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TOPICAL REVIEW

The effective temperature

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Abstract
This review presents the effective temperature notion as defined from the deviations from the equilibrium fluctuation–dissipation theorem in out-of-equilibrium systems with slow dynamics. The thermodynamic meaning of this quantity is discussed in detail. Analytic, numeric and experimental measurements are surveyed. Open issues are mentioned.

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1. Introduction

One of the core ideas of statistical mechanics is that *equilibrium states* can be accurately described in terms of a small number of thermodynamic variables, such as temperature and pressure. At present there is no equivalent framework for generic *out-of-equilibrium macroscopic systems*, and one is forced to solve their dynamics on a case-by-case basis.

Out-of-equilibrium macroscopic systems are of many different kinds. An interesting class is the one in which the relaxation is slow—with observables decaying, say, as power laws instead of exponentially. Typical instances are coarsening phenomena and generic glasses, realized as molecular, polymeric or magnetic materials, among others. Another intriguing group is the one of non-equilibrium steady states in the weak drive limit. Examples are gently vibrated granular matter and weakly sheared super-cooled liquids.

The quest for an approximate thermodynamic description of such systems or, to start with, the identification of effective parameters acting as the equilibrium ones, has a long history that we will not review in detail here. In contrast, we will focus on the development of the *effective temperature* notion that has proven to be a successful concept at least within certain limits that we will discuss.

About 20 years ago, in the context of weak turbulence, Hohenberg and Shraiman [1] proposed to define an effective temperature through the departure from the fluctuation–dissipation theorem (FDT). However, neither a detailed analysis of this quantity nor the conditions under which such a notion could have a thermodynamic meaning were given in this reference. Later, Edwards mentioned the same possibility in the context of granular matter [2]. Again, the proposal did not catch the attention of the community. In none of the ensuing studies was the importance of distinguishing different dynamic regimes sufficiently stressed and the few numerical checks performed at the time gave, consequently, confusing results. More recently, similar ideas appeared in the glassy literature [3, 4]. In this field, the possibility of solving exactly a number of schematic models [5, 6] put the definition of the effective temperature, $T_{\rm EFF}$, on a much firmer ground. The solution to these models' dynamics established the importance of reaching an asymptotic limit with slow dynamics and attributing a value of the effective temperature to each distinct dynamic regime. The solutions demonstrate the relation with the phase space volume visited dynamically [7], they illustrate in many forms the relevance of $T_{\rm EFF}$ to heat transfer and equilibration, and they set some limits to the extent of applicability of the concept. These results opened the way to studies in a wealth of more realistic cases. Beyond pure phenomenological descriptions of observed phenomena used in the past that are of limited predictive power, see e.g. [8–10], the $T_{\rm EFF}$ definition based on fluctuation–dissipation relations (FDRs) is not ambiguous and allows for direct measurements.

In this review, we sift through the effective temperature notion trying to transmit the full allure of the idea. The literature on fluctuation–dissipation violations is immense and several reviews have already been published [11–14]. We refer to these reports when appropriate. The rest of this review is structured in five sections. The next one introduces the definition of a number of observables and the effective temperature. It follows a discussion of the insight gained from the solution to mean-field glassy models in section 3. Section 4 presents the interpretation of the effective temperature definition as a *bona fide* temperature. Section 5 is devoted to a (non-exhaustive) description of numeric and experimental measurements performed so far with special emphasis on recent studies. Finally, in section 6, we present the conclusions. The reader who wants to get a quick summary of the outcome of almost 20 years of research on FDRs and effective temperatures with special emphasis on open questions can jump to this section.

2. Definitions

2.1. Canonical setting

In this review, we focus on the dynamics of a classical or quantum system coupled to a classical or quantum environment, typically in equilibrium at temperature T. In this sense, the setting is canonical. The systems relax by transferring energy to the environment. Atypical dissipation, as realized in granular matter, is referred to in section 4.9.2. Reference to some studies in isolated systems (microcanonical setting) is made in the quantum context, see section 5.13. All along this review $k_B = 1$.

2.2. Time and length scales

In equilibrium studies, one rarely specifies the initial time or implicitly takes it to $-\infty$. Out of equilibrium, a reference moment has to be defined. Time zero is usually taken to be the instant when the sample is prepared in some special conditions. In this review, we focus on infinitely rapid quenches and we define $t = 0$ as the moment when the working temperature is instantaneously reached.

We deal mainly with macroscopic systems in which the number of degrees of freedom is assumed to be the most diverging quantity, $N \to \infty$. In most of the theoretical and numerical calculations discussed in this review, time is finite with respect to N. Some reference to time that scales with the size of the system will be made.

2.3. Correlations and responses

The out-of-equilibrium dynamics are explored by measuring observables that depend on one or several times after the preparation of the sample. Observables could be of two types: those describing free evolution and those associated with responses to external perturbations. A multi-time correlation is the average over histories or initial conditions of several observables O_j that are functions of the system's degrees of freedom, evaluated at different times:

$$C_{O_1 O_2 \ldots O_n}(t_1, t_2, \ldots, t_n) = \langle O_1(t_1) O_2(t_2) \ldots O_n(t_n) \rangle \; . \tag{1}$$

$O_j(t)$ is any observable with, without loss of generality, zero mean (otherwise one simply substitutes $O_j \to O_j - \langle O_j \rangle$).

In systems with an energy function, the impulse response functions are the averaged reaction of an observable to a perturbation that modifies the potential energy as $V \to V - hO_n$ at a given instant t_n. One such response, at linear order in the perturbation, reads

$$R_{O_1 O_2 \ldots O_n}(t_1, t_2, \ldots, t_n) \equiv \left. \frac{\delta \langle O_1(t_1) \ldots O_{n-1}(t_{n-1}) \rangle}{\delta h(t_n)} \right|_{h=0} . \tag{2}$$

Others in which the perturbation(s) is (are) applied at intermediate time(s) can also be defined [15]. In dynamical systems in which the time evolution of the degrees of freedom is not related to the gradient of an energy function, one can similarly compute the response to an instantaneous kick that translates the degrees of freedom at a chosen instant as $x(t_n) \to x(t_n) + \delta x(t_n)$ [16]:

$$R_{O_1 O_2 \ldots O_n}(t_1, t_2, \ldots, t_n) \equiv \frac{\delta \langle O_1(t_1) \ldots O_{n-1}(t_{n-1}) \rangle}{\delta x(t_n)} . \tag{3}$$

Causality implies that both R's vanish when $t_n > t_j$ with $j = 1, \ldots, n-1$.

In cases with quantum fluctuations, ones needs to distinguish the order in which operators appear within the averages. The symmetric and anti-symmetric two-time correlations are

$$C_{[O_1, O_2]_\pm}(t_1, t_2) = \langle O_{H1}(t_1) O_{H2}(t_2) \rangle \pm \langle O_{H2}(t_2) O_{H1}(t_1) \rangle \tag{4}$$

with $\langle O_{Hi}(t_i) O_{Hj}(t_j) \rangle = \mathrm{Tr}[O_{Hi}(t_i) O_{Hj}(t_j) \varrho_0] / \mathrm{Tr} \varrho_0$, ϱ_0 the initial density operator, and $O_{Hi}(t_i)$ in the Heisenberg representation. The linear response to $H \to H - hO_2$ at t_2 is

$$R_{O_1 O_2}(t_1, t_2) = \left. \frac{\delta \langle O_{H1}(t_1) \rangle}{\delta h(t_2)} \right|_{h=0} = C_{[O_1, O_2]_-}(t_1, t_2)\, \theta(t_1 - t_2) \tag{5}$$

and the last identity is the Kubo formula valid in and out of equilibrium at linear order in h.

The search for a link between dissipation and fluctuations started with Einstein's derivation of a relation between the mobility and diffusion coefficient of a Brownian particle. The former is induced by an external force, and it is a response, while the latter is due to the spontaneous mean-square displacement of the particle's position, thus a correlation. Onsager's regression hypothesis and Kubo's linear response theory elaborated upon the idea of an existing relation between the two types of fluctuations near equilibrium. In the rest of this section, we present some relations between induced and spontaneous fluctuations that hold in and out of equilibrium.

2.4. Fluctuation–dissipation relations

Several FDRs between responses and correlations computed over unperturbed trajectories have been derived in recent years. We mention just two of these FDRs that hold in and out of equilibrium.

In generic Langevin dynamics with white [17] or coloured [18] equilibrium noise, and Markov processes described by a master equation with detailed balance [13], the linear response is related to a correlation function of the unperturbed system which, however, is more complex than the overlap between the two observables involved in the response. A simple example is the $1d$ white-noise Langevin equation

$$m\ddot{x}(t) + \gamma \dot{x}(t) = F[x(t)] + \xi(t) \tag{6}$$

that using the fact that $2\gamma T R_x(t,t') = \langle x(t)\xi(t')\rangle$ implies [17]

$$TR_x(t,t') = \frac{1}{2}\frac{\partial C_x(t,t')}{\partial t'} - \frac{1}{2}\frac{\partial C_x(t,t')}{\partial t} - \frac{1}{2}\langle x(t)F[x(t')] - x(t')F[x(t)]\rangle. \tag{7}$$

$F[x]$ is any deterministic force—not necessarily conservative. The generalization to other observables or field theories is straightforward. The extension to the non-Markovian case is more easily performed in the path integral formalism, by exploiting the transformation of the generating functional under time reversal [18]. This formulation allows one to prove that this relation holds for Hamiltonian dynamics as well.

In the case of an Ising variable, $s = \pm 1$, the equivalent relation reads [13]

$$TR_s(t,t') = \frac{1}{2}\frac{\partial C_s(t,t')}{\partial t'} + \frac{1}{2}\langle s(t)\sum_{s''}[s(t') - s'']w[s(t') \to s'']\rangle\ , \tag{8}$$

with w the transition probability with no perturbation applied. This expression can be recast in the same form as equation (7) since $\langle \dot{s}\rangle = \langle \sum_{s''}(s - s'')w(s \to s'')\rangle$ and the second factor in the last term plays the role of a deterministic force. Extensions to multi-valued discrete variables are simple. This type of FDR can also be generalized to intrinsically out-of-equilibrium systems with transition rates that do not obey detailed balance.

Many relations of this kind have been derived in the literature. A number of authors re-expressed them as a sum of two contributions: the ones that reduce to the FDT under equilibrium conditions and the anomalous ones that allow for many intriguing interpretations (dissipated energy flux, etc). We refer the reader to a recent review and a few articles that summarize these ideas [19] that, although certainly very interesting, are not the main focus of this review.

An FDR for dynamical systems characterized by a state variable x that reaches an invariant stationary measure, say $\rho(x)$, was proposed by Vulpiani *et al* [16]. Using the response to an impulse perturbation of type (3), one finds

$$R_x(t,t') = -\langle x(t)\frac{\delta \ln \rho[x(t')]}{\delta x(t')}\rangle\ \theta(t-t')\ . \tag{9}$$

Once again the generalization to a multi-variable system or a field theory and to other observables is immediate. Another relation of this kind was recently proposed by several authors [20]:

$$R_x(t,t') = -\langle x(t)\frac{\delta \ln \rho_h[x(t')]}{\delta h(t')}\rangle_{h=0}\ \theta(t-t')\ . \tag{10}$$

The average is computed at zero field. In order to go beyond this formal expression, one needs to know the measure ρ_h. Different assumptions (e.g. Gaussian form [16], uniform [21]) lead to different relations.

All these FDRs become the FDT under equilibrium conditions, as explained in section 2.5.

2.5. Linear FDT

The FDT states that the decay of spontaneous fluctuations cannot be distinguished from the one of forced fluctuations. Its classical and quantum formulations are explained in the following sections.

2.5.1. Classical. The classical linear and self-FDT expresses the equilibrium thermal fluctuations of an observable O in terms of the available thermal energy, T, and the linear response of O to a vanishingly small applied field linearly coupled to itself. FDT is used, for example, to infer mechanical properties of soft matter from the fluctuations in light scattering. Importantly enough, there is no condition on the scale at which the observables are defined, so it could range from the macroscopic to the microscopic. Indeed, if one monitors the dynamics of a classical system, either Newtonian or stochastic with detailed balance, and assumes that equilibrium with the bath has been reached, it follows easily that

$$R_O(t,t') = \frac{1}{T}\frac{\partial}{\partial t'} C_O(t,t')\,\theta(t-t') \ . \tag{11}$$

(This equation can be inferred from equations (8) and (9). In equilibrium, the last term on the right-hand side of equation (7) vanishes due to reciprocity and the two first ones are equal. If ρ takes the canonical form, equation (11) follows immediately from equation (9). An elegant proof for Langevin processes with generic multiplicative and coloured Gaussian noise is detailed in [18].) The integral of the response function over a time interval running from t_w to t,

$$\chi_O(t,t_w) = \int_{t_w}^{t} \mathrm{d}t'\, R_O(t,t') \ , \tag{12}$$

is a dc susceptibility that is easier to measure experimentally. The FDT implies a linear relation between χ_O and C_O:

$$\chi_O(t,t_w) = \frac{1}{T}\left[C_O(t,t) - C_O(t,t_w)\right] \ . \tag{13}$$

A parametric plot of $\chi_O/C_O(t,t)$ against $C_O/C_O(t,t)$, constructed at fixed t_w and for increasing t–t_w, is a straight line with slope $-1/T$ joining $(1,0)$ and $(0, \chi_{\rm EQ} = 1/T)$. The same result is obtained by keeping t fixed and letting t_w vary from 0 to t. A departure from the straight line (13) for any observable O signals a divergence from equilibrium.

A straightforward generalization is to the case in which the monitored observable, A, is not the same that couples to the perturbation B:

$$R_{AB}(t,t') = \frac{1}{T}\frac{\partial}{\partial t'} C_{AB}(t,t')\,\theta(t-t'). \tag{14}$$

In the body of this review and without loss of generality, we focus on the self-correlation and response, we take $C_O(t,t) = 1$, and we erase the sub-index O until discussing observable dependences in section 5. Equation (13) is then more compactly written as $\chi(C) = (1-C)/T$.

2.5.2. Quantum. When quantum fluctuations are important, one needs to take into account the statistics of the observables at hand. Let ϕ and $\phi^\dagger$ be (bosonic or fermionic) annihilation and creation operators, respectively. In the Schwinger–Keldysh closed time-contour formalism apt to deal with real-time Green functions, these are defined as

$$\mathrm{i}\hbar G^{ab}(t,t') \equiv \langle \phi^a(t)\bar\phi^b(t')\rangle = \mathrm{Tr}\big[\mathrm{T}_C\ \phi_H(t,a)\ \phi_H^\dagger(t',b)\ \varrho_{\rm H}(0,\pm)\big], \tag{15}$$

where $a, b = \pm$. $\bar\phi$ is either the complex conjugate (for bosons) or the Grassmann conjugate (for fermions) of ϕ. $\phi_{\rm H}(t,a)$ denotes the Heisenberg representation of the operator ϕ at time t on the a-branch of the Keldysh contour. $\varrho_{\mathrm{H}}(0, \pm) = \varrho(0)$ is the initial density matrix (normalized to be of unit trace) and its location on the $+$ or $-$ branch is not important, thanks to the cyclic property of the trace. In the grand-canonical ensemble, an equilibrium initial density operator reads $\varrho(0) \propto \exp(-\beta(H - \mu N))$, where N is the number operator commuting with H (in non-relativistic quantum mechanics) and μ is the chemical potential fixing the average number of particles. $\mathrm{T}_{\mathcal{C}}$ is the time-ordering operator acting with respect to the relative position of (t, a) and (t', b) on the Keldysh contour [22, 23]. A linear transformation of the fields allows one to define Keldysh, retarded and advanced Green functions:

$$\begin{aligned} G^K(t,t') &= \frac{\mathrm{i}\hbar}{2}[G^{+-}(t,t') + G^{-+}(t,t')], \\ G^R(t,t') &= [G^{+-}(t,t') - G^{-+}(t,t')]\Theta(t-t'), \\ G^A(t,t') &= [G^{+-}(t,t') - G^{-+}(t,t')]\Theta(t'-t), \end{aligned} \tag{16}$$

respectively.

The quantum FDT reads

$$G^K(\omega) = \hbar\left[\tanh\left(\frac{\beta}{2}(\hbar\omega - \mu)\right)\right]^{-\zeta} \operatorname{Im} G^R(\omega), \tag{17}$$

where Fourier transforms with respect to t–t' have been taken and $\zeta = \pm$ in the bosonic and fermionic case, respectively. One can formally recover the canonical ensemble result by setting $\mu = 0$. In particular, associating G^R with R and G^K with C in the notation of the previous classical paragraphs and setting $\mu = 0$, in the bosonic case one has

$$R(t-t') = \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\,\tanh\left(\frac{\beta\hbar\omega}{2}\right)C(\omega)\,\theta(t-t') \tag{18}$$

that in the $\beta\hbar \to 0$ limit reduces to the classical expression (14).

2.6. Eulerian and Lagrangian formalisms

A class of over-damped Markov diffusion processes take an equilibrium form—with detailed balance—in the Lagrangian (co-moving) frame of the mean local velocity. The resulting stochastic process does not contain information about the non-vanishing probability current of the original Eulerian (laboratory) frame dynamics. In practice, though, it is difficult to compute the local velocity and the passage to the co-moving frame is difficult to implement. Still, this observation allows one to derive a modified FDT in the Eulerian frame that includes the effect of the mean local velocity $\vec{v}$ as an additional term [24]:

$$TR_{AB}(t,t') = \theta(t-t')[\partial_{t'}C_{AB}(t,t') - \langle[\vec{v}(t')\cdot\vec{\nabla}B(t')]A(t)\rangle]. \tag{19}$$

In equilibrium, the second term vanishes and the FDT is recovered.

2.7. Fluctuation–dissipation ratio

Consider the following adimensional object:

$$X(t,t') \equiv \frac{TR(t,t')}{\partial_{t'}C(t,t')}, \qquad T_{\mathrm{EFF}}(t,t') \equiv \frac{T}{X(t,t')}, \tag{20}$$

where we assumed $t > t'$. When equilibrium is reached, $T_{\mathrm{EFF}} = T$. Out of equilibrium, this ratio can be used to define a, possibly two-time-dependent, effective temperature although the thermodynamic meaning of this quantity is not ensured *a priori*. Note the asymmetry between t and t' in the denominator when the dynamics is not stationary. Alternatively, experiments are usually performed in the frequency domain in which $\chi(t,\omega) = \int_0^t \mathrm{d}t''\, \mathrm{e}^{\mathrm{i}\omega\tau}\, R(t, t-\tau)$

and $C(t,\omega) = \int_0^t d\tau\, e^{i\omega\tau}\, C(t, t-\tau)$ and a time–frequency-dependent effective temperature is defined as

$$X(t,\omega) \equiv \frac{T\chi''(t,\omega)}{\omega \mathrm{Re}\, C(\omega,t)}, \qquad T_{\mathrm{EFF}}(t,\omega) \equiv \frac{T}{X(t,\omega)}. \tag{21}$$

The time-domain and frequency-domain definitions of T_{EFF} are not necessarily equivalent. If this quantity is to have a physical meaning they must coincide under reasonable conditions. We will come back to this point in section 4.3.

The asymptotic values

$$X^{\infty} = \lim_{t'\to\infty} \lim_{t\to\infty} X(t,t'), \qquad T_{\mathrm{EFF}}^{\infty} = T/X^{\infty} \tag{22}$$

(or $\lim_{\omega\to 0}\lim_{t\to\infty} X(t,\omega)$ in the frequency domain) turn out to be useful in the description of critical quenches, see section 5.3.

It is clear that X can be computed under any circumstances but its interpretation as leading to an effective temperature will not always be straightforward. In the rest of this section, we will still use the name T_{EFF}; later in the review, we will discuss when this is justified and when it is not.

2.8. Beyond linear response

Building upon the study of nonlinear responses in Langevin processes [15], Lippiello *et al* presented a unified derivation of out-of-equilibrium FDRs to arbitrary order for discrete (Ising or q-state) and continuous variables ruled by stochastic Markovian dynamics of quite generic type—single flip, Kawasaki, Langevin, etc [25]. These relations are conceptually simple although pretty lengthy to write down. They are derived from a rewriting of the multiple variations of the evolution operator with respect to the perturbation strength. In short, one ends up with a sum of an increasing number of terms that are time derivatives of higher order correlations.

The generic FDRs can also be derived from a *fluctuation principle* that simply uses the detailed balance property of the transition probabilities [15, 18, 25]. The proof goes as follows. One first relates the probability of a path, $\{\phi\} = \phi(t)$, under the effect of a perturbation $h(t)$ and conditioned to the initial value $\phi(t_I) = \phi_I$ to the probability of the reversed path, $\{\phi_R\} = \phi_R(t) = \phi(t_R)$, under the reversed perturbation and conditioned to $\phi_R(t_I) = \phi(t_F) = \phi_F$:

$$P[\{\phi\};\phi_I,\{h\}]\, e^{-\beta\int_{t_I}^{t_F} dt\, h(t)\dot{\phi}(t)} = P[\{\phi_R\};\phi_F,\{h_R\}]\, e^{\beta[H(\phi_I)-H(\phi_F)]}. \tag{23}$$

From the average over all paths weighted with a generic distribution of initial conditions, $P(\phi_I)$, the expansion in powers of $h(t)$ yields all nonlinear FDRs. This route can be followed for discrete and continuous variables as well.

These proofs do not use any equilibrium assumption and the FDRs hold in full generality (for ageing systems, in non-equilibrium steady states, etc). In equilibrium, the use of the Onsager relation and the stationary property allows one to simplify the FDRs considerably. Still, as soon as the linear regime is left, the nonlinear impulse responses are linked to sums of several derivatives of correlations. Interestingly enough, beyond linear order the FDRs depend on the microscopic dynamic rule not only out but also in equilibrium.

The interest in the generalized nonlinear FDRs is at least threefold. First, they allow one to develop efficient algorithms to compute linear and nonlinear responses without applying any perturbation. Second, they can be used to search for growing dynamic correlation lengths in glassy systems, a field of active research [26]. Third, and more importantly for the purposes of this review, they provide a way to further test the consistency of the effective temperature notion that, in the nonlinear relations between responses and correlations, should play essentially the same role as in the linear ones.

To our knowledge, quantum FDRs of this kind have not yet been derived.

3. Insight

Fully connected disordered spin models provide a mean-field description of glassy phenomenology. They capture many features of glassy thermodynamics and dynamics, at least in an approximate way. They have static and dynamic transitions at finite temperatures, T_s and T_d, that do not necessarily coincide. In a family of such models aimed to describe fragile glasses with the random first order transition (RFOT) scenario [27–29] T_s corresponds to an entropy crisis realizing the Kauzmann paradox. Below T_d, an infinite system let evolve from a high-T initial state does not come to equilibrium with the environment [5, 6] and relaxes out of equilibrium. Replica tricks [30] and the Thouless–Anderson–Palmer approach [31] give us a handle to understand equilibrium and metastable states as well as thermodynamic properties. A dynamic treatment completes the picture explaining their relaxation and the relation with the free-energy landscape. Importantly enough, a quasi-complete analytic solution exists and a clear definition of many concepts that are not as sharply grasped in finite-dimensional cases—such as metastable states—can be given. We do not intend to give here a full account of the behaviour of these models; several reviews exist [32, 33] already. We simply highlight some of their properties and their relevance to the effective temperature notion.

3.1. Slow relaxation and convergence

The energy density, as well as any other intensive observable that depends on just one time, approach a *finite* value:

$$-\infty < O_\infty = \lim_{t\to\infty} \lim_{N\to\infty} \langle O(t)\rangle < +\infty \, . \tag{24}$$

With this order of limits, the intensive quantity O_∞ is not necessarily equal to the equilibrium value $O_{\rm EQ}$. Below T_d the approach is achieved slowly, typically with a power law. If time is allowed to scale with N a different dynamic mechanism sets in and equilibrium is eventually reached, $\lim_{N\to\infty} \lim_{t\to\infty} \langle O(t)\rangle = O_{\rm EQ}$.

3.2. Separation of time scales

Below their dynamic critical temperature, T_d, mean-field glassy models relax in several time scales (see [6] for a precise definition). The correlation and linear response admit a decomposition:

$$C(t,t') = C^{(0)}(t,t') + C^{(1)}(t,t') + C^{(2)}(t,t') + \cdots \tag{25}$$

$$R(t,t') = R^{(0)}(t,t') + R^{(1)}(t,t') + R^{(2)}(t,t') + \cdots . \tag{26}$$

The first of these scales, usually labelled ST, remains finite as $t' \to \infty$, while the others diverge with t'. In mean-field models, these time scales become infinitely separated as $t' \to \infty$ due to their different functional dependences on t'. This means that when one of the terms varies, the other ones are either constant or have already decayed to zero. In particular, in the ST scale, the correlation function decays from its equal-times constant value to the Edwards–Anderson parameter $q_{\rm EA}$ that depends on bath temperature and all coupling constants. In this scale, the system follows the rules dictated by the bath and the dynamics are stationary, while in the other scales, interactions are more relevant and the relaxation is more complex. Each term contributing to the correlation scales as

$$C^{(i)}(t,t') \simeq f_C^{(i)}\left(\frac{h^{(i)}(t)}{h^{(i)}(t')}\right), \tag{27}$$

and $h^{(0)}(t) = e^{t/\tau^{(0)}} = h^{\mathrm{ST}}(t)$ with $\tau^{(0)}$ finite and $f_C^{(0)}(\infty) = q_{\mathrm{EA}}$. Loosely, one can define a time scale $\tau^{(i)}$ from

$$\frac{h^{(i)}(t)}{h^{(i)}(t')} \simeq 1 + \frac{t-t'}{\tau^{(i)}(t')} \qquad \text{with} \quad \tau^{(i)}(t') = [d\ln h^{(i)}(t')/\mathrm{d}t']^{-1}. \tag{28}$$

In practice, one can probe the different time scales by increasing the times t and t' along different directions in the t'–t plane.

One of the main ingredients in the solution to mean-field glassy models in relaxation [5, 6] is the fact that equation (13) does not hold below T_d. This does not come as a surprise since the equilibrium condition under which the FDT is proven does not apply. What really comes as a surprise is that the modification of the spontaneous/induced fluctuations relation takes a rather simple form:

$$R^{(i)}(t,t') = \frac{1}{T^{(i)}}\frac{\partial}{\partial t'}C^{(i)}(t,t')\,\theta(t-t') \qquad i = 0, 1, \ldots, \tag{29}$$

in the $t' \to \infty$ limit. The temperature of the external bath is the one of the stationary regime and the conventional FDT holds, $T^{\mathrm{ST}} = T^{(0)} = T$. We discuss the reason for this in section 4.9. In all other regimes, the $T^{(i)}$s are different from each other and from T. If one keeps t' finite, the crossover from one scale to another is smooth and a generic relation like (20) applies. The mere existence of different $T^{(i)}$ values implies, if the interpretation as a temperature is correct, the fact that some degrees of freedom cannot thermalize with others evolving in a different time scale.

So far we highlighted three characteristics: the slow relaxation, the fact that the energy density is bounded from below, and the separation of time scales. They are all fundamental for finding an effective temperature with a thermal significance. We will come back to these properties in section 5.

3.3. The parametric construction

A very instructive way to study the deviations from FDT is to construct a parametric plot of the integrated linear response against the correlation at fixed t_w, varying t between t_w and infinity. In the long t_w limit, such a function converges to a limiting curve:

$$\lim_{\substack{t_w\to\infty \\ C(t,t_w)=C}} \chi(t,t_w) = \int_C^1 \mathrm{d}C' \frac{1}{T_{\mathrm{EFF}}(C')}, \tag{30}$$

where the effective temperature [3], $T_{\mathrm{EFF}}(C)$, is a function of the correlation C. The main aims of this review are to justify the name of this function and to give a state of the art review of its measurements in model systems, numerical studies and experiments (see [11] for a previous survey of measurements of FDT violations with numerical methods and [14] for a recent discussion on effective temperatures from a different perspective). The convenience of this parametrization is that it makes it possible to compare to replica calculations and, on a more physical level, it allows for a better interpretation of fluctuations.

Let us summarize the behaviour of the asymptotic $\chi(C)$ function in different mean-field glassy models relaxing slowly with bounded dynamics after a quench from the disordered into the ordered phases. In all cases, there is a regime of short time differences $t-t_w$ in which $\chi(C)$

is a straight line of slope $-1/T$, joining $(1, 0)$ and $(q_{\rm EA}(T), [1 - q_{\rm EA}(T)]/T)$.[1] A free-energy interpretation of this regime will be given in section 4.9. At this point, the straight line breaks and the subsequent behaviour depends on the model. Three families have been identified.

Models describing domain growth such as, for example, the ferromagnetic $O(N)$ model in d dimensions in the large N limit. There is only one additional time scale in which C decays from $q_{\rm EA}$ to 0. The asymptotic parametric plot for $C \leqslant q_{\rm EA}(T)$ is flat. The susceptibility χ gets stuck at the value $[1 - q_{\rm EA}(T)]/T$ while the correlation C continues to decrease towards zero. The same result holds for the Ohta–Jasnow–Kawasaki approximation to the $\lambda\phi^4$ model of phase separation. In agreement with the fact that temperature is basically an irrelevant parameter in (at least clean) coarsening, the effective temperature diverges in the full low-temperature phase, $T^{(1)} \to \infty$.

Models of RFOT type provide a mean-field-like description of fragile glasses. Examples are the so-called F_{p-1} or type B models of the mode-coupling approach and disordered spin models with interactions among all possible $p \geqslant 3$-uples. These models also have only one additional time scale. In these cases, the χ versus C plot, for $C \leqslant q_{\rm EA}(T)$, is a straight line of negative slope larger than $-1/T$ [5], i.e. $T < T^{(1)} < \infty$. The value of $T^{(1)}$ weakly depends on the working temperature and continuously increases from T_d at the dynamic transition to a slightly higher value at $T = 0$. Another interesting case in this group is a d-dimensional directed manifold embedded in an $(N \to \infty) + d$-dimensional space under the effect of a quenched random potential with short-range correlations. The susceptibility–correlation relations for different wave-vector-dependent observables satisfy $\chi_k(C_k) = \chi_{k=0}(C_{k=0}) = \chi_{x=0}(C_{x=0})$ when times are such that they all evolve in the ageing regime characterized by a single time scale (note that $q_k^{\rm EA}$ depends on k) and, thus, the effective temperature is k-independent [36].

Mean-field spin models with quenched disorder having a genuinely continuous second-order phase transition, for example, the Sherrington–Kirkpatrick spin-glass or type A models of the mode-coupling approach. The random manifold problem with long-range potential correlations is of this type too. In these cases, the dynamics has a continuity of time scales, ordered in an ultrametric fashion. For $C \leqslant q_{\rm EA}(T)$, the $\chi(C)$ plot is a non-trivial curve with local derivative larger than $-1/T$ [6]. Each value of the effective temperature can be ascribed to a dynamic scale. The lowest value appears discontinuously ($T^{(1)} > T_d$) as one crosses T_d and can be shown to decrease with decreasing temperature. The wave-vector independence holds for the $T_{\rm EFF}$ of the random manifold as well [36, 37].

Further support to the notion of effective temperatures comes from the study of the effect of quantum fluctuations on the same family of mean-field models [38, 39]. The setting is one in which the system is in contact with a quantum environment at temperature T and the dynamics are dissipative. Below a critical surface (in the T, strength of quantum fluctuations, coupling to the bath phase diagram) that separates glassy from equilibrium phases, and in the slow dynamics regime, one finds a non-equilibrium relaxation with deviations from the quantum FDT, equation (18). These are characterized by the replacement of the bath temperature by an effective temperature $T_{\rm EFF}$. The effective temperature is again piecewise. It coincides with T when the symmetric correlation C is larger than $q_{\rm EA}$ and $T_{\rm EFF} > T$ when C goes below $q_{\rm EA}$. $\chi(C)$ recovers, in the slow regime, the structure of the classical limit. This is a signal of a time-dependent decoherence effect. The slow modes' $T_{\rm EFF}$ depends on the working parameters and is higher than T even at $T = 0$.

[1] The actual scaling of $t - t_w$ with t_w that marks the end of the regime in which the equilibrium FDT holds is difficult to establish. A power law dependence was found in a few solvable models, see e.g. [34] for the mean-field spherical ferromagnet. The more general scaling argument proposed in [35] also suggests such a dependence.

Numerical simulations, with Monte Carlo techniques or molecular dynamics of classical systems, have demonstrated that the classification can go beyond the mean-field limit at least as a good approximation. We will discuss these tests in section 5.

3.4. Pre-asymptotic effects

Expression (30) is asymptotic in time, a limit in which it is immaterial to construct the parametric plot by keeping t_w fixed and letting t increase to infinity, or by keeping t fixed and letting t_w increase up to t. However, in numerical and experimental measurements one cannot reach this long-time limit and it is important to have the best possible control of pre-asymptotic effects. The most appropriate construction at finite times is the second option above [40]. Indeed, by using the fact that $C(t, t_w)$ is monotonic with respect to t_w, one trades all t_ws by Cs and keeps t as an independent variable. Variations with respect to t_w become variations with respect to C at t fixed:

$$\frac{\partial \chi(t, t_w)}{\partial t_w} = \frac{\partial \chi(t, C)}{\partial C} = -R(t, C) = -\frac{1}{T_{\rm EFF}(t, C)} \frac{\rm d}{{\rm d}C} C = -\frac{1}{T_{\rm EFF}(t, C)}, \tag{31}$$

and the slope of the pre-asymptotic $\chi(t, C)$ plot at fixed t yields the searched pre-asymptotic $T_{\rm EFF}$, as defined in equation (20). Had we worked at fixed t_w, we would have failed to obtain $T_{\rm EFF}$ from the slope of the $\chi(C, t_w)$ plot. This is due to the asymmetry in the times involved in definition (20). The difference disappears in the long t_w and t limits with C varying in the interval [0, 1] or for stationary systems. In cases in which a normalization is needed, the choice of using $C(t, t)$ as the normalization factor does not affect the slope of the plot constructed at t fixed. The FDRs discussed in section 2.4 allow one to use this method with no extra computational effort in numerical simulations.

3.5. Cooling rate dependence and finite size effects

In the long-time dynamics of mean-field-like models, $T_{\rm EFF}$ does not depend permanently upon the cooling procedure. The same models with finite number of degrees of freedom, and more refined ones beyond the mean-field approximation, should capture a cooling rate dependence that should also become manifest in $T_{\rm EFF}$.

The dynamics of models in the RFOT class approach, in the asymptotic limit $t \to \infty$ taken after $N \to \infty$, a region of phase space, named the *threshold* [5], that is higher than equilibrium in the free-energy landscape. Further decay is not possible in finite times with respect to N since diverging barriers separate the former from the latter. This is demonstrated by the fact that the asymptotic values of averaged one-time intensive quantities, such as the energy density, take higher values than in equilibrium. Moreover, in between threshold and equilibrium, a continuous set of metastable states also separated by diverging barriers exist. Subsequent decay is achieved through activated processes in time scales scaling with N, and the relaxation of one-time quantities is expected to cross over from power law to logarithmic.

Although the full dynamic solution in the activated regime has not been derived yet (it is too difficult!), it is reasonable to imagine that the $T_{\rm EFF}$ values should be ordered with the higher on the threshold and the lower, T, in equilibrium. The effective temperature should relax, in logarithmic time scales, from the threshold value to the bath temperature. Coming back to cooling rate dependences, a slower cooling rate takes a finite size system below the threshold level and the deeper it is, the slower the rate. $T_{\rm EFF}$ should follow this variation. These claims found some support in numerical simulations, see section 5.

3.5.1. An equilibrium interlude. The replica theory of the statics of mean-field spin-glasses [30] necessitates the definition of a symmetric disorder-averaged functional order parameter, $P(q)$, that measures the probability distribution of overlaps, $Nq = \sum_{i=1}^{N} s_i\sigma_i$, between equilibrium configurations, $\{s_i\}$ and $\{\sigma_i\}$. The cumulative distribution and a further integral are

$$x(q) = \int_0^q \mathrm{d}q'\, P(q'), \qquad T\aleph(C) = \int_C^1 \mathrm{d}q \int_0^q \mathrm{d}q'\, P(q'). \tag{32}$$

The out-of-equilibrium dynamics take place in a region of phase space that is different from the one where equilibrium states lie. Still the classification of mean-field models according to $T_{\rm EFF}(C)$ (or $X(C)$) and $\chi(C)$ coincides with the one arising from the replica analysis [30] of the number and organization of equilibrium states and their implications on $x(q)$ and $\aleph(C)$ defined in equation (32). Indeed, mean-field coarsening problems have two equilibrium states related by symmetry, $2P(q) = \delta(|q| - q_{\rm EA})$ implying $x(q) = 0$ if $|q| < q_{\rm EA}$ and $x(q) = 1$ if $|q| > q_{\rm EA}$ with $q_{\rm EA}$ the square of the order parameter (replica symmetric—RS—case). Models with a RFOT are solved by a one-step replica symmetry breaking (1RSB) ansatz implying a proliferation of equilibrium states with special properties. Two possibilities exist for any two equilibrium configurations $\{s_i\}$ and $\{\sigma_i\}$: they may fall in the same or the reversed state and $q = \pm q_{\rm EA}$, respectively; or else they may fall in different states that turn out to be orthogonal and $q = 0$. The probability distribution is hence $P(q) = x_1\delta(q) + (1 - x_1)/2\delta(|q| - q_{\rm EA})$ and its integral yields $x(q) = x_1$ if $|q| \leqslant q_{\rm EA}$ and $x(q) = 1$ if $|q| > q_{\rm EA}$. Finally, models of the SK type are solved by a full RSB ansatz, have states with all kinds of overlaps, the $P(q)$ has two delta contributions at $\pm q_{\rm EA}$ and a symmetric continuous part in between. This functional form leads to an $x(q)$ taking the value 1 for $|q| > q_{\rm EA}$ and a continuous function of q for $|q|$ below $q_{\rm EA}$. In all mean-field cases, the functional forms of equilibrium and out-of-equilibrium objects are similar:

$$x(q) \leftrightarrow X(C), \qquad \aleph(C) \leftrightarrow \chi(C). \tag{33}$$

By this we mean that $\aleph$ has an FDT part in all cases and it is linear, with zero, constant and finite, and variable slope, in RS, 1RSB-RFOT and full RSB cases, respectively. In models of RFOT type, the value of the breaking point $q_{\rm EA}$ and the parameter x_1 (the slope) are not the same dynamically and statically while in models of full RSB kind, the coincidence is complete (up to a factor 2 due to the global symmetry). The same applies to all higher moments of the equilibrium overlap q and out-of-equilibrium C distribution functions [5, 6, 37]. The coincidence for full RSB models was argued to apply beyond mean-field, in finite-dimensional disordered spin systems, when the long-time limit is taken after the thermodynamic limit. Details of the reasoning that is based on the assumptions of *stochastic stability* and the convergence of the out-of-equilibrium susceptibilities to the equilibrium ones found in [41].

These ideas were developed in spin models and one would like to extend them to atomic and molecular systems. However, overlaps in continuous particle models are difficult to define in a direct measurable way. Attempts based on weakly coupled real replicas were developed in [42]. This may allow one to extend the equilibrium $\leftrightarrow$ out-of-equilibrium connection to these systems as well.

The relation between static $\aleph(C)$ and asymptotic out-of-equilibrium dynamic $\chi(C)$ could apply in much more generality than previously suspected at the price of identifying finite-time non-equilibrium, $\chi(C, t_w)$, and finite-size equilibrium, $\aleph(C, \xi(t_w))$, with the help of a time-dependent coherence length $\xi(t_w)$ [43].

4. Requirements

In this section, we list a number of conditions that the functional parameter $T_{\rm EFF}$ must satisfy to act as a temperature.

4.1. Thermometer

Any quantity to be defined as a non-equilibrium effective temperature must conform to the folklore. The first requirement is to be *measurable* with a thermometer weakly and statistically coupled to the system [3, 4]. This fact can be proven by studying the time evolution of the thermometer coupled to M identical copies of the system, all of age t_w and evolving independently. The thermometer is ruled by a Langevin equation with a non-Markovian bath with statistics given by the system's correlation and response. It thus feels the system as a complex bath with its time scales, $\tau^{(i)}(t')$, and temperatures, $T^{(i)}$. The $T^{(i)}$ to be recorded is selected by tuning the internal time scale of the thermometer to $\tau^{(i)}(t')$.

Such an experiment can be relatively easily realized in systems of particles in interaction, be them colloidal suspensions or powders. The thermometer can be a probe particle, the free and perturbed dynamics of which is followed in time. Diffusion is measured in free relaxation and mobility in the perturbed case. By comparing the two through an extended Einstein relation $T_{\rm EFF}$ of the medium, that is to say, the system of interest, is measured. Another possibility is to monitor the kinetic energy of the tracer (a quadratic variable) and associate it with the effective temperature of the environment (*via* equipartition). In both cases, by playing with the tracers' parameters, namely their mass, different regimes of relaxation are accessed and the $T^{(i)}$s are measured. Consistency with the fact that the effective temperature should be an intensive variable requires the result to be independent of the shape of the tracers. These experiments have been performed numerically and experimentally yielding very good results in the former and somehow conflicting in the latter. We will discuss them in section 5.

4.2. Role played by external baths

The fluctuation–dissipation ratio of an 'easy to equilibrate' system should acquire the temperature of its external environment. A rather simple though particularly illuminating problem that illustrates this idea is the non-Markov diffusion of a particle in a harmonic potential, simultaneously coupled to two baths, a fast one in equilibrium at temperature $T_B^{(0)} = T^{(0)}$ giving rise to white noise and instantaneous friction, and a slow one with an exponentially decaying memory kernel in equilibrium at temperature $T_B^{(1)} = T^{(1)}$. This example can be taken as a schematic model for an internal degree of freedom in a slowly driven system and, in particular, the dynamics of a (possibly confined) Brownian particle in an out-of-equilibrium medium. If the slow bath is not stationary, it can also be taken as a self-consistent equation for a variable in an ageing system—exact for mean-field disordered spin models. The particle behaves as in equilibrium at $T^{(0)}$ or $T^{(1)}$ depending on which bath it feels. The separation is made sharp by an adequate choice of the bath and potential energy parameters that push apart their own time scales. Similarly, an ageing system with multiple effective temperatures becomes stationary in all time scales with $T_B^{(i)} > T_{\rm EFF}$ and goes on ageing in time scales with $T_B^{(i)} < T_{\rm EFF}$ [4].

4.3. Observable dependences

In cases allowing for a thermodynamic interpretation, the effective temperature should behave as an intensive variable with the same value—partial equilibration—for all observables evolving in the same time scale and interacting strongly enough. A concrete check of this feature was performed within solvable glassy models. Two glasses in contact with a bath in equilibrium at temperature T, each of them with a piecewise $T_{\text{EFF}}(C)$ of the form

$$T_{\text{EFF}}^{(\text{SYST}\,1,2)}(C) = \begin{cases} T & \text{if } C \geqslant q_{\text{EA}}^{(1,2)} \\ T^{(1,2)} & \text{if } C < q_{\text{EA}}^{(1,2)} \end{cases}$$

with $T^{(1)} \neq T^{(2)}$, were chosen. The experiment of setting two observables in contact is reproduced by introducing a small linear coupling between microscopic variables of the two systems. Above a critical (though small) value of the coupling strength, the systems arrange their time scales so as to partially come to equilibrium and the effective temperatures below q_{EA} equalize. Below the critical strength, $T^{(1,2)}$ remain unaltered [4].

The observable independence of the fluctuation–dissipation ratio was investigated in a variety of out-of-equilibrium situations but no clear rationale as to when this holds was found yet. An intriguing proposal was put forward by Martens *et al* who argued that the observable independence and hence the interpretation in terms of T_{EFF} is related to the uniformity of the phase space probability distribution function (pdf) on the hyper-surface of constant energy reached dynamically [21]. These authors validated this idea in a few simple toy models relaxing in a single time scale (glassy systems excluded). Consistently, this proposal applies in mean-field disordered models. Moreover, they proposed that the observable dependence is proportional to the square root of the difference between the Shannon entropy of the dynamic state and the equilibrium one, a conjecture that deserves further investigation.

4.4. Intuitive properties

One may wonder why in all studied undriven systems, there is a two-time regime in which C decays from its equal times value to q_{EA} and FDT holds. This problem admits a physical explanation—thermal fluctuations within domains, rapid vibrations within cages, etc—that we will expose in section 5, a free-energy landscape explanation that we will discuss in section 4.9 but also a formal explanation: there exists a bound—on the difference between left- and right-hand sides of equation (13)—that vanishes in the first regime of relaxation [44].

An intuitive property of T_{EFF} is that it loosely represents the disorder level of the system. This idea translates into T_{EFF} being higher or lower than the working temperature T when the initial state is taken from an equilibrium distribution in the disordered phase or at a lower temperature than the quenching value in e.g. the $2d$ XY model [45] or elastic lines [46]. The case of a quench to a critical point in the Ising universitality class is more subtle. T_{EFF} is higher than the critical temperature T_c for initial conditions in equilibrium above and below T_c, but it takes a higher value in the former than in the latter case [47].

4.5. The fictive temperature

The latter slated property matches the 'fictive temperature', T_f, ideas that date back to the 40s at least [8] and have developed ever since, as explained in [9]. T_f is usually introduced by assuming that when a liquid falls out of equilibrium in the glass transition region its structure gets 'frozen' at a T_f that is higher than T, depends upon the cooling rate and in particular on T, and deep below the transition range approaches T_g. Several definitions in terms of the enthalpy or the thermal expansion coefficient have been given and they do not necessarily coincide. The fictive temperature is hence a phenomenological convenience and acts essentially as a parameter in an out-of-equilibrium 'equation of state'. In contrast, the effective temperature is defined in terms of fluctuations and responses, it can be measured directly, and plays a role that is closer to the thermodynamical one (although not in all possible out-of-equilibrium systems but in a class yet to be defined precisely).

4.6. Fluctuation theorem

The fluctuation theorem concerns the fluctuations of the entropy production rate in the stationary non-equilibrium state of a driven dynamical system [48]. It applies to systems that approach equilibrium when the forcing is switched off. Of interest in the context of glassy systems is to know the fate of the fluctuation theorem if the system evolves out of equilibrium even in the absence of the external drive, and whether T_{EFF} enters its modified version. These questions were addressed by Sellitto who analysed the fluctuations of entropy production in a $3d$ lattice gas model of hard core particles following the Kob–Andersen kinetically constrained rules and kept away from equilibrium by a chemical potential gradient [49], and by Crisanti and Ritort who discussed the interplay between T_{EFF} and a fluctuation theorem on heat exchange between the system and the environment in the random-orthogonal model (a case in the RFOT class) [50]. From a different perspective, closer to the one in [15], the fluctuation theorem and T_{EFF} were studied in [51] within two related problems: mean-field glassy models and a Langevin process with a number of equilibrium thermal baths with different time scales and temperatures as the one discussed in section 4.2. First, it was shown that the work done at frequency ω by conservative and non-conservative forces is weighted by the effective temperature (instead of the temperature of the bath) at the same frequency. The work of the conservative forces produces entropy if the bath is out of equilibrium since the nonlinear interaction couples modes at different frequencies which are at different temperatures, thus producing an energy flow between these modes. Second, it was proven that the entropy production rate satisfies a fluctuation theorem. Third, extensions of the Green–Kubo relations for transport coefficients were derived. Fourth, a feasible way to measure T_{EFF} by exploiting the modified fluctuation theorem was discussed. There is no satisfactory numerical test of these ideas in non-mean-field glassy systems yet but a preliminary study will be discussed in section 5.7 and a more recent analysis on coarsening systems in section 5.2.

4.7. Nonlinear effects

In a series of papers, Hayashi, Sasa *et al* [52] investigated the notion of an effective temperature in the classical non-equilibrium steady state (NESS) focusing on a strongly perturbed $1d$ white-noise Langevin system in which a Brownian particle is subject to a spatially constant driving force f and a periodic potential $U(x)$. These authors extended the definition of T_{EFF} that uses the Einstein relation between diffusion coefficient and differential mobility beyond the linear response regime, $T_{\mathrm{EFF}}(f) \equiv D(f)/\mu(f)$. They addressed the question of the physical significance of T_{EFF} in at least three different ways: by showing that it plays the role of a temperature in a large-scale description; by proving that such an out-of-equilibrium system, used as a thermostat for a Hamiltonian system, is able to transfer its effective temperature as kinetic energy; and by conducting a heat conduction experiment. The results of these tests are consistent with a thermodynamic interpretation.

4.8. Local measurements

The FDT relates the averaged local response function and fluctuations measured in equilibrium at any spatial scale. What should local measurements yield out of equilibrium?

4.8.1. Quenched disorder-induced fluctuations. In systems with quenched random interactions, spatial fluctuations in noise-averaged quantities are dictated by the local disorder.

This fact has been known for decades; for instance, the fast and slow characters of spins in disordered magnets induced by their quenched environment give rise to Griffiths phenomena (free-energy singularities, phases and slow relaxation). In the context of glassy dynamics and T_{EFF}, Montanari and Ricci-Tersenghi [53] showed that in disordered spin models defined on random graphs, spins of two types exist: paramagnetic and glassy ones with the former following fast equilibrium dynamics and the latter having a non-trivial relaxation peculiar to their environment. Strictly local deviations from FDT were characterized by an effective temperature that can be obtained with a replica calculation along the lines discussed in section 3.5.1. Pretty convincing arguments—although not completely rigorous—establish that in these models the strictly local effective temperature must diffuse to become site independent:

$$-\frac{1}{T_{\mathrm{EFF}}(C_i)} = \frac{\partial \chi_i(C_i)}{\partial C_i}. \tag{34}$$

Numerical studies of this kind of (the absence of) T_{EFF} fluctuations in the $3d$ EA model [54] will be presented in section 5.

4.8.2. Noise-induced fluctuations. In a series of papers, Chamon *et al* proposed a theoretical framework based on global time reparametrization invariance that explains the origin of dynamic fluctuations in generic—not necessarily quenched disordered—glassy systems with a separation of time scales of the kind explained in section 3.2 [55–57]. Such type of invariance had been known to exist in the out-of-equilibrium dynamics of mean-field disordered models and it was later shown to carry through to the causal asymptotic dynamics of finite d infinite-size spin-glasses, under the assumption of slow dynamics with a separation of time scales. The approach reviewed in [58] focuses on the fluctuations induced by the noise at coarse-grained length scales. The invariance acquires a physical meaning and it implies that one can easily change the clock $h(t)$ characterizing the scaling of the global correlation and linear response by applying infinitely weak perturbations that couple to the zero mode, or with a noise-induced fluctuation. An illustration of this property is the fact that the ageing relaxation dynamics of glassy systems is rendered stationary by a weak perturbing force that does not derive from a potential while the $\chi(C)$ relation in the slow regime is not much modified [59]. The same argument, applies to the fluctuations, as measured over coarse-grained regions of linear size ℓ with ℓ suitably chosen. It implies that easy fluctuations should be realized as local changes in time, $t \to h_r(t)$, with $h_r(t)$ any monotonically growing function of time associated with the coarse-graining volume centred on r:

$$\frac{h_r(t)}{h_r(t')} \simeq 1 + \frac{(t-t')}{d_{t'} \ln h_r(t')} = 1 + \frac{(t-t')}{\tau_r(t')} \tag{35}$$

that intervene in the local correlation and integrated linear response. Age measures fluctuate from point to point with younger and older pieces coexisting at the same values of the two laboratory times. The two-time coarse-grained observables, i.e. $C_r(t, t_w)$ and $\chi_r(t, t_w)$, have a slow and a fast contribution, the former characterized by scaling functions f_C and f_χ that act as 'massive variables' in the sense that they are not expected to fluctuate in the scaling limit $\delta \ll \ell \ll \xi(t, t')$ with δ the microscopic length scale (lattice-spacing or inter-particle distance), ℓ the coarse-graining length (to be chosen) and ξ the dynamically generated correlation length. Within this picture, the parametric construction $\chi_r(C_r)$ falls on the master curve for the global quantities but could be advanced or retarded with respect to the global value with a uniform effective temperature at fixed C_r value:

$$T_{\mathrm{EFF}\,r}(C_r) = T_{\mathrm{EFF}}(C_r). \tag{36}$$

This equality is non-trivial when the effective temperature is finite and it conforms to the concept that the effective temperature for different observables (the different regions in the sample in this case) must equalize if they evolve in the same time scale (i.e. take the same value of C_r). Time-reparametrization invariance is not expected to hold in cases in which $T_{\rm EFF}$ diverges [60], notably, phase ordering kinetics (see section 5.2). (Subtleties on the measurement of temperature in finite systems are reviewed in [61].)

Equation (36) with $T_{\rm EFF}(C_r)$ finite implies that the two-time variances of composite fields, the averages of which yield the local correlation and linear response, should have the same scaling with times [62]. These objects are easier to measure than the joint pdf of χ_r and C_r. Numerical results will be discussed in section 5.

4.9. Landscapes and thermodynamics

Complex systems' dynamics are sometimes interpreted as the wandering of a representative point in a phase space endowed with a complicated free-energy density landscape. The existence of an equilibrium-like relaxation at short-time differences suggests the distinction between 'transverse' and 'longitudinal' directions in the landscape, with the former being confining and close to, say, 'harmonic' and the latter giving rise to the slow out-of-equilibrium structural relaxation. Questions are posed as to what are these directions, which is the dynamic process taking the system along these directions—diffusive, activated, etc—which is their volume in phase space, and so on.

The appearance of an effective temperature suggests some form of ergodicity and it becomes tempting to relate $T_{\rm EFF}$ to some kind of microcanonic temperature defined from the volume of phase space visited during the out-of-equilibrium excursion. The construction of thermodynamics in which $T_{\rm EFF}$ played a role is the natural step to follow. In this section, we describe the successful development of this program in models of the RFOT sort and how this construction remains a suggestive picture in finite-dimensional cases.

4.9.1. Thermal systems. Describing super-cooled liquids and glasses in terms of *potential energy landscapes* dates back to Goldstein who proposed to think of a system's trajectory in phase space as a succession of steps among potential energy basins [63]. This idea developed into the inherent structure (IS) statistical mechanics framework of Stillinger and collaborators [64]. In this approach, each configuration is mapped onto a local minimum of the potential energy through a minimization process implemented, for example, by a quench to $T = 0$ (steepest descent). The IS is, then, the configuration reached asymptotically and all configurations flowing to it constitute its basin of attraction. Although *a priori* simple, this proposal hides a number of ambiguities such as the fact that the ISs depend on the microscopic dynamics (e.g. single spin flip versus cluster spin flip in a spin system), some decision making is needed in cases in which the $T = 0$ dynamics could follow different directions, etc. The proposal is to re-order and approximate the partition function as a sum over IS energy levels (including their degeneracy) times a β-dependent factor with all contributions from the rest of the configurations—associated with vibrations or the fast relaxation—basically describing the free-energy of the liquid/glass constrained to one typical basin. The assumption is that the IS partition function describes the thermodynamic properties of the state reached at very long times.

Independently, Thouless, Anderson and Palmer (TAP) [31] and de Dominicis and Young [65] showed that the *equilibrium* properties of fully connected spin disordered models can be described with a local order-parameter-dependent *free-energy landscape*. Averaged observables in equilibrium are expressed as weighted sums over the free energies of the TAP free-energy saddle-points:

$$\langle O\rangle_{\rm EQ} = \frac{\sum_\alpha O_\alpha \, {\rm e}^{-\beta F_\alpha}}{\sum_\gamma {\rm e}^{-\beta F_\gamma}} = \frac{\int {\rm d}f \; O(f)\, {\rm e}^{-\beta N[f-T\Sigma(f)]}}{\int {\rm d}f \; {\rm e}^{-\beta N[f-T\Sigma(f)]}} , \tag{37}$$

in agreement with results from replica and cavity methods. The sums in the second member run over all stationary points of the TAP free-energy landscape. They transform into integrals over free energies at the price of introducing the number of stationary points at given f, with $N\Sigma(f) = \ln \mathcal{N}(f)$ the *complexity*, or *configurational entropy* at free-energy density f. This construction is *exact* for mean-field models. In the $N \to \infty$ limit, the integral is evaluated by the saddle-point:

$$\frac{1}{T} = \left.\frac{\partial \Sigma(f)}{\partial f}\right|_{f_{\rm SP}} . \tag{38}$$

In RFOT models, the disordered state dominates above T_d and $f_{\rm EQ} = f_{\rm PM}$; in between T_d and T_s states that are not minima of the TAP free-energy control the integral since their number is sufficiently large, $f_{\rm EQ} = f_{\rm SP} - T\,\Sigma(f_{\rm SP})$; finally, at T_s the configurational entropy vanishes (entropy crisis) and the lowest lying, now glassy, states dominate.

The microcanonical vision of the effective temperature suggests us to check whether below T_d

$$\frac{1}{T_{\rm EFF}} = \left.\frac{\partial \Sigma(f)}{\partial f}\right|_{f_{\rm TH}} , \tag{39}$$

where $T_{\rm EFF}$ is the value of the slow modes effective temperature and $f_{\rm TH}$ is the free energy at the threshold, the level reached dynamically by an infinite system after a quench from high-T. This is indeed the case in RFOT-type models in the thermodynamic limit. Systems with large but finite size relax below the threshold and slowly approach equilibrium. Nieuwenhuizen conjectured that in these cases both $T_{\rm EFF}$ and $\Sigma(f)$ acquire time-dependence in such a way that equation (39) remains valid with $\Sigma(f,t)$ the complexity of the TAP states that are relevant at time t [66]. Moreover, in finite-dimensional systems with short-range interactions, barrier heights and lifetimes are finite at finite temperature and metastability becomes a matter of time scales. A recipe to compute $\Sigma(f,t)$ in these cases was given in [67].

Once a separation of modes into fast (vibrational) and slow (structural) is made in either an approximate (in finite-dimensional models with short-range interactions) or exact (in mean-field cases) way, thermodynamic potentials that involve $T_{\rm EFF}$ can be easily constructed and thermodynamic relations derived. This has been done in the IS formalism [64] and in a framework that is closer to the TAP one [66].

The inadequacy of the IS approach to describe the dynamics of coarsening and kinetically constrained models has been explained in [68]. Its limits of applicability in molecular glasses were also discussed. Nevertheless, since it is not evident how to access a free-energy landscape concretely, numerical efforts have focused on the characterization of the potential energy landscape. While increased computational facilities gave access to an exhaustive enumeration of ISs in small clusters and proteins, the calculations remain incomplete for macroscopic systems. A connection between $T_{\rm EFF}$ and the IS complexity that has to be taken with the caveats mentioned above, was discussed in [69], see section 5.

4.9.2. Athermal systems: Edwards' ensemble. The constituents of thermal systems exchange energy with the components of their environment and this exchange has an effect on their motion. The constituents of *athermal systems* are much larger than the ones of their surroundings and the energy received from the bath is irrelevant. Dissipation occurs *via* energy flow from the particles to internal degrees of freedom that are excluded from the description. Granular matter is the prototype.

In spite of the very different microscopic dynamics, the meso-/macroscopic dynamics of gently perturbed dense granular matter share many points in common with the ones of more conventional glassy systems. A detailed description of the experiments demonstrating these facts is given in [70].

In search for a statistical mechanics description of these systems, Edwards [2] proposed to use the volume, V, as the macroscopic conserved quantity, and the blocked states—defined as those in which every particle is unable to move—as the set of relevant equiprobable configurations, $P(\mathcal{C}, V) = \Omega^{-1}(V)\theta(\mathcal{C})\delta(\mathcal{V}(\mathcal{C}) - V))$, where $\theta(\mathcal{C})$ is an indicator function that equals 1 if the configuration is blocked and 0 otherwise, $\mathcal{V}(\mathcal{C})$ is the volume as a function of the configuration, and $\Omega(V) = \int d\mathcal{C}\, P(\mathcal{C}, V)$ is the volume in configuration space occupied by the blocked states. Entropy and compactivity are next defined as

$$S(V) = -\sum_{\mathcal{C}} P(\mathcal{C}, V) \ln P(\mathcal{C}, V) = \ln \Omega(V), \qquad \frac{1}{X_{\mathrm{EDW}}} = \frac{\partial S(V)}{\partial V}, \tag{40}$$

respectively. A strong hypothesis in this description is that all blocked configurations are treated on equal footing and that any distinction of dynamic origin is disregarded. Moreover, the number of conserved macroscopic variables needed to correctly describe the system is not obvious *a priori*. If the grain configurations are also characterized by an energy, one should then enlarge the description and define

$$\begin{aligned} &P(\mathcal{C}, V, E) = \Omega^{-1}(V, E)\theta(\mathcal{C})\delta(\mathcal{V}(\mathcal{C}) - V))\delta(H(\mathcal{C}) - E))\,, \\ &S(V, E) = -\sum_{\mathcal{C}} P(\mathcal{C}, V, E) \ln P(\mathcal{C}, V, E), \qquad \frac{1}{T_{\mathrm{EDW}}} = \frac{\partial S(V, E)}{\partial E}. \end{aligned} \tag{41}$$

The similarity between the entropy of blocked states and the zero-temperature complexity of glass theory is rather obvious. Indeed, the first successful check of Edwards' hypothesis was achieved in RFOT models at $T \simeq 0$ [7]. In these cases, the energy is the relevant macroscopic variable. One identifies all energy minima (the blocked configurations in a gradient descent dynamics), calculates $1/T_{\mathrm{EDW}}$ and shows that it coincides with $T_{\mathrm{EFF}} = T^{(1)}$ in the slow ageing regime ($C \leqslant q_{\mathrm{EA}}$) [5]. Moreover, all intensive observables are given by their flat average over the threshold level $e_{\mathrm{TH}} = \lim_{t\to\infty} \lim_{N\to\infty} e(t)$. At finite T, the connection can be extended at the expense of using the free energy instead of the energy, as explained in section 4.9.1.

Numerical tests in finite-dimensional kinetically constrained lattice gases [71], microscopic models of sheared granular matter including some of the subtleties of frictional forces [72], spin-glasses with athermal driving between blocked states [73] and a particle deposition model [74] gave positive results. In all these cases, Edwards' measure is able to correctly reproduce the sampling of the phase space generated by the out-of-equilibrium dynamics. Nevertheless, this description does not apply to every problem with some kind of slow dynamics. In [71], the counterexample is the domain growth of a $3d$ random field Ising model, a case in which the properties of a long-time configuration of (low) energy is not well reproduced by the typical blocked, by domain-wall pinning by disorder, configuration of the same energy. In [75], instead, the analytically solvable one-spin flip dynamics of the $1d$ Ising chain is used to display quantitative and qualitative discrepancies between the dynamic treatment and the averaging over an *a priori* probability measure of Edwards type (and refinements). At the mean-field level, the SK model and the like do not admit a simple relation between configurational entropy and T_{EFF} either [76].

A careful account of the experimental subtleties involved in trying to put Edwards' hypothesis to the test, and eventually verifying whether $T_{\mathrm{EDW}} = T_{\mathrm{EFF}}$, is given in [70]. The question remains open especially due to the difficulty in identifying the relevant extensive and intensive thermodynamic parameters [77]. All in all, the approach, very close to the IS and TAP constructions, is intriguing although not justified from first principles yet and its limits of validity remain to be set.

5. Measurements

In this section, we discuss measurements of FDT violations and tests of the effective temperature notion in a variety of physical systems out of equilibrium. Since we cannot make the description exhaustive, we simply select a number of representative cases that we hope will give a correct idea of the level of development reached in the field.

5.1. Diffusion

The dynamics of a particle in a potential and subject to a complex environment (coloured noise or baths with several time scales and temperatures) not only has a pedagogical interest but also admits an experimental realization in the form of Brownian particles immersed in, e.g., colloidal suspensions and controlled by optical tweezers.

A particle coupled to a bath in equilibrium at temperature T with noise–noise correlations of type $\langle \xi(t)\xi(t')\rangle \propto (t-t')^{-a-1}$, $0 < a < 2$, and under no external forces, performs normal or anomalous diffusion depending on a. The fluctuation–dissipation ratio, equation (20), for $t \geqslant t'$ is [78]

$$X_{xx}(t,t') = \frac{TR_{xx}(t,t')}{\partial_{t'}C_{xx}(t,t')} = \frac{D(t-t')}{D(t-t')+D(t')} , \tag{42}$$

with the diffusion coefficient $D(t) \equiv 1/2\, \mathrm{d}\langle x^2(t)\rangle/\mathrm{d}t \simeq t^a$ for $a \neq 1$ and $D(t) = \mathrm{ct}$ for $a = 1$. In such coloured noise cases, X_{xx} is a non-trivial function of time and it does not seem to admit a thermodynamic interpretation. Still, for later reference, we consider the long-time limit

$$X^\infty = \lim_{t'\to\infty}\lim_{t\to\infty} X_{xx}(t,t') \simeq \begin{cases} 0 & a<1 \quad \text{subohmic,} \\ 1/2 & a=1 \quad \text{ohmic,} \\ 1 & a>1 \quad \text{superohmic.} \end{cases}$$

Another illustrative example is the non-Markovian diffusion of a particle in a harmonic potential and subject to different external baths [4, 51, 79]. As already explained in section 4.2, this simple system allows one to show how different environments can impose their temperatures on different dynamic regimes felt by the particle. Tests of other definitions of out-of-equilibrium temperatures in this simple case confirmed that the one that appears to have the most sensible behaviour is the one stemming from the long-time limit of the relations between induced and spontaneous fluctuations [79]. All other definitions yield results that are more difficult to rationalize: in most cases, one simply finds the temperature of the fast bath and in some cases, as with the static limit in [80], one incorrectly mixes different time regimes even when their time scales are well separated.

5.2. Coarsening

When a system is taken across a second order phase transition into an ordered phase with, say, two equilibrium states related by symmetry, it tends to order locally in each of the two but, globally, it remains disordered. As time elapses, the ordered regions grow and the system reaches a scaling regime in which time dependences enter only through a typical growing length, $L(t)$. Finite-dimensional coarsening systems have been studied in great detail from the effective temperature perspective (see [13]). In this context, it is imperative to distinguish cases with a finite temperature phase transition and spontaneous symmetry breaking from those with ordered equilibrium at $T = 0$ only. Some representative examples of the former are the clean or dirty $2d$ Ising model with conserved and non-conserved order parameter. An instance of the latter is the Glauber–Ising chain and we postpone its discussion to section 5.4.

Let us focus on scalar systems with discrete broken symmetry. When time differences are short with respect to a function—typically algebraic—of the typical growing length $L(t_w)$, domain walls remain basically static and the only variation is due to thermal fluctuations on the walls and, more importantly, within the domains. This regime is stationary, and induced and spontaneous fluctuations are linked by the FDT. At longer time differences, domain walls move and observables display the out-of-equilibrium character of the system.

The correlation and total susceptibility in the $t_w \to \infty$ limit separate into two contributions $C(t, t_w) = C^{\rm ST}(t - t_w) + C^{(1)}(t, t_w)$ and $\chi(t, t_w) = \chi^{\rm ST}(t - t_w) + \chi^{(1)}(t, t_w)$. Numerical studies of $T_{\rm EFF}$ focused on the parametric construction $\chi(C, t_w)$ at fixed and finite t_w where the chosen observable is the spin itself. The resulting plot has a linear piece with slope $-1/T$, as in equation (13), that goes below $C = q_{\rm EA} = m^2$ and, consistently, beyond $\chi = [1 - m^2]/T$. The additional equilibrium contribution is due to the equilibrium response of the domain walls that exist with finite density at any finite t_w. In the truly asymptotic limit, their density vanishes and their contribution disappears. Consequently, $\lim_{t_w\to\infty} \chi(C, t_w) = C^{\rm ST} \geqslant q_{\rm EA}$ satisfies FDT and it is entirely due to fluctuations within the domains. In cases with $L(t) \simeq t^{1/z_d}$, the slow terms take the scaling forms

$$C^{(1)}(t, t_w) \simeq f_C(t/t_w) , \qquad \chi^{(1)}(t, t_w) \simeq t_w^{-a_\chi} f_\chi(t/t_w). \tag{43}$$

It would be natural to assume that $\chi^{(1)}(t, t_w)$ is proportional to the density of defects $\rho_d(t) \simeq L(t)^{-n} \simeq t^{-n/z_d}$ with $n = 1$ for scalar and $n = 2$ for vector order parameter. Although this seems plausible, a_χ is instead d-dependent. Another conjecture is [13]

$$z_d\, a_\chi = \begin{cases} n\,(d - d_L)/(d_U - d_L) & d < d_U, \\ n \quad (\text{with } \ln \text{corrections}) & d = d_U, \\ n & d > d_U. \end{cases} \tag{44}$$

d_L is the dimension at which a_χ vanishes and may coincide with the lower critical dimension. One finds $d_L = 1$ in the Ising model, $d_L = 1$ in the Gaussian approximation of Ohta, Jasnow and Kawasaki, and $d_L = 2$ in the $O(N)$ model in the large N limit. d_U is the dimension at which a_χ becomes d-independent and it does not necessarily coincide with the upper critical dimension. One finds $d_U = 3$ in the Ising model, $d_U = 2$ in the Gaussian approximation, and $d_U = 4$ in the large N $O(N)$ model. It was then suggested that d_U might be the highest d at which interfaces roughen. In all cases in which $a_\chi > 0$, $T_{\rm EFF} \to \infty$. This result was confirmed by studies of second order FDRs in the $2d$ Ising model that showed the existence of stationary contributions verifying the nonlinear equilibrium relation and ageing terms that satisfy scaling and yield $T_{\rm EFF} \to \infty$ as in the linear case [25]. The approach by Henkel *et al* based on the conjecture that the response function transforms covariantly under the group of local scale transformations fixes the form of the scaling function f_χ but not the exponent a_χ [81] and does not make predictions on $T_{\rm EFF}$. The coincidence between statics and dynamics, see section 3.5.1, holds in these cases [13].

Noise-induced spatial fluctuations in the effective temperature of clean coarsening systems were analysed in the large N $O(N)$ model with $d > 2$ [60] and with numerical simulations [82]. The first study shows that time-reparametrization invariance is not realized and that $T_{\rm EFF}$ is trivially non-fluctuating in this quasi-quadratic model. The second analysis presents a conjecture on the behaviour of the average over local (coarse-grained) susceptibility at fixed local (coarse-grained) correlation that consistently vanishes in coarsening (but is more interesting in critical dynamics as we will discuss in section 5.3).

Very recently, the analysis of heat exchanges between fast (bulk) and slow (interfacial) degrees of freedom in a finite-dimensional coarsening Ising magnet [83] gave further support to a scenario whereby the fast modes act as an equilibrium reservoir to which interfaces release heat in a way that ensures their statistical independence.

A driven phase separating model with N non-conserved order parameter (in the large N limit) was analysed in [84]. In this model, ageing is not interrupted by the drive. In the fast regime in which the relaxation is controlled by the bath temperature, the drive generates a non-trivial $\chi(C)$ curve. Instead, in the slow regime of domain wall motion, an infinite effective temperature persists under the drive.

The results gathered so far and summarized in the conjecture (44) imply that the FD ratio vanishes and thus $T_{\rm EFF}$ diverges in quenches into the ordered phase of systems above their lower critical dimension.

5.3. Critical dynamics

The non-equilibrium dynamics following a quench from the disordered state to the critical point consists in the growth of the dynamical correlation length, $\xi(t) \simeq t^{1/z_{\rm eq}}$. This length does not characterize the size of well-defined domains but the size of a self-similar structure of domains within domains, typical of equilibrium at the critical point. A continuum of *finite* time scales associated with different wave-vectors, $\tau^{(k)} \simeq k^{-z_{\rm eq}}$, exists with only the $k \to 0$ diverging. At any finite time t, critical fluctuations of large wave-vectors, $k\xi(t) \gg 1$, are in almost equilibrium, while those with small wave-vectors, $k\xi(t) \ll 1$, retain the non-equilibrium character of the initial condition. This *finite-time* separation, and the fact that the order parameter vanishes, leads to the *multiplicative* scaling forms

$$C(t,t_w) \simeq \xi(t-t_w)^{-d+2-\eta}\, f_C[\xi(t)/\xi(t_w), \xi_0/\xi(t_w)]\,,$$
$$\chi(t,t_w) \simeq \beta - \xi(t-t_w)^{-d+2-\eta}\, f_\chi[\xi(t)/\xi(t_w), \xi_0/\xi(t_w)]\,,$$

with the microscopic length ξ_0 ensuring the normalization of the correlation and the fact that χ vanishes at equal times. These forms imply that beyond the initial equilibrium part, the $\chi(C)$ plot assumes a non-trivial shape that, however, progressively disappears and approaches the equilibrium linear form at all $C > 0$. The limit $C = 0$ is distinct and the limiting parameter X^∞ should be non-trivial and it was conjectured to be universal in the sense of the renormalization group [85]. Whether this one can be interpreted as a temperature is a different issue that has been only partially discussed. For this reason, we keep the notation X^∞ (instead of $T_{\rm EFF}$) in most of this section.

The correct estimation of X^∞ has to take into account that the number of out-of-equilibrium modes decreases in the course of time (contrary to what happens in the random manifold problem in the large N limit, for example). The best determination of X^∞ is achieved by selecting the $k \to 0$ mode. A thorough review of the properties of X^∞ found with the perturbative field-theoretical approach and some exact solutions to simple models, as well as the comparison to numerical estimates, is given in [12]. At the Gaussian level, the X^∞ of local operators (e.g. powers of the field, first derivatives of the field, etc) is independent of the chosen pair—but recalls certain features of the initial condition [47] and the correlations of the environment [86]. This is not the case for non-local operators as, e.g., the energy or the tensor. Moreover, when fluctuations are taken into account with, e.g., a two-loop or first order in $\epsilon = 4 - d$ expansion, the X^∞ of local operators is found to depend upon the observables [87].

In the scalar model, using the field itself as the observable, one finds the diffusive results, equation (43), at the Gaussian level and corrections when higher orders are taken into account. For example $X^\infty = 0.30(5)$ in $d = 2$, $X^\infty = 0.429(6)$ in $d = 3$ for a quench from a disordered state, white noise and up to second order in $4 - d$. The trend of X^∞ increasing with d was found in other models too. Instead, $X^\infty \simeq 0.78$ ($d = 3$) and $X^\infty = 0.75$ ($d = 2$) if the initial state is magnetized. A larger X^∞ implies a lower $T^\infty_{\rm EFF} = T_c/X^\infty$ and the comparison between these values conforms to the intuitive idea that an ordered initial state leads to a lower effective temperature than a disordered one. However, note that as $X^\infty < 1$, $T^\infty_{\rm EFF}$ is higher than the working temperature, T_c, even for the ordered initial state. X^∞ was found to increase with N in vector models.

A different type of critical phenomena (infinite order) arise in the $2d$ XY model. The magnetic order parameter vanishes at all T but there is a low-T critical phase with quasi-long-range order (power-law decaying spatial correlations) that is destroyed at $T_{\rm KT}$ where vortices proliferate and restore a finite correlation length. Out of equilibrium, the critical scaling forms apply although with a temperature-dependent exponent, $\eta(T)$, and a growing length scale $\xi(t) \simeq (t/\ln t)^{1/2}$ (the logarithm is due to the effect of vortices). The role of the EA order parameter is played by the asymptotically vanishing function $(t_w/\ln t_w)^{-\eta(T)/2}$ and the crossover between equilibrium and out-of-equilibrium regimes takes place at a t_w-dependent value of the correlation. The $\chi(C, t_w)$ plot at finite t_w is curved, it does not reach a non-trivial master curve for $t_w \to \infty$, but $T_{\rm EFF}(t, t_w) = f_X[\xi(t)/\xi(t_w)]$. Quenches from the disordered phase, $T_0 > T_{\rm KT}$ and heating from a $T_0 = 0$ ground state to $T < T_{\rm KT}$ demonstrate that the slow modes' $T_{\rm EFF}$ depends on the initial state and it is higher (lower) when $T_0 > T$ ($T_0 < T$) [45]. We allow ourselves to use the name $T_{\rm EFF}$ in this case since these results point in the direction of justifying its thermodynamic meaning. Similar results were obtained for $1 + 1$ elastic manifolds with and without quenched disorder (see [46] and references therein). As the dynamic–static link is concerned, Berthier *et al* evinced that the extension to finite-times finite-sizes works, at least at not too high-Ts where free vortices inherited from the initial condition are still present.

The exact calculation of the joint probability distribution of the finite-size correlation and linear response in the spherical ferromagnet quenched to its critical temperature was given in [88]. The results prove that these fluctuations are not linked in a manner akin to the relation between the averaged quantities, as proposed in [58], see section 4.8.2, for glassy dynamics. The correlation-susceptibility fluctuations in non-disordered finite-dimensional ferromagnets quenched to the critical point were examined in [82] where it was shown that the restricted average of the susceptibility, at fixed value of the two-time overlap between system configurations, obeys a scaling form. Within the numerical accuracy of the study, the slope of the scaling function yields, in the asymptotic limit of mostly separated times, the value X^∞.

The first experiments testing fluctuation–dissipation deviations in a liquid crystal quenched to its critical point appeared recently and the results are consistent with what has been discussed above [89].

The coexistence of a single time scale in the ageing regime together with a smooth and time-dependent $\chi(C, t_w)$ plot arises naturally in a critical regime and it is due to the lack of sharp time-scale separation.

Although many evaluations of X^∞ in a myriad of models tend to confirm that it mostly behaves as a critical property [12], the thermodynamic nature of this parameter has not been explored in full extent yet. Measurements with thermometers and connections to microcanonical definitions have not been performed at critical points.

5.4. Quenches to the lower critical dimension

The kinetic Glauber–Ising spin chain is the protoype of a dynamic model at its lower critical dimension. Taking advantage of the fact that this is one of the very few exactly solvable models of non-equilibrium statistical mechanics, several issues concerning the effective temperature interpretation have been addressed in this case, notably the observable dependences.

After a quench from $T_0 \to \infty$ to $T = 0$, the factor $X_s(t, t')$, associated with the spin correlation and susceptibility, is smaller than or equal to 1 and its value X_s^∞ in the limit $C_s \to 0$ evolves smoothly from 1/2 (as in models characterized by simple diffusion such as the random walk or the Gaussian model [17]) to 1 (equilibrium) as $t/\tau_{\rm eq}$ grows from 0 to ∞ ($1/\tau_{\rm eq} = 1 - \tanh(2J/T)$ is the smallest eigenvalue of the master equation operator). Moreover, X_s is an exclusive function of the auto-correlation C_s as in more complex instances of glassy behaviour [91].

The value for the long-wavelength analogue, the fluctuating magnetization, X_m^∞, is identical to the local value X_s^∞. The physical origin of the local-global correspondence, which can also be obtained by field-theoretic arguments [12], is that the long-wavelength Fourier components dominate the long-time behaviour of both quantities. In contrast, observables that are sensitive to the domain wall motion have $X_d^\infty = 0$ [90], the difference residing on the interplay between criticality and coarsening, a peculiar feature of models with $T_c = 0$ [13, 90].

The dependence on the initial condition is also interesting. A non-zero initial magnetization does not change the value of X_s^∞ at $T = 0$. Instead, demagnetized initial conditions with strong correlations between spins so that only a finite number of domain walls exist in the system yield $X_s^\infty = 0$ (the same result is found in the spherical ferromagnet) [40]. The deviations from nonlinear FDTs have not been fully analysed yet.

The static–dynamic connection [41] described in section 3.5.1 does not hold in the $1d$ Ising chain [13] and the non-trivial $\chi(C)$ cannot be used to infer the properties of the equilibrium state. Indeed, the ageing part of the response is finite asymptotically while the equilibrium $P(q)$ has a double-delta (RS) structure as in higher dimensions. The reason for the failure is that the hypotheses used to derive the connection are not fulfilled.

The large N $O(N)$ model in $d = 2$ shares many common features with the phenomenology described above [60] although it has not been studied in as much detail.

To sum up, a quench to $T = 0$ at the lower critical dimension does not seem to be the dimensional continuation of a line of critical quenches in the (T, d) plane (as often implicitly assumed), but the continuation of a line of $T = 0$ quenches: the system behaves as in the coarsening regime, although $X^\infty \neq 0$ for observables that do not focus on the domain wall dynamics [13].

5.5. Relaxation in structural glasses

In particle glassy systems, a separation of time scales exists although it is not as sharp as in mean-field models or coarsening systems, at least within simulational and experimental time scales. In atomic glasses, the existence of an FDT part implies that the rapid particle vibrations within the cages occur in equilibrium while the structural relaxation is of a different out-of-equilibrium kind, and it is not necessarily ruled by the temperature of the bath. Tests of the thermodynamic origin of fluctuation–dissipation violations in the ageing regime of these systems were carried through in much greater detail and we summarize them below.

5.5.1. Simulations of microscopic models. Mono-atomic and binary Lennard–Jones mixtures, soft sphere systems and the BKS potential for silica are standard models for glass-forming liquids. Both Monte Carlo and molecular dynamics simulations [92–95] suggest that the three first cases belong to the RFOT class of systems defined in section 3.3 with $T_{\rm EFF} = T^{(1)}$ constant in the ageing regime. $T^{(1)}$ depends weakly on the bath temperature and systems' parameters but it does not on the preparation protocol as demonstrated by measurements after quenches and crunches [93] or the microscopic dynamics [94]. Tests of partial equilibration between fluctuations at different wave-vectors gave positive results [92]. Importantly enough, these models have a well-defined equilibrium behaviour and their energy density is naturally bounded. Of special interest is the numerical method devised to compute linear responses in molecular systems with high precision that allowed Berthier to resolve the paradoxical behaviour previously reported for silica [95].

Numerical evidence for a slow decrease in time of the configurational temperature, as defined in equation (39), although with the IS complexity, is in agreement with the idea of the system's representative point penetrating below the threshold in the (free)-energy landscape [69].

The ratchet effect of an asymmetric intruder in an ageing glass was studied numerically in [96]. The energy flowing from slow to fast modes is rectified to produce directed motion. The (sub)velocity of the intruder grows monotonically with T_{EFF}/T and this current could be used to measure T_{EFF}.

5.5.2. Kinetically constrained models. Kinetically constrained models are toy models of the glassy phenomenon [97, 98]. Their equilibrium measure is just the Boltzmann factor of independent variables and correlations only reflect the hard core constraint. Still, many dynamic properties of glass-forming liquids and glasses are captured by these models, due to the sluggishness introduced by the constrained dynamic rules. The literature on kinetically constrained models is vast; a recent review with tests of T_{EFF} is [98]. After the successful study of T_{EFF} in the Kob–Andersen model [99], non-monotonic low-temperature response functions found in various spin-facilitated models were taken as evidence against the existence of effective temperatures in these systems. The confusion arose from the incorrect construction of the $\chi(C)$ plot by using t_w instead of t fixed (see section 3.4) that led to the incorrect treatment of the transient regime. Still, even taking this into account, a large number of observables have negative fluctuation–dissipation ratios; this might be related to the fact that these models do not have a proper thermodynamics.

5.5.3. Experiments. Grigera and Israeloff were the first to measure FDT violations in glasses by comparing dielectric susceptibility and polarization noise in glycerol at $T = 179.8$ K, i.e. relatively close to $T_g \simeq 196$ K [100]. At fixed measuring frequency $\omega \simeq 8$ Hz, they found an effective temperature that slowly diminishes from $T_{\mathrm{EFF}} \simeq 185$ K to roughly 180 K in 10^5 s, that is to say in the order of days! This pioneering experiment in such a traditional glass former has not had a sequel yet.

Particle-tracking experiments in a colloidal suspension of PMMA particles revealed an effective temperature of the order of double the ambient one from the mobility—diffusivity relation [101].

In the soft matter realm, an aqueous suspension of clay, Laponite RG, in its colloidal glass phase is a favourite. During ageing, because of electrostatic attraction and repulsion, Laponite particles form a house-of-cards-like structure. After a number of rather confusing reports, the status of T_{EFF} in this system can be summarized as follows. The surprisingly high-T_{EFF} found with dielectric spectroscopy combined with spontaneous polarization noise measurements was later ascribed to violent and intermittent events possibly linked to the presence of ions in the solution which may be the actual source of FDT violation. For the moment, dielectric degrees of freedom are invalidated as a good test ground for T_{EFF} in this sample [102]. Using other methods, several groups found that T_{EFF} detaches from the bath temperature. Strachan *et al* [103] measured the diffusion of immersed probe particles of different sizes via dynamic light scattering and simultaneous rheological experiments and found a slightly higher T_{EFF} than T. With micro-rheology, Abou and Gallet observed that T_{EFF} increases in time from T to a maximum and then decreases back to T [104]. Using a passive micro-rheology technique and extracting T_{EFF} from the energy of the probe particle *via* equipartition, Greinert *et al* also observed that T_{EFF} increases in time [105]. In parallel, a series of global mechanical tests, and passive and active micro-rheological measurements that monitor the displacement and mobility of probe Brownian particles were performed by Ciliberto's and Bonn's groups, both finding no violation of FDT over a relatively wide frequency range [102, 106, 107]. In a very detailed article, Jop *et al* explain many subtleties in the experimental techniques employed and, especially, the data analysis used to extract T_{EFF} that could have biased the results quoted above. A plausible reason for the lack of out-of-equilibrium signal in some experiments using Laponite as well as other colloidal glasses is that the range of frequency–time explored may not enter the ageing regime. Moreover, none of these works studied the degrees of freedom of the Laponite disks themselves but, instead, the properties of the solvent molecules or probe particles. More recently, Maggi *et al* combined dynamic light scattering measurements of the correlation function of the colloid rotations with those of the refringence response [108] and a $\chi(C, t_w)$ plot that is rather constant as a function of C and slowly recovers the equilibrium form as the arrested phase is approached (t_w ranges from 90 to 1200 min and the violations are observed for time differences between 0.1 and 1 ms, i.e. frequencies between 10 and 1 kHz). T_{EFF} is at most a factor of 5 larger than T. The actual behaviour of Laponite remains mysterious—and not only where T_{EFF} is concerned!

Oukris and Israeloff measured local dielectric response and polarization noise in polyvinyl acetate with electric-force microscopy [109]. They probed long-lived nanoscale fluctuations just below T_g, achieved a good signal-to-noise ratio down to very low frequencies, constructed a parametric plot by keeping t_w fixed and found a non-trivial asymptotic form with no t_w dependence within the available accuracy. The data combine into the parametric plot $T_{\mathrm{EFF}}(C) \simeq TC^{-0.57}$ in the ageing regime.

5.6. Relaxation in frustrated magnetic systems

Disordered and frustrated magnets behave collectively at low temperatures and developed ordered phases that although not fully understood are accepted to exist. As macroscopic glassy systems, they present a separation of time scales in their low-temperature dynamics and are good candidates to admit a thermodynamic interpretation of the FDT violations.

5.6.1. Remarks on model systems. The physics of spin-glasses is a controversial subject. Some authors push an Ising domain-growth interpretation of their dynamics—slowed down by domain wall pinning by disorder—also known as the *droplet picture* [110]. If the scheme discussed in section 5.2 were reproduced under strong disorder, the asymptotic $\chi(C)$ plot would have a linear piece of slope $-1/T$ and a sharp transition at q_{EA} to a flat ageing piece. The domain-growth interpretation is not accepted by other authors and more complex scenarios based on the static [30] and dynamic [6] solutions to the SK model are envisaged, with a non-trivial $\chi(C)$ as a result. Much effort has been put in trying to interpret numerical and experimental data as validating one description at the expense of the other. Unfortunately, it is very difficult to distinguish between the two. A third possibility is that, in a loose sense, the spin-glass be like the low-T phase in the $2d$ XY model, with quasi-long-range order. Yet another proposal is that actual spin-glass samples are of Heisenberg-type and that chirality might be decoupled from spin with a chiral-glass order arriving at a higher critical temperature than the spin-glass ordering [111].

Trap models [112] were devised to describe slow dynamics in systems with weak ergodicity breaking. The simplest version comprises an ensemble of uncoupled particles exploring a spatially unstructured landscape of (free) energy traps by thermal activation at a given thermodynamic temperature. For the specific choice of an exponential prior distribution of trap depths, the model shows a glass transition at the inverse decay constant T_g below which an equilibrium Boltzmann state cannot exist. This approach was applied, notably, to describe experiments in spin-glasses. In trap models with energetic or even entropic barriers, the $\chi(C)$ was found to have a slope that varies continuously (with $X^\infty = 0$) even though there is a single scaling of relaxation times with age, it depends non-trivially on the observable and one cannot use it to define a meaningful $T_{\rm EFF}$ [40]. The reason for this failure seems to be the unbounded nature of the energy and the fact that an equilibrium distribution does not exist below T_g. Still, Fielding and Sollich showed that *neutral* observables, that is to say those uncorrelated with the averaged energy, share the same non-trivial $\chi(C)$ relation.

5.6.2. Simulations. Monte Carlo simulations of the $3d$ Edwards–Anderson (EA) model were carried out by several groups. One of the hallmarks of the dynamics of the SK model, dynamic ultrametricity [6], is absent from all numerical and experimental data analysed so far. Magnetic correlation and susceptibility relax in two scales, the by now usual stationary one for finite time differences and an ageing one in which the data are well described by a simple t/t_w scaling. This ageing scaling does not conform with the droplet picture either, which predicts an asymptotic $\ln t/\ln t_w$ form. In all studies so far, the parametric plot was constructed by keeping t_w fixed and the curves drift towards increasing values of χ for longer t_ws as in a transient or critical system. In simple coarsening problems, the drift with increasing t_w goes in the opposite direction of rendering the ageing part of the curves flatter; this remark suggests us to discard a simple droplet picture. The outcome $\chi(C)$ found for the longest t_w reached was interpreted as being non-constant [113]—as in the SK model—although this is, in our opinion, not that clear from the data that could be described by a straight line. The simultaneous t/t_w scaling, the lack of unambiguous evidence for a stable plateau at $q_{\rm EA}$, and a curved $\chi(C)$ in the ageing regime are not what would be expected from an analogy with the SK model. Instead, it would be consistent with critical dynamics and the $2d$ XY model similitude. A number of caveats on the numerical analysis should, however, be lifted before reaching a firm conclusion.

The finite-time finite-length relation between static $\aleph(C, \xi(t_w))$ and long-time out-of-equilibrium dynamic $\chi(C, t_w)$ (section 3.5.1) [43] was put to the test in the $2d$ and $3d$ EA models at finite T. The notable coincidence of the two functions found in the $2d$ case, in which there is no complex equilibrium structure, suggests that the claimed coincidence of $\chi(C)$ and $\aleph(C)$ in $3d$ [113] might also be valid *just* in the transient regime.

Simulations of the $3d$ Heisenberg spin-glass model with weak anisotropy suggest that $T_{\rm EFF}$ associated with the spin degrees of freedom is constant and about twice the critical temperature for spin-glass ordering [114]. As far as we know, chiral degrees of freedom have not been used to estimate $T_{\rm EFF}$.

As regards fluctuations, the two kinds discussed in section 4.8 were measured in the $3d$ EA spin-glass. Disordered induced ones [54], in which one computes strictly local noise-averaged correlations and linear responses, demonstrate the existence of two types of spins in each sample: rapid paramagnetic ones and slow ones. The former satisfy FDT while the latter evolve in two time regimes with a fast one satisfying FDT and a slow one in which $\chi_i(C_i)$ looks quite flat as in coarsening systems. The simulation suggests that the two ensembles behave independently of each other and are strongly correlated with the backbone of the ground state configurations. The average over all sites (at finite t_w) gives rise to a curve with a non-constant slope. These results suggest a still different picture for the spin-glass dynamics in which a rather compact set of spins undergoes coarsening of the backbone equilibrium configurations while the other ones behave paramagnetically. This intriguing idea needs to be put to further test.

The analysis of noise-induced fluctuations suggests that equation (36) is valid although better numerical data would be needed to have definitive evidence for this statement. A more detailed discussion can be found in [58]. Very recent studies of nonlinear fluctuations that take advantage of FDRs to compute higher order responses point in the direction of the TRI scenario [62] with a finite $T_{\rm EFF}$.

5.6.3. Experiments. On the experimental side, the first attempt to quantify FDT violations in spin-glasses was indirect [115]. Simultaneous measurements of global magnetic noise and susceptibility in the thiospinel insulating spin-glass were later performed by Hérisson and Ocio [116]. The data confirm deviations from the FDT with a $\chi(C, t_w)$ plot of relatively curved form although still evolving during the experimental time window. The authors interpreted it as evidence for the full RSB scenario, *via* the association $\chi(C) \leftrightarrow \aleph(C)$. However, as with numerical data [113], dynamic ultrametricity fails to show off, the asymptotic limit of the parametric construction is still far, and a clear-cut distinction between a curved and a linear $\chi(C)$ is difficult to assess.

More recent experiments exploit two novel techniques, Hall-sensor-based magnetometer and giant magnetoresistance technology to detect signals from very small samples [117]. The use of these probes opens the way to perform a systematic study of FDT violations in magnetic systems of different kinds (spin-glasses, super-spin glasses, disordered ferromagnets, etc). The first of these measurements appeared recently [118] in a super-spin glass, a system of magnetic nanoparticles suspended in fluid glycerol with a single-domain magnetic structure that behaves as one large spin, the orientation of which is the only degree of freedom. The large magnetic moment facilitates the observation of magnetic noise. For ageing times of the order of 1 h, the ratio of $T_{\rm EFF}$ to the bath temperature T grows from 1 to 6.5 when T is lowered from T_g to 0.3 T_g, regardless of the noise frequency.

Artificial spin ice is yet another material in which the $T_{\rm EFF}$ notion has been tested [119].

5.7. Driven liquids and glasses

In [120], the molecular dynamics of a binary Lennard–Jones mixture under a steady and homogeneous shear flow was studied. The deviation from FDT is similar to the one found analytically in disordered spin models of RFOT type with asymmetric couplings that mimic non-conservative forces [59, 121]. Moreover, it does not depend on the observable. The tracer particle experiment was also realized. When the tracers' Einstein frequency is smaller than the inverse relaxation time of the fluid, a non-equilibrium equipartition theorem holds with $m_{\rm TR} v_z^2 = T_{\rm EFF}$, where v_z is the velocity in the direction transverse to the flow. For increasing $m_{\rm TR}$ the effective temperature very slowly crosses over from T to the slow mode value, in perfect agreement with the notion of a temperature measured by a thermometer sensible to the scale. $T_{\rm EFF}$ also captures the essential phenomenological idea that when a system is sheared more vigorously its effective temperature increases.

O'Hern *et al* also studied FDRs in shear fluids [80]. This group defined an effective temperature through the 'static limit' $\lim_{t\to\infty} \chi(t-t_w)/C(t,t)$, a kind of average of the slope of the $\chi(C)$ plot over the full range of $C(t-t_w)$ that mixes different time scales (in particular, the high and low frequency ones). A more thorough discussion of the comparison between this definition and the one described in this review was given by Ilg and Barrat [79] within a fully solvable model that demonstrates the importance of *not* mixing time scales to get physically sensible results.

A first study of the fluctuations of entropy production in a Lennard–Jones fluid above and below T_g under a shear flow appeared in [122] and the need to take into account $T_{\rm EFF}$, as obtained from the modification of the FDT below T_g, was signalled in this paper. A more detailed analysis of the time-scale-dependent effective temperature would be needed to fully test the proposal in [51].

Another prominent example is the current-driven motion of vortices in type II superconductors. Disorder reduces dissipation and is responsible for non-equilibrium transport and magnetic properties. The external force induces two dynamic phase transitions separating plastic flow, smectic flow and a frozen transverse solid. A low-frequency $T_{\rm EFF}$ that decreases with increasing driving force and reaches the equilibrium melting temperature when the dynamic transverse freezing occurs was computed from the transverse motion in the fluid moving phase [123].

Sollich *et al* proposed an extension of the trap model [112] that includes strain degrees of freedom to describe the rheological, i.e. deformation and flow, properties of soft materials, such as foams, pastes and dense emulsions [124]. This is the so-called soft glass material model. In the steadily driven regime, stationarity is restored. Interestingly enough, the same (curved) FDRs for observables that are uncorrelated with the averaged energy (to within logarithmic corrections) are found in the trap model with and without drive [125]. In other words, as in mean-field disordered models [59], the relations between correlations and linear responses are the same for aged and weakly driven glasses, as modelled by trap models and observed with such neutral observables. The correspondence does not apply to observables that do not satisfy this condition.

5.8. Granular matter

Several studies of the effective temperature of granular matter have been pursued theoretically, numerically and experimentally. In the latter front, D'Anna *et al* immersed a torsion oscillator in a granular system fluidized by strong high frequency external vibrations to realize the 'thermometer' experiment. They found $T_{\rm EFF} \propto \Gamma^2$ with Γ the adimensional measure of vibrational intensity, and quite independently of ω [126]. Wang *et al* [127] visualized the dynamics of tracer particles embedded in a $3d$ granular ensemble slowly sheared by the rotating inner wall of a Couette cell. $T_{\rm EFF}$, as obtained from the comparison between the tracer's diffusion and mobility perpendicular to the applied rate of strain, is independent of the shear rate used and the tracer's properties but does depend on the packing density of the system. Tests of the thermodynamic properties of $T_{\rm EFF}$ have not been carried through in this system yet. The dependence on the direction of the applied stress was studied by Twardos and Dennin in a plastic bead raft close to jamming [128]. As expected, the correlations and linear responses in the direction of flow do not decay slowly and $\chi(C)$ does not have the same properties as in the transverse direction (cf [129] and [72]). Gei and Behringer stressed the fact that in a granular assembly, the outcome of a mobility measurement depends on whether one imposes the velocity or the external force [130].

In the powders literature reference is often made to the 'granular temperature', a measure of the temperature of the fast modes, as given by the kinetic energy of the grains $T_K \equiv \frac{2}{d} E_K \equiv \frac{2}{d}\langle v^2\rangle$. Importantly enough, T_K is a high frequency measure that does not really access the structural properties of the sample and, in a sense, plays the role of the environmental temperature in thermal systems. T_K is generically smaller than $T_{\rm EFF}$, as in thermal systems where $T_K = T$, the temperature of the bath.

5.9. Activated dynamics

Activated processes often occur in systems that are out of equilibrium, in the sense that their response to an external drive is strongly nonlinear or that their phase space distribution is not the Gibbs–Boltzmann one. The question as to whether an Arrhenius law governs the activation rate, possibly with an effective temperature, and how the latter compares to the one defined from the deviations from FDT has been addressed recently [131]. Ilg and Barrat studied the effect of an out-of-equilibrium flowing environment, a weakly sheared super-cooled liquid, on the activated dynamics between the two stable conformations of dumbbell particles. The transition rate is well described by an Arrhenius law with a temperature that crosses over from the one of the equilibrium bath to a higher value close to the T_{EFF} of the slow modes of the driven fluid. The crossover roughly occurs at the value of the rate that corresponds to the inverse of the α relaxation time of the fluid.

Three related studies are also worth mentioning. An effective temperature, also consistent with the one stemming from fluctuation–dissipation measurements, appears in a phenomenological Arrhenius law that describes transverse jumps between channels in the driven motion of vortex lattices with random pinning [129]. Haxton and Liu showed that in the shear-dominated regime, the stress of a 2d sheared fluid follows an Arrhenius law with the effective temperature [132]. A study of activation and T_{EFF} in a 2d granular system close to jamming appeared in [133].

5.10. Biological systems

In biologically inspired problems, the relevance of T_{EFF} was stressed to reveal the active process in hair bundles [135] and model cells [136]. Morozov *et al* [137] studied a model of the cytoskeletal network made of semi-flexible polymers subject to thermal and motor-induced fluctuations and found a T_{EFF} that exceeds the environmental temperature T only in the low-frequency domain where motor agitation prevails over thermal fluctuations. Simple gene network models were studied from the T_{EFF} perspective in [134]. Fluctuation–dissipation ratios were used to quantify the degree of frustration, due to the existence of many metastable disordered states, in the formation of viral capsids and the crystallization of sticky discs, two self-assembly processes [138]. Fluctuations and responses of blood cell membranes for varying ATP concentration were measured very recently [139]. The measured T_{EFF} approaches the bath temperature at high frequencies and increases at low frequencies reaching four to ten times the ambient temperature.

Ratchets are simple models of molecular motors, out-of-equilibrium systems with directed dissipative transport in the absence of any external bias. Harada and Sasa proposed to use the violations of FDT in flashing ratchets as a means to measure the energy input per unit time in molecular motors—an otherwise difficult quantity to access [140]. Kolton showed that the rectified transverse velocity of a driven particle in a geometric ratchet is equivalent to the response of a 1d flashing ratchet at a drive-dependent T_{EFF}, as defined from the generalized Einstein relation [141].

Active matter is driven out of equilibrium by internal or external energy sources. Its constituents absorb energy from their environment or from internal fuel tanks and dissipate it by carrying out internal movements that lead to translational or rotational motion. A typical example are self-propelled particle assemblies in bacterial colonies. The role played by T_{EFF} in the stability of dynamic phases of motorized particle systems was stressed by Shen and Wolynes [142]. Multiple measurements of T_{EFF} were carried out with molecular dynamic simulations of motorized spherical as well as linear molecules in interaction [143]. All measurements (from fluctuation–dissipation ratio and using tracers) yield a constant low-frequency $T_{\mathrm{EFF}} > T$ when the effect of the motors is not correlated with the structural rearrangements they induce. Instead, T_{EFF} takes a slightly lower value than T when susceptible motors are used, as argued in [142]. Such an 'inversion' also occurs in relaxational systems in which the initial configuration is chosen to be one of equilibrium at a lower T than the working one [45–47]. In the case of uncorrelated motors, T_{EFF}/T was found to follow the empirical law $T_{\mathrm{EFF}}/T \simeq 1+\gamma f^2$ with f the active force relative to the mean potential force and $\gamma \sim 15$ a parameter. Palacci *et al* [144] investigated T_{EFF} by following Perrin's analysis of the density profile in the steady state of an active colloidal suspension under gravity. The active particles used—JANUS particles—are chemically powered colloids and the suspension was studied with optical microscopy. The measurements show that the active colloids are hotter than in the passive limit with a T_{EFF} that increases as the square of the parameter that controls activation, the Peclet number, a dependence that is highly reminiscent of the f^2-dependence of the simulations mentioned above. Other theoretical studies of T_{EFF} in active matter appeared in [145].

Joly *et al* [146] used numerical techniques to study the non-equilibrium steady state dynamics of a heated crystalline nanoparticle suspended in a fluid. This problem models an active colloid that acts as a local heat source and generates a temperature gradient around it. By comparing the mobility to the velocity correlation function, they found that the FDT approximately holds at short-time lags with a temperature value that coincides with the kinetic one. In contrast, at long-time lags data are compatible with the temperature estimated by using the Einstein relation.

Certainly, many more studies of effective temperatures will appear in this very active field of research, essentially out of equilibrium, in the near future.

5.11. Plasticity

Langer *et al* extended the traditional phenomenological defect-flow theory to the shear transformation zone (STZ) theory of large-scale plastic deformation in amorphous materials [147]. The new theory incorporates effective temperature ideas. It is a picture of plastic deformation in molecular glasses in which a 'disorder temperature' characterizes the steady state of the system and controls the slow processes. The relation between the disorder temperature and a configurational entropy, under the assumption of a sharp separation of time scales between structural and vibrational processes, as well as other thermodynamic properties of it—along the lines discussed in section 4.9—were addressed. As far as we know, there have been no tests to compare the STZ disorder temperature and the effective temperature computed from fluctuation–dissipation measurements; this is the reason why we keep distinct names for the two quantities. The STZ theory suggests an explanation of shear-banding instabilities that have been put to the numerical test in [148].

Another approach to plasticity consist in adapting the RFOT-replica approach to this problem [149]. Numerical simulations of binary soft sphere mixtures at low temperature prove that the stress relaxation and its response to a strain step are also linked by a modified FDT with a single valued T_{EFF} in the ageing regime [150].

5.12. Turbulent fluids

Last but not least, the experimental quest for effective temperatures in the steady state of turbulent flows has recently restarted. As far as we know, two experiments on turbulent flows appeared in the literature after the role played by different time scales was stressed in the investigation of effective temperatures in macroscopic systems with slow dynamics.

The transverse fluctuations and the response of a string held at its ends and at constant tension (a mechanical probe) in the inertial regime of a stationary turbulent air jet flow were compared to obtain the effective temperature. The ranges of wave numbers, and Reynolds number ($7.4 \times 10^4 \leqslant Re \leqslant 1.7 \times 10^5$), accessed in this experiment are pretty wide [151]. All measurements are compatible with $T_{\mathrm{EFF}} \propto k^{-11/3}$. The k dependence confirms that there is no equilibrium between Fourier modes due to the energy flux between scales. The particular exponent is explained in [151] with a simple model derived from Kolmogorov 1941.

In a more recent experiment focus was set on an anisotropic non-homogeneous axisymmetric von Kármán flow at large Reynolds number [152]. The measurements are, though, somehow indirect and yield an effective temperature that depends on the observable. This feature needs to be better characterized.

5.13. Quantum models

There is growing interest in the dynamics of quantum systems. Of particular importance in this field is to distinguish cases in which the system of interest is isolated and the dynamics occur at constant energy from those in which the system is coupled to an environment and the dynamics are dissipative. The out-of-equilibrium dynamics are typically induced in two ways: the system is driven out of equilibrium by, for instance, a coupling to electric or heat current sources, or it is strongly perturbed by time-dependent external fields; the system evolves after a quench meaning that a parameter in its Hamiltonian is changed with some protocol. All these cases are easily realized in the laboratory nowadays and their potential applicability is being explored. Theoretical searches for effective temperatures in these problems, defined in different ways, are starting to appear in the literature.

As recalled in section 2, linear responses and correlation functions in equilibrium quantum systems are related by FDTs that take a slightly different form depending on the bosonic or fermionic character of the observables. In an out-of-equilibrium situation, one can compare the linear response and the correlation and test whether, at least in some dynamic regime, a parameter replacing the environmental temperature appears. This route was first taken within mean-field quantum glassy models quenched from their disordered into their ordering phase [38, 39, 153]. In these models, the quantum FDT holds within the rapid stationary scale. Instead, when the dynamics gets slow and the ageing regime is attained, the relation between linear response and correlation takes the classical form, with an effective temperature that is higher than the one of the bath and different from zero. As in the classical case, the static properties of these models can be solved with the replica and TAP approaches and a connection with the dynamic solution can be established [154].

More recently, the analysis of mesoscopic quantum models commenced. A metallic ring threaded by a time-dependent magnetic field and coupled to a lead in equilibrium at temperature T and chemical potential μ was studied in [155]. The numerical solution to the Schwinger–Keldysh equations for the free but driven fermions, in the limit of small dissipation, suggests that their Green functions satisfy an FDT with constant T_{EFF}. More recently, Arrachea *et al* attacked the problem of a wire connected to left and right reservoirs (in equilibrium at the same chemical potential and temperature) and driven out of equilibrium by different ac pumps locally connected to the wire [156]. The local effective temperature was computed from the modified FDT and by requiring that there be no heat flow to nor from macroscopic probes, i.e. thermometers, weakly coupled to chosen sites on the device. For weak driving and environmental temperatures lower than the Fermi energy of the electrons, these two measurements coincide on each site. T_{EFF} has spatial $2k_F$-Friedel-like oscillations (allowing for local cooling) but its spatial average is higher than the temperature of the reservoirs when the pump is applied. Moreover, the direction of heat flow between the device and each one of the leads is dictated by the temperature gradient at the contact, defined as the difference between T_{EFF} at the contact and the temperature of the lead. For conveniently chosen pumps, the device can extract energy from one lead and transport it to the opposite one.

The theory of isolated quantum models is a very active field of research, pushed by the large number of experiments on cold-atom systems that are being realized worldwide. Questions on thermalization after a quantum quench, in which a parameter in the Hamiltonian is suddenly changed, are being posed, extending this very much investigated issue in classical statistical mechanics to quantum systems. At present the belief is that non-integrable quantum systems do reach equilibrium while integrable ones do not. Rigol *et al* proposed to use extended Gibbs–Boltzmann measures in the latter case with as many constants of motion as necessary fixed by the initial condition [157]. Currently, this proposal is being debated and studied in particular cases. For instance, Rossini *et al* complemented it with the idea that non-local observables (when expressed in terms of quasi-particles) should equilibrate in integrable systems while local ones should not under general conditions [158]. They based this conjecture on their mixed analytic/numeric solution of a quantum spin chain. Studies of effective temperatures (as defined from deviations from FDT) in this context are underway [159]. Another quantum setting in which an effective temperature was proposed is the entanglement entropy of a sub-system [160].

6. Conclusions

In this review, we discussed the notion of an effective temperature [1–4] that stemmed from the deviations from the fluctuation–dissipation theorem found in the analytic solution to simple glass models [5, 6] and self-consistent approximations to more realistic glassy systems [32, 33]. Although phenomenological definitions of out-of-equilibrium temperatures were not lacking in the literature, see e.g. [8, 10], the effective temperature discussed in this report lies, in our opinion, on a firmer physical basis.

Fluctuation–dissipation violations can be searched for in all out-of-equilibrium systems. Whether the outcome can be interpreted as giving origin to an effective temperature is a different and more delicate issue. In this review, we tried to distinguish cases in which the latter can be done from cases in which it is not possible. A necessary condition seems to be that the system should evolve slowly, irrespectively of whether it relaxes or is in a NESS, in a small entropy production regime [4]. Although not rigorously proven, it seems natural to require that it approaches an approximately flat measure on a region of phase space, the entropy of which (also called complexity) should give an alternative, microcanonic-like, access to the effective temperature [2, 7, 21, 66]. A consequence of the last condition is that the energy density, and more generally the averages of one-time-dependent observables converge to *finite* values. All these features are satisfied exactly in some solvable mean-field-like glasses such as the celebrated p-spin disordered model and the low-T mode-coupling approximations. Numerical simulations suggest that they are also, at least within a given time regime, in a large number of glassy systems with short-range interactions, including Lennard–Jones mixtures and others. In all these cases, the systems have a rather complex collective behaviour with a separation of time scales fast–slow that has an influence on the values that the effective temperature takes.

As far as it has been checked, in all the above-mentioned cases, the effective temperature conforms to the common prejudices one has of a temperature and, more importantly, it has the most welcome property of being measurable directly, hence being open to straightforward—though difficult to implement—experimental tests. Central to the correct identification is the realization that the relaxation time scales have to be correctly identified and that measurements have to focus on each of them separately. Simulations in atomic glasses are very complete and yield pretty convincing support to the effective temperature ideas [92–96, 120–122]. Experimental subtleties prove to be difficult to keep under control, especially in colloidal suspensions [101–108], but successful measurements of fluctuation–dissipation violations in structural glasses [100, 109] have been surveyed. Very recent numerical [143] and experimental [144] studies of active matter are also very promising. Simulations of spin-glass [113] clearly demonstrate the violation of FDT but are not as precise as to determine beyond doubt the actual functional form of the modified relation. In our opinion, pre-asymptotic effects are still to be disentangled from the asymptotic regime in a satisfactory way. The same applies to the first experiments in an insulator spin-glass [116], the outcome of which could be greatly improved by the use of innovative techniques [117, 118]. A careful analysis of the thermodynamic properties of the fluctuation–dissipation ratio remains to be done in such frustrated and disordered magnets.

Quite naturally, the idea that effective temperatures could be relevant to other out-of-equilibrium systems was explored in the last 15 years. Critical quenches in dissipative classical systems [12], zero temperature dynamics at the lower critical dimension [13], granular matter [71–77] and turbulent fluids [151, 152] are just some examples in which FDRs have been studied. The question remains as to whether the outcome admits a thermodynamic interpretation in, at least, some limits. In the case of critical quenches, this was proposed to be the case when the long two-time limit prescription of Godrèche and Luck is taken [85] (although perturbative corrections yield undesired observable dependences yet to be understood [87]). Simulations of weakly perturbed granular matter are very encouraging [72] but experiments have turned out to be much harder to realize [70] (see, though, [127]). In turbulent fluids, the question remains open experimentally.

Out-of-equilibrium quantum systems are receiving enormous attention nowadays, boosted by experiments in cold atoms and nanotechnology. In this realm, fundamental questions of thermalization arise, especially in isolated samples submitted to a quench. The effective temperature has been studied in a few dissipative quantum systems out of equilibrium, both mean-field [38, 39] and low-dimensional [155, 156]. The analysis of whether it also plays a role in quantum quenches of isolated samples is underway [159].

This review presents a vast panorama of fluctuation–dissipation deviations in non-equilibrium classical, and to a much smaller extent, quantum systems, and their interpretation, in some cases, in terms of effective temperatures. Many questions of fundamental interest remain open in this field. One of the main open challenges in the context of glassy systems is to find the microscopic origin of these modifications. From a wider viewpoint, the validity of the effective temperature concept should be more clearly delimited.

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