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Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients

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Abstract

Dynamical systems described by equations of motion with the first-order time derivative (dissipative) terms of even and odd powers, and coefficients varying either in time or in space, are considered. Methods to obtain standard and non-standard Lagrangians are presented and used to identify classes of equations of motion that admit a Lagrangian description. It is shown that there are two general classes of equations that have standard Lagrangians and one special class of equations that can only be derived from non-standard Lagrangians. In addition, each general class has a subset of equations with non-standard Lagrangians. Conditions required for the existence of standard and non-standard Lagrangians are derived and a relationship between these two types of Lagrangians is introduced. By obtaining Lagrangians for several dynamical systems and some basic equations of mathematical physics, it is demonstrated that the presented methods can be applied to a broad range of physical problems.

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1. Introduction

The fact that all fundamental equations of modern physics can be derived from corresponding Lagrangians is well-known and strongly emphasized in many textbooks and monographs [1–7]. A lesser-known fact is that practically all of those Lagrangians were not part of an *a priori* process that originally led to the equations, and that their explicit forms were usually obtained in ad hoc fashions instead of being strictly derived from first principles [4–7]. The main purpose of this paper is to present general methods to derive standard and non-standard Lagrangians, and use these methods to identify classes of equations of motion that can be derived from the Lagrangians.

We consider Lagrangians to be standard if they can be expressed as differences between 'kinetic energy terms' and 'potential energy terms'; these Lagrangians may, or may not, explicitly depend on time through either an exponential factor or another function, and they are also referred to as S-equivalent Lagrangians [8]. All other forms of Lagrangians discussed in this paper are called non-standard Lagrangians [6]; note that Arnold [9] refers to them as non-natural Lagrangians.

Dynamical systems discussed in this paper are described by second-order ordinary differential equations with the first-order time derivative (dissipative) terms of even and odd powers, and coefficients varying either in time or in space. We refer to these systems as 'dissipative' because of the first-order time derivative (velocity) terms in their equations of motion [10, 11]. One must keep in mind that this general definition encompasses both non-conservative systems with the linear (or higher odd powers of) velocity terms and conservative systems with the quadratic (or higher even powers of) velocity terms; the latter are also called 'damping-like' terms [10] because the resulting equations of motion are invariant with respect to time reversal, which means that the systems are conservative.

It is well-known that Newtonian mechanics can be applied to both conservative and non-conservative systems; however, the Lagrangian and Hamiltonian formulations of mechanics are limited to conservative systems [1–5]. The validity of the latter is guaranteed by a corollary resulting from well-known mathematical theorems first proved by Bauer [12]. The corollary shows that it is impossible to apply the Lagrangian formulation and Hamilton's variational principle to a linear dissipative system described by a single equation of motion with constant coefficients; this is a problem related to the time reversibility of the action principle, which does not allow selecting a special direction in time [6]. However, Bateman [13] found loopholes in Bauer's results and constructed Lagrangians for dissipative systems.

One of Bateman's techniques is to add to a system under consideration another one that is reversed in time and has negative friction. The method leads to two equations of motion, and the resulting Hamiltonian gives extraneous solutions that must be suppressed [6, 7]. An interesting modification of this method was done by Dekker [14] who introduced two first-order equations that were complex conjugate of each other and showed how to combine them to obtain one real, second-order equation of motion. Bateman's other technique is to consider a Lagrangian that depends explicitly on time through an exponential factor and to obtain the desired equation of motion by ignoring the time-dependent term [6, 7]. The problem with this approach is that the resulting momentum and Hamiltonian may not be physically meaningful.

A generalized method to deal with non-conservative systems was developed by Riewe [15, 16], who formulated Lagrangian and Hamiltonian mechanics by using fractional derivatives. His main result is that non-conservative forces can be calculated from potentials that contain fractional derivatives. After the method was applied to several systems [17, 18], it became clear that its broad applications are limited by the complexity of fractional calculus. Another problem with the method is that its equations are acasual and that the procedure to change these equations into casual ones is not well defined [19]. In addition, the method cannot directly be used to quantize linear dissipative systems.

In the literature the inverse problem of finding a Lagrangian for a given equation of motion is called the Helmholtz problem, which requires solving Helmholtz equations [20, 21]. An alternative approach was developed by Musielak *et al* [22], who introduced a general method to derive a standard Lagrangian for a nonlinear dynamical system with a quadratic first-order time derivative term and coefficients variable in the space coordinates.

In this paper, the method [22] is used to determine forms of equations of motion that have standard Lagrangians. Another method is developed to identify equations of motion that can be derived from non-standard Lagrangians. It is shown that there are two general classes of equations of motion with standard Lagrangians and one special class of equations of motion with non-standard Lagrangians. An interesting result is that each general class has also a subset of equations of motion with non-standard Lagrangians. Conditions required for the existence of standard and non-standard Lagrangians are derived and a relationship between these two types of Lagrangians is introduced. By obtaining Lagrangians for several dynamical systems and some basic equations of mathematical physics, it is demonstrated that the presented methods can be applied to a broad range of physical problems.

The outline of the paper is as follows: a method to derive standard Lagrangians and basic equations of motion are presented in section 2, equations of motion with time and space-dependent coefficients that can be derived from either standard or non-standard Lagrangians are determined in sections 3 and 4, physical implications of the obtained results are discussed in section 5 and conclusions are given in section 6.

2. Method and basic equations

To introduce the method that is used in this paper to derive standard Lagrangians for dynamical systems with variable coefficients, we consider two second-order ordinary differential equations with different variable coefficients:

$$\ddot{x} + C(t)G(x) = 0 \tag{1}$$

and

$$\ddot{x} + c(x)g(x) = 0, \tag{2}$$

where $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$, $C(t)$ is an arbitrary function of t only, and $G(x)$, $g(x)$ and $c(x)$ are arbitrary functions of x and they do not depend explicitly on t.

Depending on the physical meaning of the independent variable t and the dependent variable x, and on the form of the coefficients $C(t)$ and $c(x)$ and the functions $G(x)$ and $g(x)$, equations (1) and (2) may describe many different physical systems. The best known examples of such systems are the simple linear oscillator, with $C(t) = c(x) = \omega_{\mathrm{o}}^2 = \mathrm{const}$ and $G(x) = g(x) = x$, and the nonlinear pendulum, with $C(t) = c(x) = \omega_{\mathrm{o}}^2 = \mathrm{const}$ and $G(x) = g(x) = \sin x$, where ω_{o} is the natural frequency of the oscillatory system.

Other examples of different linear and nonlinear oscillators that are represented by equations (1) and (2) are given in two monographs [23, 24]. Moreover, equation (1) can be cast into the form of either the Airy, Mathieu or Hill equation [10, 11], or the time-independent Klein–Gordon equation with constant [1] and variable [25] coefficients or the time-independent Schrödinger equation [1]; note that for the Klein–Gordon and Schrödinger equations, the variables t and x have different physical meanings.

Writing Lagrangians $L_C(\dot{x}, x, t)$ and $L_c(\dot{x}, x)$ for equations (1) and (2), respectively, is a straightforward procedure [22] as long as $G(x)$, $g(x)$ and $c(x)$ are continuous and integrable functions. Assuming that this is the case, we have

$$L_C(\dot{x}, x, t) = \frac{1}{2}\dot{x}^2 - C(t)\int^x G(\tilde{x})\,\mathrm{d}\tilde{x}, \tag{3}$$

$$L_c(\dot{x}, x) = \frac{1}{2}\dot{x}^2 - \int^x c(\tilde{x})g(\tilde{x})\,\mathrm{d}\tilde{x}. \tag{4}$$

According to Courant [26], the above integrals are indefinite because any value of the lower limit of these integrals can be chosen. Since different indefinite integrals of the same function differ only by an additive constant, the lower limit of the integrals can be chosen in such a way that the resulting constant is always zero. Because of this property, the integrals are written without their lower limits.

For $G(x)$, $g(x)$ and $c(x)$ being continuous functions, the derivatives of the above integrals with respect to x are given by their integrands [26]. This property is essential to demonstrate that equations (1) and (2) are obtained when the above Lagrangians are substituted into the Euler–Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{C,c}}{\partial \dot{x}}\right) - \frac{\partial L_{C,c}}{\partial x} = 0. \tag{5}$$

Since $L_C(\dot{x}, x, t)$ and $L_c(\dot{x}, x)$ are expressed as the difference between the 'kinetic' and 'potential' energy terms (see equations (3) and (4)), we refer to them as standard Lagrangians (see section 1). For the Airy, Mathieu and Hill equations, and the time-independent Klein–Gordon and Schrödinger equations, the integral in equation (3) can be evaluated and the explicit form of the Lagrangian $L_C(\dot{x}, x, t)$ can be obtained. However, this may not be the case for physical systems described by the Lagrangian $L_c(\dot{x}, x)$; depending on the form of $c(x)$, the integral in equation (4) may lead to a non-local Lagrangian [22].

Let us now add 'dissipative terms' to equations (1) and (2), and write them down in the following general form:

$$\ddot{x} + B(t)F(\dot{x}) + C(t)G(x) = 0, \tag{6}$$

$$\ddot{x} + b(x)f(\dot{x}) + c(x)g(x) = 0, \tag{7}$$

where $F(\dot{x})$ and $f(\dot{x})$ are arbitrary functions that solely depend on $\dot{x}$, and the coefficients $B(t)$ and $b(x)$ depend only on t and x, respectively.

The basic idea of this method is to find transformations that remove the dissipative terms from equations (6) and (7), and allow casting these equations in the form of equations (1) and (2); obviously, the existence of such transformations depends on specific forms of the functions $F(\dot{x})$ and $f(\dot{x})$. Once the transformations are found, the corresponding Lagrangians for the transformed variables can easily be derived from equations (3) and (4). Then, the standard Lagrangians for the original variables are also obtained.

We assume that the functions $F(\dot{x})$ and $f(\dot{x})$ are given as power laws, which means that equations (6) and (7) can be written as

$$\ddot{x} + B(t)\dot{x}^m + C(t)G(x) = 0 \tag{8}$$

and

$$\ddot{x} + b(x)\dot{x}^n + c(x)g(x) = 0, \tag{9}$$

where m and n are integers $1, 2, 3, \ldots$. The choice of the power laws for the functions $F(\dot{x})$ and $f(\dot{x})$ is justified by numerous physical systems described by the above equations.

Examples of force-free dynamical systems with linear damping ($m = n = 1$) include a damped pendulum whose length increases in time [27], a damped Duffing oscillator [28], a damped Duffing–van der Pol system [29] and Liénard and other types of oscillators [8, 29–31]; specifically, the Liénard equation is obtained when $n = 1$ and $c(x) = 1$. Numerous dynamical systems with nonlinear damping ($m = n = 2$) have been considered in the literature [8, 24, 32–38]. Dynamical systems with higher powers of m and n have also been discussed [39], and an important example of such systems is the nonlinearly damped double-well Duffing oscillator [40].

Having described the method and specified the basic equations of motion used in this paper, we now determine which of those equations can be derived from standard and non-standard Lagrangians. Let us begin with the equations of motion with time-dependent coefficients.

3. Equations with time-dependent coefficients

3.1. Existence of standard Lagrangians

The general form of the equation of motion with time-dependent coefficients is given by equation (8). To determine the value of m for which the first-order derivative term can be removed from the equation, we use the following integral transformation:

$$x(t) = x_1(t)\,\mathrm{e}^{I_1(t)}, \tag{10}$$

where

$$I_1(t) = \int^t \phi_1(\tilde{t})\,\mathrm{d}\tilde{t}, \tag{11}$$

and ϕ_1 is an arbitrary function to be determined; the restriction on this function is that it must be integrable and differentiable. In addition, if ϕ_1 is a continuous function, then the indefinite integral $I_1(t)$ always possesses a derivative such that $\mathrm{d}I_1(t)/\mathrm{d}t = \phi_1(t)$ [26]. Also note that the above transformation guarantees that the variables $x(t)$ and $x_1(t)$ have the same zeros.

The transformed equation of motion can be written as

$$\ddot{x}_1 + 2\dot{x}_1\phi_1 + \left(\dot{\phi}_1 + \phi_1^2\right)x_1 + (\dot{x}_1 + \phi_1 x_1)^m B(t)\,\mathrm{e}^{(m-1)I_1(t)} + C(t)G(x_1\mathrm{e}^{I_1(t)})\,\mathrm{e}^{-I_1(t)} = 0. \tag{12}$$

Since the terms with $\dot{x}_1$ have different powers, they cannot be removed by the function $\phi_1(t)$ unless $m = 1$; note that only for $m = 1$ the power of the two terms with $\dot{x}_1$ is the same, which means that one can find such $\phi_1(t)$ that allows removing these terms. This is an important result as it shows that the integral transformation used here is limited to the equations of motion with the linear damping term.

Taking into account the fact that $m = 1$, equation (12) becomes

$$\ddot{x}_1 + [B(t) + 2\phi_1(t)]\dot{x}_1 + \left[\dot{\phi}_1 + \phi_1^2 + B(t)\phi_1\right]x_1 + C(t)G(x_1\,\mathrm{e}^{I_1(t)})\,\mathrm{e}^{-I_1(t)} = 0. \tag{13}$$

This shows that the condition required to remove the term with $\dot{x}$ is $\phi_1(t) = -B(t)/2$; note that substitution of this condition into equations (10) and (11) gives a well-known transformation [28].

Using the above condition, we write equation (13) as

$$\ddot{x}_1 - \tfrac{1}{2}\left(\dot{B} + \tfrac{1}{2}B^2\right)x_1 + C(t)G(x_1\,\mathrm{e}^{-I_B(t)})\,\mathrm{e}^{I_B(t)} = 0, \tag{14}$$

where $I_B(t) = \int^t B(\tilde{t})\,\mathrm{d}\tilde{t}/2$. Since the form of equation (14) is similar to that given by equation (1), we use equation (3) to write the Lagrangian $L_1(\dot{x}_1, x_1, t)$ in the following form:

$$L_1(\dot{x}_1, x_1, t) = \frac{1}{2}\dot{x}_1^2 + \frac{1}{4}\left(\dot{B} + \frac{1}{2}B^2\right)x_1^2 - C(t)\,\mathrm{e}^{I_B(t)}\int^{x_1} G(\tilde{x}_1\mathrm{e}^{-I_B(t)})\,\mathrm{d}\tilde{x}_1. \tag{15}$$

It is easy to show that substitution of $L_1(\dot{x}_1, x_1, t)$ into the Euler–Lagrange equation (see equation (5)) leads to the equation of motion given by equation (14). Also note that based on our definition given in section 1, $L_1(\dot{x}_1, x_1, t)$ is a standard Lagrangian.

Having obtained the Lagrangian for the transformed variables $\dot{x}_1$ and x_1, we now express $L_1(\dot{x}_1, x_1, t)$ in terms of the original variables $\dot{x}$ and x and derive the standard Lagrangian $L(\dot{x}, x, t)$. Its explicit form is given by the following proposition.

Proposition 1. *The equation of motion*

$$\ddot{x} + B(t)\dot{x} + C(t)G(x) = 0, \tag{16}$$

with $B(t)$, $C(t)$ and $G(x)$ being continuous, differentiable and integrable functions, admits a Lagrangian description and the resulting standard Lagrangian is

$$L(\dot{x}, x, t) = \frac{1}{2}\left[\dot{x}^2 + B\dot{x}x + \frac{1}{2}(\dot{B} + B^2)x^2\right]\mathrm{e}^{2I_B(t)} - C(t)\,\mathrm{e}^{2I_B(t)}\int^x G(\tilde{x})\,\mathrm{d}\tilde{x}, \tag{17}$$

where

$$I_B(t) = \frac{1}{2}\int^t B(\tilde{t})\,\mathrm{d}\tilde{t}. \tag{18}$$

Proof. Substitution of the Lagrangian $L(\dot{x}, x, t)$ into the Euler–Lagrange equation (see equation (5)) gives equation (16), which validates proposition 1. □

An important result obtained here is that the presented method of writing a standard Lagrangian is limited to the equation of motion with the time-dependent coefficients and the linear ($m = 1$) damping term. The explicit dependence of $L(\dot{x}, x, t)$ on t will be discussed in section 3.2, where the results described in proposition 1 are applied to selected physical systems and to some equations of mathematical physics.

3.2. Applications of standard Lagrangians

To apply the results of proposition 1 to physical systems, we begin with a pendulum whose length increases with time like in Poe's thrilling tale 'The Pit and the Pendulum'. Actually, we assume that the pendulum is nonlinear and damped and its length $l(t)$ increases with time t at the constant rate a so that $l(t) = l_0 + at$, where l_0 is the initial length of the pendulum. If $\theta(t)$ is a displacement and g is gravity, then the equation of motion for this system [27] is given by

$$\ddot{\theta} + \frac{2a}{l_0 + at}\dot{\theta} + \frac{g}{l_0 + at}\sin\theta = 0. \tag{19}$$

By comparing this equation to equation (16) and using equation (17), we may immediately write the Lagrangian $L(\dot{\theta}, \theta, t)$. The result is

$$L(\dot{\theta}, \theta, t) = \frac{1}{2}\left(\dot{\theta}^2 + \frac{2a}{l_0 + at}\dot{\theta}\theta\right)(l_0 + at)^2 + \frac{1}{2}a\theta^2 + g(l_0 + at)\cos\theta. \tag{20}$$

Our second example is the Lane–Emden equation, which describes a self-gravitating and spherically symmetric object composed of a fluid with the polytropic index κ. This equation is used in stellar astrophysics to study internal structures of stars [41], and its general form is

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} + \frac{2}{\xi}\frac{\mathrm{d}\psi}{\mathrm{d}\xi} + \psi^\kappa = 0, \tag{21}$$

where ψ is dimensionless stellar density and ξ is also dimensionless and related to stellar radius.

The Lagrangian $L(\psi', \psi, t)$, with $\psi' = \mathrm{d}\psi/\mathrm{d}\xi$, can be written by using equation (17), which gives

$$L(\psi', \psi, t) = \frac{1}{2}\left[(\psi')^2 + \frac{2}{\xi}\psi'\psi + \frac{1}{\xi^2}\psi^2 - \frac{2}{\kappa + 1}\psi^{\kappa+1}\right]\xi^2. \tag{22}$$

To discuss other applications, we take $G(x) = x$ and write the equation of motion (see equation (16)) as

$$\ddot{x} + B(t)\dot{x} + C(t)x = 0. \tag{23}$$

For this equation, the integral in the standard Lagrangians $L(\dot{x}, x, t)$ can be evaluated (see equation (17)), and the Lagrangian becomes

$$L(\dot{x}, x, t) = \tfrac{1}{2}\left[\dot{x}^2 + B\dot{x}x - \tfrac{1}{2}(2C - \dot{B} - B^2)x^2\right]\mathrm{e}^{2I_B(t)}. \tag{24}$$

Depending on the form of $B(t)$ and $C(t)$, equation (23) can be cast in the form of the Sturm–Liouville equation if $B(t) = \dot{p}(t)/p(t)$ and $C(t) = q(t) + \lambda r(t)$ [41], which means that many important equations of mathematical physics can be represented by equation (23). Examples include the Bessel, Legendre, Laguerre, Hermite, Chebyshev, Jacobi, hypergeometric and confluent hypergeometric equations [1, 42]. An important result of this analysis is that for each of these equations, the standard Lagrangian $L(\dot{x}, x, t)$ can formally be obtained from equation (24). To demonstrate this, we apply the results to the Bessel and Laguerre equations.

We write the Bessel equation in the following form:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y} + \frac{1}{y}\frac{\mathrm{d}u}{\mathrm{d}y} + \left(1 + \frac{\nu^2}{y^2}\right) u = 0, \tag{25}$$

where ν is a parameter. Note that the physical meaning of the independent variable y and the dependent variable u depends on a specific problem that is considered. Using equation (17), we obtain the standard Lagrangian $L(u', u, y)$, with $u' = du/\mathrm{d}y$, from which the above Bessel equation can be derived. The Lagrangian is given by

$$L(u', u, y) = \frac{1}{2}\left[(u')^2 + \frac{1}{y}u'u - \left(1 + \frac{\nu^2}{y^2}\right) u^2\right] y. \tag{26}$$

Similarly, we may obtain the standard Lagrangian for the Laguerre equation, which can be written as

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z} + \left(\frac{1}{z} - 1\right)\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{k}{z}v = 0, \tag{27}$$

where k is a parameter. The standard Lagrangian $L(v', v, z)$, with $v' = \mathrm{d}u/\mathrm{d}z$, becomes

$$L(v', v, z) = \frac{1}{2}\left[(v')^2 + \left(\frac{1}{z} - 1\right) v'v\right] z\,\mathrm{e}^{-z} - \frac{1}{2}\left[\frac{k}{z} + \frac{1}{2}\left(\frac{2}{z} - 1\right)\right] v^2 z\,\mathrm{e}^{-z}. \tag{28}$$

It must be noted that among the four different Lagrangians derived above, only $L(v', v, z)$ depends explicitly on the independent variable z through an exponential factor; this type of dependence is characteristic for Lagrangians first discussed by Bateman [13]. Clearly, the other three Lagrangians are of different types as their dependence on the independent variable is not exponential.

The above examples demonstrate that the method presented in this paper can be used to obtain standard Lagrangians for many equations of mathematical physics as well as for equations of motion describing oscillatory and other mechanical systems with linear damping terms.

3.3. Existence of non-standard Lagrangians

It was demonstrated that the nonlinear second-order Riccati equation admits a Lagrangian formulation [31] and that its non-standard Lagrangian is given by

$$\mathcal{L}(\dot{x}, x, t) = \frac{1}{\dot{x} + \alpha u(x, t)}, \tag{29}$$

where $\alpha = \text{const}$ and $u(x, t) = v_0(t) + v_1(t)x + v_2(t)x^2$, with the functions $v_0(t)$, $v_1(t)$ and $v_2(t)$ to be determined.

By generalizing this result, the following form of non-standard Lagrangian is considered:

$$\mathcal{L}(\dot{x}, x, t) = \frac{1}{P(t)\dot{x}^m + Q(t)H(x) + R(t)}, \tag{30}$$

where the functions $P(t)$, $Q(t)$, $R(t)$ and $H(x)$ are to be determined; note that it is required that these functions are differentiable.

We use the Euler–Lagrange equation (equation (5)) to derive a general equation of motion resulting from the above non-standard Lagrangian. The result is

$$A_1(\dot{x}, x, t)\ddot{x} + A_2(\dot{x}, x, t)\dot{x}^m + A_3(x, t) = 0, \tag{31}$$

where

$$A_1(\dot{x}, x, t) = m(m+1)P^2\dot{x}^{2(m-1)} + (QH + R)P\dot{x}^{m-2}, \tag{32}$$

$$A_2(\dot{x}, x, t) = (1+2m)PQH' + 2m(\dot{Q}H + \dot{R})P\dot{x}^{-1} + mP\dot{P}\dot{x}^{(m-1)} - m(QH + R)\dot{P}\dot{x}^{-1} \tag{33}$$

and

$$A_3(x, t) = (QH + R)QH'. \tag{34}$$

In addition, $H'(x) = \mathrm{d}H(x)/\mathrm{d}x$ and $\dot{P}(t)$, $\dot{Q}(t)$ and $\dot{R}(t)$ are time-derivatives of these functions.

In order to determine the functions $P(t)$, $Q(t)$, $R(t)$ and $H(x)$, the coefficients $A_1(\dot{x}, x, t)$, $A_2(\dot{x}, x, t)$ and $A_3(x, t)$ must be compared to the corresponding coefficients in equation (8). The comparison shows that A_1 has to be 1, or any other constant, and A_2 has to be a function of t only. To satisfy these two conditions, we must take $m = 1$ and also assume that $P(t) = \text{const}$; without any loss of generality, we simply take $P(t) = 1$.

With these assumptions, the resulting coefficients are: $A_1 = 2$, $A_2(x, t) = 3Q(t)H'(x)$ and $A_3(x, t) = 2(\dot{Q}H + \dot{R}) + QH'(QH + R)$. Since $A_2(x, t)$ must be a function of t only, it is required that $H' = 1$, which means that $H(x) = x$. Thus, equation (31) becomes

$$\ddot{x} + \tfrac{3}{2}Q(t)\dot{x} + \left[\dot{Q}(t) + \tfrac{1}{2}Q^2(t)\right]x + \tfrac{1}{2}Q(t)R(t) + \dot{R}(t) = 0. \tag{35}$$

A comparison of this equation to the original equation of motion given by equation (8) with $m = 1$ shows that $Q(t) = 2B(t)/3$ and

$$\left[\dot{Q}(t) + \tfrac{1}{2}Q^2(t)\right]x + \tfrac{1}{2}Q(t)R(t) + \dot{R}(t) = C(t)G(x). \tag{36}$$

To satisfy equation (36), we take $G(x) = x$ and $R(t) = 0$. Then, we have

$$C(t) = \dot{Q}(t) + \tfrac{1}{2}Q^2(t). \tag{37}$$

The main result is that there is only one special form of the equation of motion, with $m = 1$, $G(x) = x$ and $C(t)$ given by equation (37), for which a Lagrangian formulation with a non-standard Lagrangian is possible. The following proposition summarizes this important result.

Proposition 2. *The equation of motion*

$$\ddot{x} + B(t)\dot{x} + C(t)x = 0, \tag{38}$$

with $B(t)$ and $C(t)$ being continuous and differentiable functions of t, admits a Lagrangian description with the following non-standard Lagrangian:

$$\mathcal{L}(\dot{x}, x, t) = \frac{1}{\dot{x} + Q(t)x}, \tag{39}$$

if, and only if,

$$C(t) = \dot{Q}(t) + \tfrac{1}{2}Q^2(t) \tag{40}$$

and

$$Q(t) = \tfrac{2}{3}B(t). \tag{41}$$

Proof. Substitution of the Lagrangian $\mathcal{L}(\dot{x}, x, t)$ into the Euler–Lagrange equation (see equation (5)) yields equation (38) with $C(t)$ and $Q(t)$ given by equations (40) and (41), respectively; thus, proposition 2 is proved. □

3.4. Equation with standard and non-standard Lagrangians

Using equations (40) and (41), we write the explicit form of equation (38) as

$$\ddot{x} + B(t)\dot{x} + \tfrac{2}{3}\left[\dot{B}(t) + \tfrac{1}{3}B^2(t)\right]x = 0. \tag{42}$$

According to proposition 2, this equation of motion can be derived from the following non-standard Lagrangian:

$$\mathcal{L}(\dot{x}, x, t) = \frac{1}{\dot{x} + \frac{2}{3}B(t)x}. \tag{43}$$

However, if $C(t) = 2(\dot{B} + B^2/3)/3$ and $G(x) = x$, then the form of equation (42) is the same as that of equation (16). This means that the above equation of motion has also a standard Lagrangian. To derive this Lagrangian, we use equation (17) and obtain

$$L(\dot{x}, x, t) = \tfrac{1}{2}\left[\dot{x}^2 + B\dot{x}x + \tfrac{1}{6}\left(\tfrac{5}{3}B^2 - \dot{B}\right)x^2\right]\mathrm{e}^{2I_B(t)}. \tag{44}$$

An interesting result obtained here is that among all equations of motion with the time-dependent coefficients considered in this paper (see equation (8)), only equation (42) can be derived from either the standard or the non-standard Lagrangian.

4. Equations with space-dependent coefficients

4.1. Existence of standard Lagrangians

We now consider equation (9) and determine values of n for which the damping term can be removed by the following integral transformation:

$$x(t) = x_2(t)\,\mathrm{e}^{I_2(x_2)}, \tag{45}$$

where

$$I_2(x_2) = \int^{x_2} \phi_2(\tilde{x}_2)\,\mathrm{d}\tilde{x}_2, \tag{46}$$

and ϕ_2 is an arbitrary function to be determined; other restrictions on ϕ_2 and the rules for taking derivatives are the same as those already discussed for ϕ_1 below equation (11).

The transformed equation of motion can be written as

$$\ddot{x}_2 + \frac{1}{1 + x_2\phi_2}\left[2\phi_2 + x_2\phi_2^2 + x_2\left(\frac{\mathrm{d}\phi_2}{\mathrm{d}x_2}\right)\right]\dot{x}_2^2 + (1 + x_2\phi_2)^{n-1} b(x_2\,\mathrm{e}^{I_2(x_2)})\,\mathrm{e}^{(n-1)I_2(x_2)}\dot{x}_2^n$$
$$+ \frac{\mathrm{e}^{-I_2(x_2)}}{1 + x_2\phi_2} c(x_2\,\mathrm{e}^{I_2(x_2)}) g(x_2\mathrm{e}^{I_2(x_2)}) = 0. \tag{47}$$

The condition that ϕ_2 must satisfy is

$$\left[2\phi_2 + x_2\phi_2^2 + x_2\left(\frac{\mathrm{d}\phi_2}{\mathrm{d}x_2}\right)\right]\dot{x}_2^2 + \left[(1 + x_2\phi_2)^n b(x_2\,\mathrm{e}^{I_2(x_2)})\,\mathrm{e}^{(n-1)I_2(x_2)}\right]\dot{x}_2^n = 0. \tag{48}$$

Noting that $(1 + x_2\phi_2) \neq 0$ (see equation (47)), one sees that there is no ϕ_2 that satisfies equation (48) and depends solely on x_2, unless $n = 2$. The latter case is exceptional because such ϕ_2 can indeed be found [22]. To show this, we take $n = 2$ and write the condition given by equation (48) as

$$2\phi_2 + x_2\phi_2^2 + x_2\frac{\mathrm{d}\phi_2}{\mathrm{d}x_2} + (1 + x_2\phi_2)^2 b(x_2\,\mathrm{e}^{I_2(x_2)})\,\mathrm{e}^{I_2(x_2)} = 0, \tag{49}$$

and ϕ_2 becomes a function of x_2 only as required by the integral transformation (see equation (45)).

Since there is $\phi_2(x_2)$ that satisfies equation (49), we write equation (47) as

$$\ddot{x}_2 + \Omega_2^2(x_2) = 0, \tag{50}$$

with

$$\Omega_2^2(x_2) = \frac{\mathrm{e}^{-I_2(x_2)}}{1 + x_2\phi_2} c(x_2\,\mathrm{e}^{I_2(x_2)})g(x_2\,\mathrm{e}^{I_2(x_2)}). \tag{51}$$

Using the results obtained in section 2, the standard Lagrangian $L_2(\dot{x}_2, x_2)$ for equation (50) is

$$L_2(\dot{x}_2, x_2) = \frac{1}{2}\dot{x}_2^2 - \int^{x_2} \Omega_2^2(\tilde{x}_2)\,\mathrm{d}\tilde{x}_2. \tag{52}$$

After evaluating $\phi_2(x_2)$ and replacing the transformed variables by the original variables [22], one may obtain the standard Lagrangian $L(\dot{x}, x)$. The explicit form of $L(\dot{x}, x)$ is given by the following proposition.

Proposition 3. *The equation of motion*

$$\ddot{x} + b(x)\dot{x}^2 + c(x)g(x) = 0, \tag{53}$$

with $b(x)$, $c(x)$ and $g(x)$ being continuous, differentiable and integrable functions of x, admits a Lagrangian description and its standard Lagrangian is

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2\,\mathrm{e}^{2I_b(x)} - \int^{x} c(\tilde{x})g(\tilde{x})\,\mathrm{e}^{2I_b(\tilde{x})}\,\mathrm{d}\tilde{x}, \tag{54}$$

where

$$I_b(x) = \int^{x} b(\tilde{x})\,\mathrm{d}\tilde{x}. \tag{55}$$

Proof. The Lagrangian $L(\dot{x}, x)$ yields the equation of motion (see equation (53)) after $L(\dot{x}, x)$ is substituted into the Euler–Lagrange equation (see equation (5)); this shows that the proof of proposition 3 is straightforward. □

An important property of the standard Lagrangian $L(\dot{x}, x)$ is that it does not diverge as the system evolves in time. Also note that in the special case of $g(x) = x$, $L(\dot{x}, x)$ reduces to the form originally derived by Musielak *et al* [22].

4.2. Applications of standard Lagrangians

We now apply the result of proposition 3 to some physical systems. Our first example is a well-known problem in classical mechanics [9, 21]. Consider a bead sliding without friction along a wire that is bent in the shape of a parabola $z = ar^2$, where $a = \text{const}$. The wire rotates about its vertical symmetry axis with the angular velocity ω. The equation of motion for this problem is typically derived by calculating the kinetic and potential energy, writing the Lagrangian and using the Euler–Lagrange equation (see equation (5)). The result is

$$\ddot{r} + \frac{4a^2 r}{1 + 4a^2r^2}\dot{r}^2 + \frac{2ga - \omega^2}{1 + 4a^2r^2} r = 0, \tag{56}$$

where g is gravity.

Let us reverse this process by assuming that the above equation of motion is given, so we can use proposition 3 to derive the standard Lagrangian $L(\dot{r}, r)$. By comparing equation (56) to equation (53), calculating $I_b(r)$ and evaluating the integrals in equations (54) and (55), we obtain

$$L(\dot{r}, r) = \tfrac{1}{2}(\dot{r}^2 + 4a^2r^2\dot{r}^2 + \omega^2 r^2) - gar^2, \tag{57}$$

which shows that the obtained Lagrangian is the same as the original one derived directly from the kinetic and potential energy [9, 21].

Now, consider the following equation of motion:

$$\ddot{x} - \frac{\lambda x}{1+\lambda x^2}\dot{x}^2 + \frac{\alpha^2}{1+\lambda x^2}x = 0, \tag{58}$$

which has been studied in the literature as an example of a nonlinear oscillator [35–38]. Despite its similarity to equation (56), the standard Lagrangians for both equations are different. To show this, we use equation (54) and write the standard Lagrangian $L(\dot{x}, x)$ as

$$L(\dot{x}, x) = \frac{1}{2}\left(\frac{1}{1+\lambda x^2}\right)\left(\dot{x}^2 + \frac{\alpha^2}{\lambda}\right). \tag{59}$$

A comparison of this Lagrangian to that given by Carinena *et al* [37] shows that their Lagrangian is incorrect (see their equation (2)).

These results show that it is a straightforward procedure to derive standard Lagrangians for the equations of motion discussed above. As long as an equation of motion is of the form given by equation (53), its standard Lagrangian can always be derived directly from equation (54). However, it must be noted that depending on the form of the functions $c(x)$ and $g(x)$, some Lagrangian may become non-local [22].

4.3. Existence of non-standard Lagrangians

Previous studies [30, 31] showed that the non-standard Lagrangian

$$\mathcal{L}_1(\dot{x}, x) = \frac{1}{\dot{x} + \alpha x^2}, \tag{60}$$

where $\alpha =$ const, gives the following equation of motion:

$$\ddot{x} + 3\alpha x\dot{x} + \alpha^2 x^3 = 0. \tag{61}$$

For the purpose of this paper, we generalize the above results and consider the following form of the non-standard Lagrangian:

$$\mathcal{L}(\dot{x}, x) = \frac{1}{p(x)\dot{x}^n + q(x)h(x) + r(x)}, \tag{62}$$

where the functions $p(x), q(x), r(x)$ and $h(x)$ are to be determined; note that it is required that these functions are differentiable. Clearly, the Lagrangian $\mathcal{L}_1(\dot{x}, x)$ is a special case of the general Lagrangian $\mathcal{L}(\dot{x}, x)$.

To determine whether the non-standard Lagrangian $\mathcal{L}(\dot{x}, x)$ can be used to obtain the equations of motion given by equation (9), we substitute $\mathcal{L}(\dot{x}, x)$ into the Euler–Lagrange equation (see equation (5)) and obtain

$$a_1(\dot{x}, x)\ddot{x} + a_2(\dot{x}, x)\dot{x}^n + a_3(x) = 0, \tag{63}$$

where

$$a_1(\dot{x}, x) = n(n+1)p^2\dot{x}^{2(n-1)} - n(n-1)(qh+r)p\dot{x}^{n-2} \tag{64}$$

$$a_2(\dot{x}, x) = (n+1)pp'\dot{x}^n + (2n+1)(q'h + qh' + r')p - (n-1)(qh+r)p' \tag{65}$$

and

$$a_3(x) = (q'h + qh' + r')(qh+r), \tag{66}$$

where p', q', r' and h' are the derivatives of these functions with respect to x.

By comparing equation (63) to the original equation of motion (see equation (9)), one finds that a_1 must be 1, or any other constant, and that a_2 must be a function of x only. To satisfy these conditions, we take $n = 1$ and $p(x) =$ const, or simply $p(x) = 1$. Let us begin with these assumptions and then consider another case when $n = 1$ and $p(x)$ is arbitrary.

4.3.1. First case: $n = 1$ and $p(x) = 1$. In this case, equation (63) can be written as

$$\ddot{x} + a_2(x)\dot{x} + a_3(x) = 0, \tag{67}$$

where

$$a_2(x) = \tfrac{3}{2}(q'h + qh' + r') \tag{68}$$

and

$$a_3(x) = \tfrac{1}{2}(q'h + qh' + r')(qh + r). \tag{69}$$

Without any loss of generality, we may take $r(x) = 0$ and write the following conditions:

$$a_2(x) = \tfrac{3}{2}(q'h + qh') = b(x) \tag{70}$$

and

$$a_3(x) = \tfrac{1}{2}(q'h + qh')qh = c(x)g(x), \tag{71}$$

from which the functions $q(x)$ and $h(x)$ can be determined. The result is

$$q(x)h(x) = 3\frac{c(x)}{b(x)}g(x). \tag{72}$$

Knowing $q(x)h(x)$, we use equation (62) to obtain the non-standard Lagrangian $\mathcal{L}(\dot{x}, x)$. Its explicit form is given by the following proposition.

Proposition 4. *The equation of motion*

$$\ddot{x} + b(x)\dot{x} + c(x)g(x) = 0, \tag{73}$$

with $b(x)$ and $c(x)$ being continuous and differentiable functions of x, admits a Lagrangian description with the following non-standard Lagrangian:

$$\mathcal{L}(\dot{x}, x) = \frac{1}{\dot{x} + 3\frac{c(x)}{b(x)}g(x)}, \tag{74}$$

if, and only if,

$$\frac{2}{9}b(x) = \left[\frac{c(x)}{b(x)}\right]' g(x) + \left[\frac{c(x)}{b(x)}\right] g'(x). \tag{75}$$

Proof. We substitute the Lagrangian $\mathcal{L}(\dot{x}, x)$ into the Euler–Lagrange equation (see equation (5)) and obtain equation (73) only after using the condition given by equation (75); this validates proposition 4. □

4.3.2. Application. The results of proposition 4 have immediate application to the equation of motion given by equation (61). According to this equation, $b(x) = 3\alpha x$, $c(x) = \alpha^2$ and $g(x) = x^3$, with $\alpha = \text{const}$. We write equation (75) as

$$\frac{b'(x)}{b(x)}c(x)g(x) + \frac{2}{9}b^2(x) = [c(x)g(x)]' \tag{76}$$

and check that the condition is satisfied by the above functions. As a result, the non-standard Lagrangian $\mathcal{L}(\dot{x}, x)$ can be obtained directly from equation (74) and its form is

$$\mathcal{L}(\dot{x}, x) = \frac{1}{\dot{x} + \alpha x^2}. \tag{77}$$

This is the same Lagrangian as that given by equation (60), which means that our method gives correct non-standard Lagrangian for equation (61). To determine whether there are other equations of motion of the form given by equation (73) would require finding such functions $b(x)$, $c(x)$ and $g(x)$ that would satisfy equation (76); the problem is, however, out of the scope of this paper.

4.3.3. Second case: $n = 1$ and arbitrary $p(x)$. Assuming that $p(x) \neq 0$, equation (63) becomes

$$\ddot{x} + a_1(x)\dot{x}^2 + a_2(x)\dot{x} + a_3(x) = 0, \tag{78}$$

where

$$a_1(x) = \frac{p'}{p}, \tag{79}$$

$$a_2(x) = \frac{3}{2p}(q'h + qh' + r') \tag{80}$$

and

$$a_3(x) = \frac{1}{2p^2}(q'h + qh' + r')(qh + r). \tag{81}$$

A comparison of these equations to equations (67) through (69) shows that $a_1(x)$ is a new coefficient that depends only on $p(x)$ and its derivative, and that the coefficients $a_2(x)$ and $a_3(x)$ are reduced to those given by equations (68) and (69) if $p(x) = 1$.

A new equation of motion that has a non-standard Lagrangian is obtained when $a_2(x) = 0$, which requires that $q'h + qh' + r' = 0$. Since the coefficients $a_2(x)$ and $a_3(x)$ are related (see equations (80) and (81)), the condition $a_2(x) = 0$ implies that $a_3(x) = 0$. Thus, equation (78) is reduced to only two terms and it can be written as

$$\ddot{x} + a_1(x)\dot{x}^2 = 0. \tag{82}$$

Noting that the form of this equation is the same as that given by equation (53) with $c(x) = 0$, we determine $p(x)$ by taking $a_1(x) = b(x)$. The result is

$$p(x) = \mathrm{e}^{I_b(x)}, \tag{83}$$

with $I_b(x)$ given by equation (55).

To obtain a non-standard Lagrangian that can be used to derive equation (82), we must determine the functions $q(x)$, $h(x)$ and $r(x)$ from the condition $a_2(x) = 0$ or $q'h+qh'+r' = 0$. Without any loss of generality, we take $r(x) = 0$ and obtain $q'h + qh' = 0$. Obviously, the trivial solution $q(x) = 0$ or $h(x) = 0$ satisfy this condition but there are also two pairs of non-trivial solutions, namely, $q(x) = x$ and $h(x) = 1/x$, and $q(x) = 1/x$ and $h(x) = x$. Since both pairs give the same non-standard Lagrangian, it does not matter which one is chosen. Hence, proposition 5 is now in order.

Proposition 5. *The equation of motion*

$$\ddot{x} + b(x)\dot{x}^2 = 0, \tag{84}$$

with $b(x)$ being a continuous, differentiable and integrable function of x, admits a Lagrangian description with the following non-standard Lagrangian:

$$\mathcal{L}(\dot{x}, x) = \frac{1}{\dot{x}\,\mathrm{e}^{I_b(x)} + 1}, \tag{85}$$

where

$$I_b(x) = \int^x b(\tilde{x})\,\mathrm{d}\tilde{x}. \tag{86}$$

Proof. The equation of motion (see equation (84)) is obtained after the non-standard Lagrangian $\mathcal{L}(\dot{x}, x)$ is substituted into the Euler–Lagrange equation (see equation (5)); this validates proposition 5. □

4.4. Equation with standard and non-standard Lagrangians

In section 3.4, we discussed one equation of motion (see equation (42)) that can be derived from either the standard or the non-standard Lagrangian. The same is true for the equation of motion given by equation (84) for which the following standard Lagrangian (see equation (54)) is obtained:

$$L(\dot{x}, x) = \tfrac{1}{2}\dot{x}^2\, \mathrm{e}^{2I_b(x)}. \tag{87}$$

There is an interesting relationship between the non-standard and standard Lagrangians given by equations (85) and (87), respectively. By making these Lagrangians equal, one obtains a general solution of equation (84). We write $L(\dot{x}, x) = \mathcal{L}(\dot{x}, x)$ or explicitly

$$\frac{1}{2}\dot{x}^2\mathrm{e}^{2I_b(x)} = \frac{1}{\dot{x}\,\mathrm{e}^{I_b(x)} + 1}, \tag{88}$$

which is satisfied when $\dot{x}\,\mathrm{e}^{I_b(x)} = 1$. The fact that this condition is a solution to equation (84) can be seen by taking its derivative with respect to t. The general form of the solution derived from this condition is

$$\int_{x_0}^{x} \mathrm{e}^{I_b(\tilde{x})}\,\mathrm{d}\tilde{x} = t. \tag{89}$$

To understand the physical meaning of this solution, we consider two examples in which $b(x) = 1/x$ and $b(x) = -1/x$. Specific solutions corresponding to these two cases are $x(t) = \sqrt{2t}$ and $x(t) = \mathrm{e}^t$ respectively. The solutions show that neither the term $\dot{x}^2/x$ nor the term $-\dot{x}^2/x$ are 'dissipative'. Instead the terms make both systems unstable because $x \to \infty$ as $t \to \infty$.

The origin of this instability can be determined by writing the autonomous system of equations for equation (84). We introduce $y = \dot{x}$ and $\dot{y} = \ddot{x}$, and obtain

$$\dot{x} = y \qquad \text{and} \qquad \dot{y} = -b_a(x)y^2. \tag{90}$$

Since the right-hand sides of these equations are zero for $y = 0$, there is no restriction on x and, as a result, no stable critical point exists in the phase space (x, y). This lack of stable critical point is responsible for the diverging solutions.

5. Discussion

The obtained results show that there are at least three distinct classes of equations of motion that admit a Lagrangian description and that two of these classes can be classified as general and one as special. All equations of motion that belong to the two general classes have standard Lagrangians; however, the equations of the special class can only be derived from non-standard Lagrangians. An interesting result is that each general class has also a subset of equations with non-standard Lagrangians. In the following, the three classes of equations of motion and the conditions required for the existence of standard and non-standard Lagrangians are discussed.

We considered the equations of motion with time-dependent coefficients and different powers of dissipative terms, and showed that only those equations that are of the form $\ddot{x} + B(t)\dot{x} + C(t)G(x) = 0$ have standard Lagrangians (see proposition 1). This means that there is one general class of equations of motion with time-dependent coefficients that admit a Lagrangian description. If the coefficients $B(t)$ and $C(t)$ are related by $C(t) = 2[\dot{B}(t) + B^2(t)/3]/3$ and the function $G(x) = x$, then all equations of motion that satisfy these conditions also have non-standard Lagrangians (see proposition 2) and form a subset within the general class. The relationship between $B(t)$ and $C(t)$ and the form of the function $G(x)$ become the conditions for the existence of non-standard Lagrangians for this class of equations of motion.

Among all equations of motion with space-dependent coefficients considered in this paper, only those that are described by $\ddot{x} + b(x)\dot{x}^2 + c(x)g(x) = 0$ have standard Lagrangians (see proposition 3). Hence, these equations form the general class of equations of motion with space-dependent coefficients. Within this class, there is a subset of equations with $c(x) = 0$ for which non-standard Lagrangians exist (see proposition 5); note that the condition $c(x) = 0$ is necessary for the existence of non-standard Lagrangians. An interesting result is that there is a relationship between the standard and non-standard Lagrangians and that this relationship can be used to obtain a general solution to any equation of this subset.

The equations of motion that form the special class are represented by $\ddot{x} + b(x)\dot{x} + c(x)g(x) = 0$. According to proposition 4, these equations have only non-standard Lagrangians and the condition for the existence of these Lagrangians is $b'cg/b + 2b^2/9 = (cg)'$, where $'$ denotes the derivative with respect to x. The relationship between the coefficients $b(x)$ and $c(x)$, and the function $g(x)$, seems to be highly restricted and yet there is at least one equation of motion that satisfy it (see section 4.2).

The fact that some equations of motion can be derived from standard and/or non-standard Lagrangians has important physical implications. The main advantage of a Lagrangian description is that it guarantees a well-formulated dynamical problem and, therefore, it is central to any physical theory of particles or fields that is developed from Hamilton's principle, also known as the action principle [5].

For a physical theory with a given Lagrangian, a complete set of equations of motion with dynamical variables that represent positions and momenta, like in Newtonian dynamics, is automatically obtained from the Euler–Lagrange equations. The theory displays symmetries and continuity conditions and becomes the starting point of any theoretical analysis [6]. In other words, derivation of a Lagrangian from which an equation of motion can be obtained by means of the action principle is the most concise description of a dynamical system.

The Lagrangian formulation is also advantageous for including additional forces, studying the stability of solutions, applying perturbation theory, establishing the existence of resonances and calculating Lyapunov exponents [1–3, 5]. In addition, dynamical systems admitting the Lagrange description are endowed with the preserved energy function [6, 31].

Another important benefit of the Lagrangian formulation is that once the Lagrangian and the momenta of a given dynamical system are known, then the Hamiltonian can be determined and the system becomes amenable to the techniques of quantum mechanics; this cannot be done by using Newtonian dynamics [17]. Many attempts have been made to apply variational principles to quantization of conservative and non-conservative dynamical systems [43–50]. The results obtained in this paper clearly identify the equations of motion for which the quantization techniques can be directly used.

All the above benefits of the Lagrangian formalism provide compelling arguments for identifying equations of motion that can be derived from the action principle. As is shown in this paper, there are the two general classes of equations of motion that admit standard Lagrangians and one special class of equations of motion that has non-standard Lagrangians. The general classes include many standard oscillatory systems [10, 11, 23, 24, 36], some basic equations of mathematical physics such as Bessel, Legendre, Laguerre, Hermite, Chebyshev, Jacobi, hypergeometric and confluent hypergeometric equations [1, 42], and the Lane–Emden equation [41]. These diverse equations describe a broad range of problems in different areas of classical and quantum physics. The obtained results show that all these equations admit the Lagrangian description and, most importantly, that the methods developed in this paper can be used to derive their corresponding Lagrangians.

6. Conclusions

The considered dynamical systems are described by equations of motion with the first-order time derivative (dissipative) terms of even and odd powers, and coefficients varying either in time or in space. Methods to obtain standard and non-standard Lagrangians were presented and used to identify classes of equations of motion that can be derived from these Lagrangians. It was shown that at least three distinct classes of equations of motion exist and that two of these classes can be classified as general and one as special. All equations of motion that belong to the two general classes have standard Lagrangians; however, the equations of the special class can only be derived from non-standard Lagrangians. An interesting result is that each general class has also a subset of equations with non-standard Lagrangians.

The obtained results show that the existence of standard Lagrangians is limited to the equations of motion with either time-dependent coefficients and linear dissipative terms or space-dependent coefficients and quadratic dissipative terms. However, the equations of motion that can be derived from non-standard Lagrangians range from the equations with time-dependent coefficients and linear dissipative terms to the equations that have space-dependent coefficients and either linear or quadratic dissipative terms. A new result is that the forms of equations with non-standard Lagrangians are restricted by the conditions that must be satisfied by the coefficients and functions of these equations.

The main results were applied to several dynamical systems and to some basic equations of mathematical physics. Specific examples included a nonlinear and damped pendulum whose length increases in time, the Lane–Emden equation, the Bessel and Laguerre equations and some nonlinear oscillators. By deriving Lagrangians for these examples, it was shown that the methods to obtain standard and non-standard Lagrangians can be applied to a broad range of physical problems.

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