of integration are carefully chosen), this does not mean that the sphaleron is energy-minimising. Because $\sqrt{U(\phi)}$ becomes imaginary for $\phi > a^2$, the integrated quadratic expression in (12) can be less than zero. In one direction, the unstable deformation of the sphaleron locally makes $\phi > a^2$ and lowers the energy. By symmetry, and rather curiously, the reversed deformation, where ϕ is everywhere between 0 and a value less than a^2, also lowers the energy.

Another interesting sphaleron–the quartic sphaleron–occurs for the potential

$$U(\phi) = \frac{4}{a^2 b^2}\phi^2\left(a^2-\phi\right)\left(b^2-\phi\right), \tag{48}$$

with $b > a > 0$. The sphaleron runs from $\phi_0 = 0$ to $\phi_1 = a^2$ and back. It represents the barrier for tunnelling from the zero-energy false vacuum at $\phi = 0$ to the true negative-energy vacuum lying at the potential minimum, between a^2 and b^2 [4, 6].

Although the potential here is quartic, it is analytically similar to the cubic potential (41). The square root of the quartic potential has branch points at a^2 and b^2, whereas for the cubic potential they are at a^2 and infinity. To deal with the quartic potential, we use the fractional linear transformation

$$\Phi = \frac{\left(b^2-a^2\right)\phi}{b^2-\phi}. \tag{49}$$

This maps $\phi = 0, a^2, b^2$ to $\Phi = 0, a^2, \infty$. The quartic sphaleron solution can now be calculated in terms of Φ using the results for the cubic sphaleron. We find, simply,

$$\Phi(x) = \frac{a^2}{\cosh^2 x}, \tag{50}$$

and in terms of the original variable ϕ, the solution is

$$\phi(x) = \frac{a^2 b^2}{(b^2-a^2)\cosh^2 x + a^2}. \tag{51}$$

Finding the energy is algebraically more complicated than for the cubic sphaleron, as there are extra factors from the fractional linear transformation. However, the energy integrand is rational on the Riemann surface Σ, and the total energy is calculated to be

$$E = \frac{a^4}{12}\frac{1}{z^5}\left[6z - 4z^3 + 6z^5 - 3\left(1-z^2-z^4+z^6\right)\log\left(\frac{1+z}{1-z}\right)\right], \tag{52}$$

where $z = \frac{a}{b}$. With the chosen normalisation factors, the quartic potential (48) for $0 \leqslant \phi \leqslant a^2$ reduces to the cubic potential (41) as $b \to \infty$, and one can easily see that the sphaleron solution (51) reduces to (45) in this limit. For the energy, the limit is less obvious, but after expanding the logarithmic contribution in (52) for small z, one observes several cancellations and that the leading term in the square brackets is of order z^5. The quartic sphaleron energy is $\frac{16}{15}a^4$ as $b \to \infty$, as expected.

The quartic sphaleron has an illuminating alternative form. Let $a^2 = c^2 \tanh s$ and $b^2 = \frac{c^2}{\tanh s}$. Then (51) becomes

$$\phi(x) = \frac{c^2 \sinh s \cosh s}{\cosh^2 x + \sinh^2 s}, \tag{53}$$

which can be reexpressed as

$$\phi(x) = \frac{c^2}{2}\left(\tanh(x+s) - \tanh(x-s)\right), \tag{54}$$

that is, as a superposition of a ϕ^4 kink centred at $-s$ and a ϕ^4 antikink at s. The kink and antikink are in unstable equilibrium because the potential is a tilted version of the standard, symmetric potential of ϕ^4-theory. After a small perturbation they can either annihilate into the false vacuum, or separate, generating an increasing region of true vacuum.

6. Examples of kink chains

Kink chains are spatially periodic solutions $\phi(x)$ of the types of field equation we have been considering. More precisely, they are chains of alternating kinks and antikinks, with ϕ spatially oscillating between neighbouring simple zeros of $U(\phi)$. Because U has simple zeros, Σ is connected, and the kink chain is semi-BPS. We shall not discuss kink chain solutions in much detail. Most of them involve elliptic or hyperelliptic integrals.

The simplest kink chain arises from the quadratic potential (figure 7)

$$U(\phi) = 1 - \phi^2 . \tag{55}$$

This is interesting, even though the Euler–Lagrange equation (2) is the elementary linear equation $\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \phi = 0$. The kink chain solution is

$$x = \int \frac{\mathrm{d}\phi}{\sqrt{1-\phi^2}} = \sin^{-1}\phi\,, \tag{56}$$

so

$$\phi(x) = \sin x\,, \tag{57}$$

with spatial period 2π (figure 8). Kinks are centred at $x = 0 \mod 2\pi$ and antikinks at $x = \pi \mod 2\pi$. The energy per period is

$$E = \oint \sqrt{1-\phi^2}\,\mathrm{d}\phi = \pi\,, \tag{58}$$

where the integral is along a closed loop in Σ, enclosing both branch points. Like the cubic sphaleron, this kink chain is rather unphysical because the potential is unbounded below. However, it often occurs for limiting parameter values of more realistic potentials.

A more physical kink chain occurs for the potential [8]

$$U(\phi) = \left(1-\phi^2\right)\left(1-k^2\phi^2\right)\,, \tag{59}$$

with $0 < k < 1$. The solution of the field equation is the elliptic integral

$$x = \int \frac{\mathrm{d}\phi}{\sqrt{(1-\phi^2)(1-k^2\phi^2)}}\,, \tag{60}$$

with the integral circulating around the branch points at $\phi = \pm 1$ as x increases, If we fix $\phi = 0$ and $\frac{d\phi}{\mathrm{d}x} > 0$ at $x = 0$, then

$$x = \int_0^{\phi} \frac{\mathrm{d}\phi}{\sqrt{(1-\phi^2)(1-k^2\phi^2)}}\,. \tag{61}$$

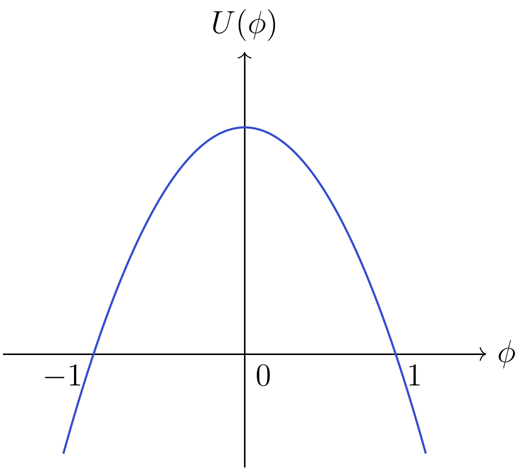

Figure 7. Quadratic potential with simple zeros.

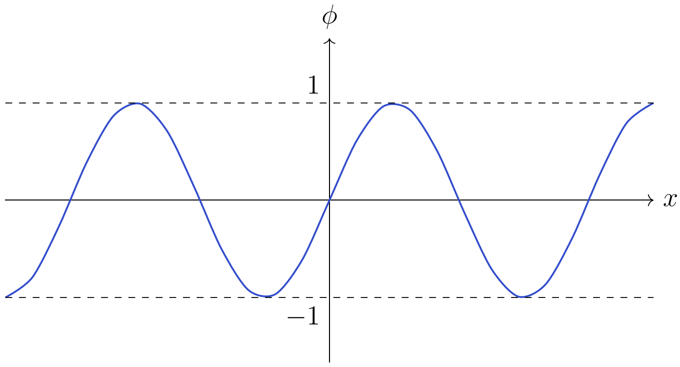

Figure 8. Kink chain for quadratic potential.

This Legendre elliptic integral of the first kind can be inverted using the Jacobi elliptic function sn with modulus k, giving the kink chain

$$\phi(x) = \mathrm{sn}(x,k)\,. \tag{62}$$

ϕ oscillates between -1 and 1, as for the quadratic potential (55), and has equally-spaced kinks and antikinks. The full spatial period is

$$X = 4\int_0^1 \frac{\mathrm{d}\phi}{\sqrt{(1-\phi^2)(1-k^2\phi^2)}} = 4K(k)\,, \tag{63}$$

where

$$K(k) = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \cdots\right) \tag{64}$$

is the complete elliptic integral of the first kind. The energy per period of the kink chain is

$$E = 4\int_0^1 \sqrt{(1-\phi^2)(1-k^2\phi^2)}\,\mathrm{d}\phi = \frac{4}{3k^2}\left(\left(1+k^2\right)E(k) - \left(1-k^2\right)K(k)\right), \tag{65}$$

where

$$E(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \cdots\right) \tag{66}$$

is the complete elliptic integral of the second kind. As $k \to 0$, the potential U approaches the simple quadratic form (55), and unsurprisingly, the kink chain solution $\mathrm{sn}(x,k)$ approaches $\sin x$, its period X approaches 2π and its energy per period approaches π.

Kink chains are not stable. The translation zero mode (the spatial derivative of the chain) oscillates with the same period as the chain itself, and has two nodes per period. An infinitely-long chain therefore has infinitely many unstable modes. Manton and Samols [8], discussed the modes of instability for a kink chain (62) of finite length, having N complete periods and periodic boundary conditions. There are $2N-1$ modes of instability, all arising from the breakdown of the equal spacing of the N kinks and N antikinks. The kinks and antikinks tend to approach each other and annihilate.

7. Trigonometric potentials

In a number of scalar field theories, in particular, the well-known sine-Gordon theory, the potential $U(\phi)$ is a periodic, trigonometric function of ϕ. We will consider here the class of potentials $U(\phi) = \cos^n\phi$, with n a positive integer, and will focus on the simplest cases $n = 1,2,4$ and 8. For all n, $U = 1$ at $\phi = 0$, and the closest zeros of U are at $\phi = \pm\frac{\pi}{2}$. There are static solutions interpolating between these latter values of ϕ. The $n = 1$ potential $\cos\phi$ has simple zeros at $\pm\frac{\pi}{2}$, so the solution is a semi-BPS kink chain. For $n = 2k$, $U(\phi)$ has the real square root $V(\phi) = \cos^k\phi$ which can be straightforwardly integrated to obtain a superpotential $W(\phi)$, and here the solution is a BPS kink. The minima of U at $\pm\frac{\pi}{2}$ are quadratic for $n = 2$, so the kink has tails that are short-range. This is, in fact, a variant of the sine-Gordon kink [24, 25]. For larger, even n, the minima at $\pm\frac{\pi}{2}$ are of higher order, so the kink has long-range tails [26].

For all n, it is possible to use the rational parameter $t = \tan\frac{\phi}{2}$, with t between -1 and 1. This sometimes converts the integrals specifying the solutions and their energies into integrals that have appeared earlier. Generally, for $U(\phi) = \cos^n\phi$, the kink or kink chain solution is given by

$$x = \int \frac{\mathrm{d}\phi}{\cos^{\frac{n}{2}}\phi}. \tag{67}$$

Using $t = \tan\frac{\phi}{2}$ this becomes

$$\frac{x}{2} = \int \frac{\left(1+t^2\right)^{\frac{n}{2}-1}}{\left(1-t^2\right)^{\frac{n}{2}}}\,\mathrm{d}t, \tag{68}$$

which has a rational integrand when n is even.

For $n = 1$ the solution (68) is the lemniscate elliptic integral of the first kind

$$\frac{x}{2} = \int \frac{\mathrm{d}t}{\sqrt{1-t^4}}\,, \tag{69}$$

so $t = \mathrm{sl}\left(\frac{x}{2}\right)$, where sl is the lemniscate elliptic function, related to Jacobi functions via

$$\mathrm{sl}\,(z) = \mathrm{sn}\,(z,i) = \mathrm{sc}\left(z,\sqrt{2}\right)\,. \tag{70}$$

The explicit semi-BPS kink chain solution is therefore [24]

$$\tan\frac{\phi}{2} = \mathrm{sl}\left(\frac{x}{2}\right)\,, \tag{71}$$

which oscillates between $\phi = \pm\frac{\pi}{2}$.

The spatial period and energy per period of this kink chain are best found using the Beta function, whose trigonometric integral expression and value are

$$B(z_1,z_2) = 2\int_0^{\frac{\pi}{2}} (\sin\phi)^{2z_1-1}(\cos\phi)^{2z_2-1}\,\mathrm{d}\phi = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1+z_2)}\,. \tag{72}$$

The period X and energy per period E are then

$$\begin{aligned} X &= 4\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{\cos\phi}} = 2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\,, \\ E &= 4\int_0^{\frac{\pi}{2}} \sqrt{\cos\phi}\,\mathrm{d}\phi = 2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}\,. \end{aligned} \tag{73}$$

Their approximate values are $X \simeq 10.488$, $E \simeq 4.793$, and there is the curious exact relation $XE = 16\pi$.

For $n = 2$, the potential is $U(\phi) = \cos^2\phi = \frac{1}{2}(\cos(2\phi)+1)$, a variant of the sine-Gordon potential [24]. The sine-Gordon kink solution in this case is

$$x = \int \frac{\mathrm{d}\phi}{\cos\phi} = 2\int \frac{\mathrm{d}t}{1-t^2} = 2\tanh^{-1} t\,, \tag{74}$$

and therefore

$$t = \tan\frac{\phi}{2} = \tanh\frac{x}{2}\,. \tag{75}$$

More simply,

$$\tan\phi = \sinh x\,. \tag{76}$$

The kink connects $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ as x runs from $-\infty$ to ∞, and the tails are short-range. Here, $V(\phi) = \cos\phi$, so the superpotential is $W(\phi) = \sin\phi$. The sine-Gordon kink is therefore BPS, and its energy is

$$E = W\left(\frac{\pi}{2}\right) - W\left(-\frac{\pi}{2}\right) = 2\,. \tag{77}$$

For the $n = 4$ potential $U(\phi) = \cos^4\phi$, the BPS kink solution and its energy are

$$x = \int \frac{\mathrm{d}\phi}{\cos^2\phi}\,, \quad E = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\phi\,\mathrm{d}\phi\,. \tag{78}$$

Both integrals are straightforward, and give results [27]

$$x = \tan\phi\,, \quad E = \frac{1}{2}\pi\,, \tag{79}$$

so $\phi(x) = \tan^{-1}x$. The tail behaviours are of the long-range form $\phi \sim \pm\frac{\pi}{2} - \frac{1}{x}$ as $x \to \pm\infty$. In this example, the rational parametrisation gives

$$x = 2\int \frac{1+t^2}{(1-t^2)^2}\,\mathrm{d}t = \frac{2t}{1-t^2} = \tan\phi\,. \tag{80}$$

The integral here is the same as in (32), defining the algebraic BPS kink solution that connects two quartic zeros without log terms.

Similarly, for the $n = 8$ potential $U(\phi) = \cos^8\phi$, the kink solution and its energy are

$$x = \int \frac{\mathrm{d}\phi}{\cos^4\phi}\,, \quad E = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\phi\,\mathrm{d}\phi\,, \tag{81}$$

so

$$x = \tan\phi + \frac{1}{3}\tan^3\phi\,, \quad E = \frac{3}{8}\pi\,, \tag{82}$$

where we used the identity $\sec^4\phi = (1+\tan^2\phi)\sec^2\phi$. The kink solution here is implicit, unless one solves a cubic equation. The tails are dominated by the $\tan^3\phi$ term, and approach the limiting values more slowly than in the $n = 4$ case. With further effort, solutions for any n can be determined.

8. Conclusions

We have reviewed the topologically interesting, static solutions of scalar field theories in one dimension having a single scalar field ϕ. These solutions are kinks, sphalerons and kink chains, whose nature depends critically on the multiplicities of the zeros of the potential $U(\phi)$ appearing in the first-order field equation (the first integral of the static Euler–Lagrange equation, with the constant of integration absorbed into U).

Although the Bogomolny rearrangement of the field energy (the completion of the square) is formally valid for all these types of solution, we have made a novel distinction between BPS and semi-BPS solutions. If $\sqrt{U(\phi)}$ is defined on the ϕ-plane, i.e. if $\sqrt{U}$ is a simple function of ϕ with no explicit square roots remaining, then a kink solution can be regarded as a BPS kink. If $\sqrt{U}$ has branch points and is well-defined only on the double-cover of the ϕ-plane, then we refer to a kink solution as semi-BPS. The field equation is generally more difficult to solve in the semi-BPS case, and the energy is more tricky to calculate. Sphaleron and kink chain solutions require U to have simple zeros (or possibly, zeros of higher odd orders); $\sqrt{U}$ therefore has branch points and these solutions are inevitably semi-BPS. BPS and semi-BPS kinks are stable, or at least metastable, but sphaleron and kink chain solutions have modes of instability. We have presented a broad range of examples of these solution types. Most are well-known, but not all.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

This work has been partially supported by consolidated Grant ST/T000694/1 from the UK Science and Technology Facilities Council. I am grateful to the organisers of the 2023 SIG XI workshop in Krakow, Poland, and in particular to Andrzej Wereszczynski for discussions about the distinction between BPS and semi-BPS solutions. Alonso-Izquierdo *et al* [17] was in preparation at the same time as this paper, and I am grateful for receiving an early draft.

ORCID iD

N S Manton https://orcid.org/0000-0002-2938-156X

References

[1] Rajaraman R 1982 *Solitons and Instantons* (Elsevier Science)
[2] Manton N and Sutcliffe P 2004 *Topological Solitons* (Cambridge University Press)
[3] Shnir Y M 2018 *Topological and Non-Topological Solitons in Scalar Field Theories* (Cambridge University Press)
[4] Callan C G Jr and Coleman S 1977 Fate of the false vacuum. II. First quantum corrections *Phys. Rev.* D **16** 1762
[5] Manton N S and Klinkhamer F 1984 A saddle-point solution in the Weinberg–Salam theory *Phys. Rev.* D **30** 2212
[6] Avelar A T, Bazeia D, Losano L and Menezes R 2008 New lump-like structures in scalar-field models *Eur. Phys. J.* C **55** 133
[7] Manton N S and Romańczukiewicz T 2023 Simplest oscillon and its sphaleron *Phys. Rev.* D **107** 085012
[8] Manton N S and Samols T M 1988 Sphalerons on a circle *Phys. Lett.* B **207** 179
[9] Bogomolny E B 1976 The stability of classical solutions *Sov. J. Nucl. Phys.* **24** 449
[10] González J A and Estrada-Sarlabous J 1989 Kinks in systems with degenerate critical points *Phys. Lett.* A **140** 189
[11] Lohe M A 1979 Soliton structures in $P(\varphi)_2$ *Phys. Rev.* D **20** 3120
[12] Khare A, Christov I C and Saxena A 2014 Successive phase transitions and kink solutions in ϕ^8, ϕ^{10} and ϕ^{12} field theories *Phys. Rev.* E **90** 023208
[13] Bazeia D, Menezes R and Moreira D C 2018 Analytical study of kinklike structures with polynomial tails *J. Phys. Commun.* **2** 055019
[14] Radomskiy R V, Mrozovskaya E V, Gani V A and Christov I C 2017 Topological defects with power-law tails *J. Phys.: Conf. Ser.* **798** 012087
[15] Gani V A, Marjaneh A M and Blinov P A 2020 Explicit kinks in higher-order field theories *Phys. Rev.* D **101** 125017
[16] Blinov P A, Gani T V, Malnev A A, Gani V A and Sherstyukov V B 2022 Kinks in higher-order polynomial models *Chaos Solitons Fractals* **165** 112805
[17] Alonso-Izquierdo A, Navarro-Obregón S, Oles K, Queiruga J, Romanczukiewicz T and Wereszczynski A 2023 Semi-BPS sphaleron and its dynamics (arXiv:2308.14420)
[18] Freund P G O 1986 *Introduction to Supersymmetry* (Cambridge University Press)
[19] Hardy G H 1916 *The Integration of Functions of a Single Variable* 2nd edn (Cambridge University Press)
[20] Dashen R F, Hasslacher B and Neveu A 1974 Nonperturbative methods and extended-hadron models in field theory. II. Two-dimensional models and extended hadrons *Phys. Rev.* D **10** 4130
[21] Polyakov A M 1974 Particle spectrum in quantum field theory *JETP Lett.* **20** 194
[22] Christ N and Lee T D 1975 Quantum expansion of soliton solutions *Phys. Rev.* D **12** 1606

[23] Dorey P, Gorina A, Romańczukiewicz T and Shnir Y 2023 Collisions of weakly-bound kinks in the Christ–Lee model *J. High Energy Phys.* **09** 045
[24] Perring J K and Skyrme T H R 1962 A model unified field equation *Nucl. Phys.* **31** 550
[25] Rubinstein J 1970 Sine-Gordon equation *J. Math. Phys.* **11** 258
[26] Mello B A, González J A, Guerrero L E and López-Atencio E 1998 Topological defects with long-range interactions *Phys. Lett.* A **244** 277
[27] Mohammadi M, Riazi N and Azizi A 2012 Radiative properties of kinks in the $\sin^4(\varphi)$ system *Prog. Theor. Phys.* **128** 615