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On flows of viscoelastic fluids under threshold-slip boundary conditions

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Abstract. We investigate a boundary-value problem for the steady isothermal flow of an incompressible viscoelastic fluid of Oldroyd type in a 3D bounded domain with impermeable walls. We use the Fujita threshold-slip boundary condition. This condition states that the fluid can slip along a solid surface when the shear stresses reach a certain critical value; otherwise the slipping velocity is zero. Assuming that the flow domain is not rotationally symmetric, we prove an existence theorem for the corresponding slip problem in the framework of weak solutions. The proof uses methods for solving variational inequalities with pseudo-monotone operators and convex functionals, the method of introduction of auxiliary viscosity, as well as a passage-to-limit procedure based on energy estimates of approximate solutions, Korn's inequality, and compactness arguments. Also, some properties and estimates of weak solutions are established.

1. Introduction

We consider the system of nonlinear partial differential equations governing steady-state flows of an incompressible viscoelastic fluid of Oldroyd type [1] in a bounded domain $\Omega \subset \mathbf{R}^3$ with sufficiently smooth boundary $\partial\Omega$ under the Fujita threshold-slip boundary condition [2]:

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}\mathbf{S} + \nabla\pi = \rho\mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{S} = \mathbf{E} + (1-\omega)\mu\mathbf{D}(\mathbf{u}) \quad \text{in } \Omega, \tag{1.2}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\mathbf{E} + \lambda(\mathbf{u}\cdot\nabla)\mathbf{E} = \omega\mu\mathbf{D}(\mathbf{u}) \quad \text{in } \Omega, \tag{1.4}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

$$|(\mathbf{Sn})_{\tan}| \le q \quad \text{on } \partial\Omega, \tag{1.6}$$

$$|(\mathbf{Sn})_{\tan}| < q \implies \mathbf{u}_{\tan} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{1.7}$$

$$|(\mathbf{Sn})_{\tan}| = q \implies \mathbf{u}_{\tan} \upharpoonleft\!\downharpoonright (\mathbf{Sn})_{\tan} \quad \text{on } \partial\Omega, \tag{1.8}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the velocity at the point $\mathbf{x}\in\Omega$, $\mathbf{S} = \mathbf{S}(\mathbf{x})$ is the extra-stress tensor, $\pi = \pi(\mathbf{x})$ is the pressure, $\mathbf{f} = \mathbf{f}(\mathbf{x})$ denotes the body force, $\mathbf{E} = \mathbf{E}(\mathbf{x})$ is the elastic part of the stress tensor, $\mathbf{D}(\mathbf{u})$ is the strain-rate tensor,

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}),$$

$\mu > 0$ is the viscosity coefficient, $\rho > 0$ is the density of the fluid, $\lambda > 0$ denotes the stress relaxation time, $\omega \in (0, 1)$ is a dimensionless parameter, $q = q(\mathbf{x}) \geq 0$ is the threshold of the tangential stress, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ denotes the outward-directed unit normal vector to $\partial\Omega$, and $(\,\cdot\,)_{\tan}$ stands for the tangential component of a vector, i.e.,

$$\mathbf{u}_{\tan} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}.$$

The symbol $\updownarrows$ is used to denote oppositely directed vectors. In other words, for any $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$

$$\mathbf{a} \updownarrows \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} + |\mathbf{a}||\mathbf{b}| = 0,$$

where $|\cdot|$ denotes the Euclidean norm.

The unknowns in system (1.1)–(1.8) are $\mathbf{u}$, $\mathbf{S}$, $\mathbf{E}$, and π, while all other quantities are assumed to be given.

In the present paper, we introduce the concept of weak solutions of problem (1.1)–(1.8) by the use of variational inequalities. Under appropriate conditions on the data, we prove a theorem on the existence and properties of weak solutions to (1.1)–(1.8).

The main novelty of our results is that the phenomenon of *threshold slip* at solid surfaces is taken into account. This study continues the series of articles written by the author [3–6], devoted to slip problems for flows of viscoelastic Oldroyd fluids.

Remark 1. Starting from pioneering works of M. Renardy [7] and C. Guillopé & J.-C. Saut [8], the equations of motion of Oldroyd-type fluids have been studied by many specialists (see, e.g., [9–19] and the references therein). A detailed analysis of mathematical results for the Oldroyd model and other similar non-Newtonian models can be found in the review article [20].

Remark 2. If we make $\lambda = \omega = 0$ formally, system (1.1)–(1.8) reduces to the classical Navier–Stokes equations with threshold-slip boundary conditions. For such type of slip problems, well-posedness results were obtained by H. Fujita [2]; in this regard, see also the papers [21, 22].

2. Notations

Denote by $\mathbf{R}^{3\times 3}_{\mathrm{sym}}$ the space symmetric matrices of size 3×3 with the following scalar product:

$$\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{3} A_{ij} B_{ij}$$

for the matrices $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$.

We use the standard notation

$$\mathbf{L}^p(\Omega, \mathbf{R}^d), \quad \mathbf{H}^m(\Omega, \mathbf{R}^d) = \mathbf{W}^{m,2}(\Omega, \mathbf{R}^d)$$

for the Lebesgue and Sobolev spaces of functions defined on Ω and with values in $\mathbf{R}^d$. For a detailed treatment of these spaces, consult, e.g., [23].

As is well known, it is possible to ascribe a value at the boundary (the trace) to vector functions from the space $\mathbf{H}^1(\Omega, \mathbf{R}^d)$. For that we use the trace operator [24]

$$\gamma_0\colon \mathbf{H}^1(\Omega, \mathbf{R}^d) \to \mathbf{H}^{1/2}(\partial\Omega, \mathbf{R}^d)$$

such that $\gamma_0 \mathbf{w} = \mathbf{w}|_{\partial\Omega}$ when $\mathbf{w} : \bar{\Omega} \to \mathbf{R}^d$ is a smooth vector function. In the sequel, for $\mathbf{v} \in \mathbf{H}^1(\Omega, \mathbf{R}^d)$, we shall write simply $\mathbf{v}|_{\partial\Omega}$ instead of $\gamma_0 \mathbf{v}$.

Denote by $\mathbf{C}^\infty_0(\Omega, \mathbf{R}^{3\times 3}_{\mathrm{sym}})$ the set of infinitely differentiable functions with support contained in Ω and with values in $\mathbf{R}^{3\times 3}_{\mathrm{sym}}$.

Let us introduce the function spaces needed below:

$$\mathbf{H}_0^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}}) = \text{the closure of the set } \mathbf{C}_0^\infty(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}}) \text{ in the space } \mathbf{H}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}}),$$

$$\mathbf{Q}(\Omega,\mathbf{R}^3) = \{\mathbf{v}\in\mathbf{C}^\infty(\bar\Omega,\mathbf{R}^3):\ \ \nabla\cdot\mathbf{v}=0,\ \ \mathbf{v}|_{\partial\Omega}\cdot\mathbf{n}=0\},$$
$$\mathbf{X}(\Omega,\mathbf{R}^3) = \text{the closure of the set } \mathbf{Q}(\Omega,\mathbf{R}^3) \text{ in the space } \mathbf{H}^1(\Omega,\mathbf{R}^3),$$
$$\mathbf{L}^2_+(\partial\Omega,\mathbf{R}) = \{v\in L^2(\partial\Omega,\mathbf{R}):\ \ v(\mathbf{x})\geq 0 \ \text{ for a.e. } \mathbf{x}\in\partial\Omega\}.$$

Assuming that $\Omega\subset\mathbf{R}^3$ is not a body of revolution, we define the norm in $\mathbf{X}(\Omega,\mathbf{R}^3)$ by the formula
$$\|\mathbf{v}\|^2_{\mathbf{X}(\Omega,\mathbf{R}^3)} = \int\limits_\Omega |\mathbf{D}(\mathbf{v})|^2\,\mathbf{dx}.$$
It follows from Korn's inequality (see the Appendix, Proposition 2 (a)) that the norm $\|\cdot\|_{\mathbf{X}(\Omega,\mathbf{R}^3)}$ is equivalent to the norm induced from the Sobolev space $\mathbf{H}^1(\Omega,\mathbf{R}^3)$.

3. Weak (variational) formulation of problem (1.1)–(1.8)

Assume that the following conditions hold:

$$0<\lambda,\quad 0<\mu,\quad 0<\omega<1,\quad \mathbf{f}\in\mathbf{L}^2(\Omega,\mathbf{R}^3),\quad q\in\mathbf{L}^2_+(\partial\Omega,\mathbf{R}). \tag{3.1}$$

Definition. We say that a triplet

$$(\mathbf{u},\mathbf{S},\mathbf{E})\in\mathbf{X}(\Omega,\mathbf{R}^3)\times\mathbf{L}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}})\times\mathbf{L}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}})$$

is a *weak solution* of problem (1.1)–(1.8) if the condition (1.2) is valid and

$$\begin{aligned}&-\rho\sum_{i=1}^3\int\limits_\Omega u_i\mathbf{u}\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,\mathbf{dx}+\int\limits_\Omega\mathbf{E}:\mathbf{D}(\mathbf{w})\,\mathbf{dx}-\frac{1}{\omega\mu}\int\limits_\Omega|\mathbf{E}|^2\,\mathbf{dx}\\ &+(1-\omega)\mu\int\limits_\Omega\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{w}-\mathbf{u})\,\mathbf{dx}+\int\limits_{\partial\Omega}q|\mathbf{w}|\,ds-\int\limits_{\partial\Omega}q|\mathbf{u}|\,ds\\ &\geq\rho\int\limits_\Omega\mathbf{f}\cdot(\mathbf{w}-\mathbf{u})\,\mathbf{dx}\quad\forall\,\mathbf{w}\in\mathbf{X}(\Omega,\mathbf{R}^3),\end{aligned} \tag{3.2}$$

$$\int\limits_\Omega\mathbf{E}:\mathbf{F}\,\mathbf{dx}-\lambda\sum_{i=1}^3\int\limits_\Omega u_i\mathbf{E}:\frac{\partial\mathbf{F}}{\partial x_i}\,\mathbf{dx}=\omega\mu\int\limits_\Omega\mathbf{D}(\mathbf{u}):\mathbf{F}\,\mathbf{dx}\quad\forall\,\mathbf{F}\in\mathbf{H}_0^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}}). \tag{3.3}$$

Remark 3. Let us explain how relations (3.2) and (3.3) arise. Suppose that $(\mathbf{u},\mathbf{S},\mathbf{E},\pi)$ is a classical solution to problem (1.1)–(1.8) and $\mathbf{w}\in\mathbf{X}(\Omega,\mathbf{R}^3)$. If we take the scalar product of both sides of (1.1) by the vector function $\mathbf{w}-\mathbf{u}$ and integrate by parts over the domain Ω, then we obtain

$$-\rho\sum_{i=1}^3\int\limits_\Omega u_i\mathbf{u}\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,\mathbf{dx}+\int\limits_\Omega\mathbf{S}:\mathbf{D}(\mathbf{w}-\mathbf{u})\,\mathbf{dx}-\int\limits_{\partial\Omega}(\mathbf{Sn})\cdot(\mathbf{w}-\mathbf{u})\,ds=\rho\int\limits_\Omega\mathbf{f}\cdot(\mathbf{w}-\mathbf{u})\,\mathbf{dx}.$$

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Substituting (1.2) into the second term of the last equality, we get

$$-\rho\sum_{i=1}^{3}\int_{\Omega} u_i\mathbf{u}\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,\mathbf{dx}+\int_{\Omega}\mathbf{E}:\mathbf{D}(\mathbf{w}-\mathbf{u})\,\mathbf{dx}+(1-\omega)\mu\int_{\Omega}\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{w}-\mathbf{u})\,\mathbf{dx}$$

$$-\int_{\partial\Omega}(\mathbf{Sn})\cdot(\mathbf{w}-\mathbf{u})\,ds=\rho\int_{\Omega}\mathbf{f}\cdot(\mathbf{w}-\mathbf{u})\,\mathbf{dx}. \tag{3.4}$$

Further, taking the scalar product of (1.4) with $\mathbf{E}$ and integrating over the domain Ω, we obtain

$$\int_{\Omega}|\mathbf{E}|^2\,\mathbf{dx}=\omega\mu\int_{\Omega}\mathbf{D}(\mathbf{u}):\mathbf{E}\,\mathbf{dx}, \tag{3.5}$$

where we used the following equality

$$\sum_{i=1}^{3}\int_{\Omega}u_i\frac{\partial\mathbf{E}}{\partial x_i}:\mathbf{E}\,\mathbf{dx}=0.$$

Combining (3.4) and (3.5), we find that

$$-\rho\sum_{i=1}^{3}\int_{\Omega} u_i\mathbf{u}\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,\mathbf{dx}+\int_{\Omega}\mathbf{E}:\mathbf{D}(\mathbf{w})\,\mathbf{dx}-\frac{1}{\omega\mu}\int_{\Omega}|\mathbf{E}|^2\,\mathbf{dx}$$

$$+(1-\omega)\mu\int_{\Omega}\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{w}-\mathbf{u})\,\mathbf{dx}-\int_{\partial\Omega}(\mathbf{Sn})\cdot(\mathbf{w}-\mathbf{u})\,ds=\rho\int_{\Omega}\mathbf{f}\cdot(\mathbf{w}-\mathbf{u})\,\mathbf{dx}. \tag{3.6}$$

It can be shown that the system of boundary conditions (1.6)–(1.8) is equivalent to the following system:

$$|(\mathbf{Sn})_{\tan}|\le q \quad \text{on } \partial\Omega,$$

$$(\mathbf{Sn})_{\tan}\cdot\mathbf{u}_{\tan}+q|\mathbf{u}_{\tan}|=0 \quad \text{on } \partial\Omega.$$

Employing these relations and the Cauchy–Schwarz inequality, we deduce

$$-\int_{\partial\Omega}(\mathbf{Sn})\cdot(\mathbf{w}-\mathbf{u})\,ds=\int_{\partial\Omega}(\mathbf{Sn})_{\tan}\cdot(\mathbf{u}-\mathbf{w})_{\tan}\,ds$$

$$=-\int_{\partial\Omega}q|\mathbf{u}_{\tan}|+(\mathbf{Sn})_{\tan}\cdot\mathbf{w}_{\tan}\,ds\le-\int_{\partial\Omega}q|\mathbf{u}|\,ds+\int_{\partial\Omega}q|\mathbf{w}|\,ds.$$

If we combine this with (3.6), we obtain (3.2).

On taking the scalar product of both the left-hand and right-hand sides of (1.4) with a vector function $\mathbf{F}\in\mathbf{H}_0^2(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}})$ and integrating over Ω, we get (3.3).

Finally, it should be also noted that if a weak solution $(\mathbf{u},\mathbf{S},\mathbf{E})$ of problem (1.1)–(1.8) is sufficiently regular, then there exists a function π such that $(\mathbf{u},\mathbf{S},\mathbf{E},\pi)$ is a classical solution to problem (1.1)–(1.8).

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4. Main results

We are now in a position to state our main results.

Theorem. *Assume that $\Omega \subset \mathbf{R}^3$ is not a body of revolution and relations* (3.1) *hold. Under these conditions, we have*

(i) *boundary-value problem* (1.1)–(1.8) *has at least one weak solution* $(\mathbf{u}, \mathbf{S}, \mathbf{E})$ *such that*

$$(1-\omega)\mu \int\limits_{\Omega} |\mathbf{D}(\mathbf{u})|^2 \, \mathbf{dx} + \frac{1}{\omega\mu} \int\limits_{\Omega} |\mathbf{E}|^2 \, \mathbf{dx} + \int\limits_{\partial\Omega} q|\mathbf{u}| \, ds \leq \rho \int\limits_{\Omega} \mathbf{f} \cdot \mathbf{u} \, \mathbf{dx}; \tag{4.1}$$

(ii) *the set of weak solutions to problem* (1.1)–(1.8) *is sequentially weakly closed in the space*

$$\mathbf{X}(\Omega, \mathbf{R}^3) \times \mathbf{L}^2(\Omega, \mathbf{R}^{3\times3}_{\mathrm{sym}}) \times \mathbf{L}^2(\Omega, \mathbf{R}^{3\times3}_{\mathrm{sym}});$$

(iii) *if* $(\mathbf{u}_0, \mathbf{S}_0, \mathbf{E}_0)$ *is a weak solution of problem* (1.1)–(1.8) *such that*

$$(\mathbf{u}_0, \mathbf{S}_0, \mathbf{E}_0) \in \mathbf{X}(\Omega, \mathbf{R}^3) \times \mathbf{L}^2(\Omega, \mathbf{R}^{3\times3}_{\mathrm{sym}}) \times \mathbf{H}^1(\Omega, \mathbf{R}^{3\times3}_{\mathrm{sym}}),$$

then

$$(1-\omega)\mu \int\limits_{\Omega} |\mathbf{D}(\mathbf{u}_0)|^2 \, \mathbf{dx} + \frac{1}{\omega\mu} \int\limits_{\Omega} |\mathbf{E}_0|^2 \, \mathbf{dx} + \int\limits_{\partial\Omega} q|\mathbf{u}_0| \, ds = \rho \int\limits_{\Omega} \mathbf{f} \cdot \mathbf{u}_0 \, \mathbf{dx}; \tag{4.2}$$

(iv) *if there exists a potential* $h \in \mathbf{H}^1(\Omega, \mathbf{R})$ *such that* $\mathbf{f} = \nabla h$, *then* $(\mathbf{u}_0, \mathbf{S}_0, \mathbf{E}_0) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

5. Sketch of the proof of the theorem

The proof of statement (i) is derived in four steps.

Step 1. For a fixed $\mathbf{u} \in \mathbf{X}(\Omega, \mathbf{R}^3)$, we consider the following linear problem depending on a positive parameter ε:

Find $\mathbf{E} \in \mathbf{H}^2_0(\Omega, \mathbf{R}^{3\times3}_{\mathrm{sym}})$ *such that*

$$\varepsilon \int\limits_{\Omega} \Delta\mathbf{E} : \Delta\mathbf{F} \, \mathbf{dx} + \int\limits_{\Omega} \mathbf{E} : \mathbf{F} \, \mathbf{dx} - \lambda \sum_{i=1}^{3} \int\limits_{\Omega} u_i \mathbf{E} : \frac{\partial \mathbf{F}}{\partial x_i} \, \mathbf{dx}$$

$$= \omega\mu \int\limits_{\Omega} \mathbf{D}(\mathbf{u}) : \mathbf{F} \, \mathbf{dx} \quad \forall\, \mathbf{F} \in \mathbf{H}^2_0(\Omega, \mathbf{R}^{3\times3}_{\mathrm{sym}}). \tag{5.1}$$

For any $\varepsilon > 0$ and $\mathbf{u} \in \mathbf{X}(\Omega, \mathbf{R}^3)$, problem (5.1) has a unique solution. Denote by $\mathbf{T}_\varepsilon(\mathbf{u})$ the solution of this problem. It can be proved that the operator

$$\mathbf{T}_\varepsilon \colon \mathbf{X}(\Omega, \mathbf{R}^3) \to \mathbf{H}^2_0(\Omega, \mathbf{R}^{3\times3}_{\mathrm{sym}})$$

is completely continuous.

Step 2. Let us consider one more auxiliary problem:

Find $\mathbf{u} \in \mathbf{X}(\Omega, \mathbf{R}^3)$ *such that*

$$-\rho \sum_{i=1}^{3} \int\limits_{\Omega} u_i \mathbf{u} \cdot \frac{\partial \mathbf{w}}{\partial x_i} \, \mathbf{dx} + \int\limits_{\Omega} \mathbf{T}_\varepsilon(\mathbf{u}) : \mathbf{D}(\mathbf{w} - \mathbf{u}) \, \mathbf{dx} + (1-\omega)\mu \int\limits_{\Omega} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w} - \mathbf{u}) \, \mathbf{dx}$$

$$+ \int\limits_{\partial\Omega} q|\mathbf{w}| \, ds - \int\limits_{\partial\Omega} q|\mathbf{u}| \, ds \geq \rho \int\limits_{\Omega} \mathbf{f} \cdot (\mathbf{w} - \mathbf{u}) \, \mathbf{dx} \quad \forall\, \mathbf{w} \in \mathbf{X}(\Omega, \mathbf{R}^3). \tag{5.2}$$

By $[\mathbf{X}(\Omega, \mathbf{R}^3)]^*$ denote the dual space of $\mathbf{X}(\Omega, \mathbf{R}^3)$ and introduce the following operators:

$$\mathbf{M}\colon \mathbf{X}(\Omega, \mathbf{R}^3) \to [\mathbf{X}(\Omega, \mathbf{R}^3)]^*,$$

$$\langle \mathbf{M}(\mathbf{u}), \mathbf{w} \rangle = (1-\omega)\mu \int\limits_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w})\, \mathbf{dx},$$

$$\mathbf{K}_\varepsilon\colon \mathbf{X}(\Omega, \mathbf{R}^3) \to [\mathbf{X}(\Omega, \mathbf{R}^3)]^*,$$

$$\langle \mathbf{K}_\varepsilon(\mathbf{u}), \mathbf{w} \rangle = -\rho \sum_{i=1}^3 \int\limits_\Omega u_i \mathbf{u} \cdot \frac{\partial \mathbf{w}}{\partial x_i}\, \mathbf{dx} + \int\limits_\Omega \mathbf{T}_\varepsilon(\mathbf{u}) : \mathbf{D}(\mathbf{w})\, \mathbf{dx},$$

$$J\colon \mathbf{X}(\Omega, \mathbf{R}^3) \to \mathbf{R},$$

$$J(\mathbf{u}) = \int\limits_{\partial\Omega} q|\mathbf{u}|\, ds.$$

In these notations, problem (5.2) can be written as the following variational inequality

$$\langle \mathbf{M}(\mathbf{u}) + \mathbf{K}_\varepsilon(\mathbf{u}) - \rho\mathbf{f}, \mathbf{w} - \mathbf{u} \rangle + J(\mathbf{w}) - J(\mathbf{u}) \geq 0 \quad \forall\, \mathbf{w} \in \mathbf{X}(\Omega, \mathbf{R}^3). \tag{5.3}$$

Observe that $\mathbf{M}$ is a monotone operator and

$$\langle \mathbf{M}(\mathbf{u}), \mathbf{u} \rangle = (1-\omega)\mu \|\mathbf{u}\|^2_{\mathbf{X}(\Omega, \mathbf{R}^3)}.$$

Moreover, it can be shown that the sum $\mathbf{M} + \mathbf{K}_\varepsilon$ is a pseudo-monotone operator and

$$\frac{\langle \mathbf{M}(\mathbf{u}) + \mathbf{K}_\varepsilon(\mathbf{u}), \mathbf{u} \rangle + J(\mathbf{u})}{\|\mathbf{u}\|_{\mathbf{X}(\Omega, \mathbf{R}^3)}} \to +\infty$$

as $\|\mathbf{u}\|_{\mathbf{X}(\Omega, \mathbf{R}^3)} \to +\infty$. Then from the existence results for variational inequalities with pseudo-monotone operators and convex functionals (see the Appendix, Proposition 1), it follows that for any $\varepsilon > 0$ inequality (5.3) has a solution $\mathbf{u}_\varepsilon \in \mathbf{X}(\Omega, \mathbf{R}^3)$.

Step 3. Let $\varepsilon_n > 0$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. We denote by $\mathbf{u}_{\varepsilon_n}$ the solution of (5.3) corresponding to $\varepsilon = \varepsilon_n$. Set $\mathbf{E}_{\varepsilon_n} = \mathbf{T}_{\varepsilon_n}(\mathbf{u}_{\varepsilon_n})$. Obviously, we have

$$\varepsilon_n \int\limits_\Omega \Delta \mathbf{E}_{\varepsilon_n} : \Delta \mathbf{F}\, \mathbf{dx} + \int\limits_\Omega \mathbf{E}_{\varepsilon_n} : \mathbf{F}\, \mathbf{dx} - \lambda \sum_{i=1}^3 \int\limits_\Omega u_{\varepsilon_n i} \mathbf{E}_{\varepsilon_n} : \frac{\partial \mathbf{F}}{\partial x_i}\, \mathbf{dx}$$

$$= \omega\mu \int\limits_\Omega \mathbf{D}(\mathbf{u}_{\varepsilon_n}) : \mathbf{F}\, \mathbf{dx} \quad \forall\, \mathbf{F} \in \mathbf{H}^2_0(\Omega, \mathbf{R}^{3\times 3}_{\mathrm{sym}}), \tag{5.4}$$

$$-\rho \sum_{i=1}^3 \int\limits_\Omega u_{\varepsilon_n i} \mathbf{u}_{\varepsilon_n} \cdot \frac{\partial \mathbf{w}}{\partial x_i}\, \mathbf{dx} + \int\limits_\Omega \mathbf{E}_{\varepsilon_n} : \mathbf{D}(\mathbf{w} - \mathbf{u}_{\varepsilon_n})\, \mathbf{dx}$$

$$+(1-\omega)\mu \int\limits_\Omega \mathbf{D}(\mathbf{u}_{\varepsilon_n}) : \mathbf{D}(\mathbf{w} - \mathbf{u}_{\varepsilon_n})\, \mathbf{dx} + \int\limits_{\partial\Omega} q|\mathbf{w}|\, ds - \int\limits_{\partial\Omega} q|\mathbf{u}_{\varepsilon_n}|\, ds$$

$$\geq \rho \int\limits_\Omega \mathbf{f} \cdot (\mathbf{w} - \mathbf{u}_{\varepsilon_n})\, \mathbf{dx} \quad \forall\, \mathbf{w} \in \mathbf{X}(\Omega, \mathbf{R}^3). \tag{5.5}$$

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Using (5.4) and (5.5), we establish estimates independent of ε_n:

$$\|\mathbf{u}_{\varepsilon_n}\|_{\mathbf{X}(\Omega,\mathbf{R}^3)} \le \frac{\rho}{(1-\omega)\mu}\|\mathbf{f}\|_{[\mathbf{X}(\Omega,\mathbf{R}^3)]^*}, \tag{5.6}$$

$$\|\mathbf{E}_{\varepsilon_n}\|_{\mathbf{L}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}})} \le \rho\sqrt{\frac{\omega}{1-\omega}}\|\mathbf{f}\|_{[\mathbf{X}(\Omega,\mathbf{R}^3)]^*}. \tag{5.7}$$

Step 4. We want to pass to the limit $n \to \infty$ in (5.4) and (5.5). Taking into account the estimates (5.6) and (5.7), we can assume without loss of generality that

$$\lim_{n\to\infty} \mathbf{u}_{\varepsilon_n} = \mathbf{u}_* \quad \text{weakly in } \mathbf{X}(\Omega,\mathbf{R}^3), \tag{5.8}$$

$$\lim_{n\to\infty} \mathbf{E}_{\varepsilon_n} = \mathbf{E}_* \quad \text{weakly in } \mathbf{L}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}}) \tag{5.9}$$

for some $\mathbf{u}_* \in \mathbf{X}(\Omega,\mathbf{R}^3)$ and $\mathbf{E}_* \in \mathbf{L}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}})$.

The weak convergence (5.8) and the compactness of the embeddings

$$\mathbf{i}\colon \mathbf{X}(\Omega,\mathbf{R}^3) \to \mathbf{L}^4(\Omega,\mathbf{R}^3), \quad \gamma_0\colon \mathbf{X}(\Omega,\mathbf{R}^3) \to \mathbf{L}^2(\partial\Omega,\mathbf{R}^3)$$

imply that

$$\lim_{n\to\infty} \mathbf{u}_{\varepsilon_n} = \mathbf{u}_* \quad \text{strongly in } \mathbf{L}^4(\Omega,\mathbf{R}^3), \tag{5.10}$$

$$\lim_{n\to\infty} \mathbf{u}_{\varepsilon_n}|_{\partial\Omega} = \mathbf{u}_*|_{\partial\Omega} \quad \text{strongly in } \mathbf{L}^2(\partial\Omega,\mathbf{R}^3). \tag{5.11}$$

Putting $\mathbf{F} = \mathbf{E}_{\varepsilon_n}$ in (5.4), we see that

$$\int\limits_\Omega \mathbf{D}(\mathbf{u}_{\varepsilon_n}) : \mathbf{E}_{\varepsilon_n}\,\mathbf{dx} \ge \frac{1}{\omega\mu}\int\limits_\Omega |\mathbf{E}_{\varepsilon_n}|^2\,\mathbf{dx}.$$

If we combine this inequality with (5.5), we obtain

$$\begin{aligned}&-\rho\sum_{i=1}^3\int\limits_\Omega u_{\varepsilon_n i}\mathbf{u}_{\varepsilon_n}\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,\mathbf{dx} + \int\limits_\Omega \mathbf{E}_{\varepsilon_n} : \mathbf{D}(\mathbf{w})\,\mathbf{dx}\\ &-\frac{1}{\omega\mu}\int\limits_\Omega |\mathbf{E}_{\varepsilon_n}|^2\,\mathbf{dx} + (1-\omega)\mu\int\limits_\Omega \mathbf{D}(\mathbf{u}_{\varepsilon_n}) : \mathbf{D}(\mathbf{w}-\mathbf{u}_{\varepsilon_n})\,\mathbf{dx}\\ &+\int\limits_{\partial\Omega} q|\mathbf{w}|\,ds - \int\limits_{\partial\Omega} q|\mathbf{u}_{\varepsilon_n}|\,ds \ge \rho\int\limits_\Omega \mathbf{f}\cdot(\mathbf{w}-\mathbf{u}_{\varepsilon_n})\,\mathbf{dx} \quad \forall\,\mathbf{w}\in\mathbf{X}(\Omega).\end{aligned} \tag{5.12}$$

Using (5.8)–(5.11) and the following inequalities

$$\int\limits_\Omega |\mathbf{D}(\mathbf{u}_*)|^2\,\mathbf{dx} \le \liminf_{n\to\infty}\int\limits_\Omega |\mathbf{D}(\mathbf{u}_{\varepsilon_n})|^2\,\mathbf{dx}, \tag{5.13}$$

$$\int\limits_\Omega |\mathbf{E}_*|^2\,\mathbf{dx} \le \liminf_{n\to\infty}\int\limits_\Omega |\mathbf{E}_{\varepsilon_n}|^2\,\mathbf{dx}, \tag{5.14}$$

we pass to the lower limit in (5.12). The result is

$$-\rho\sum_{i=1}^3\int\limits_\Omega u_{*i}\mathbf{u}_*\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,\mathbf{dx} + \int\limits_\Omega \mathbf{E}_* : \mathbf{D}(\mathbf{w})\,\mathbf{dx}$$

$$-\frac{1}{\omega\mu}\int\limits_{\Omega}|\mathbf{E}_*|^2\,\mathbf{dx}+(1-\omega)\mu\int\limits_{\Omega}\mathbf{D}(\mathbf{u}_*):\mathbf{D}(\mathbf{w}-\mathbf{u}_*)\,\mathbf{dx}$$

$$+\int\limits_{\partial\Omega}q|\mathbf{w}|\,ds-\int\limits_{\partial\Omega}q|\mathbf{u}_*|\,ds\geq\rho\int\limits_{\Omega}\mathbf{f}\cdot(\mathbf{w}-\mathbf{u}_*)\,\mathbf{dx}\quad\forall\,\mathbf{w}\in\mathbf{X}(\Omega,\mathbf{R}^3). \tag{5.15}$$

Further, fix an arbitrary vector function $\mathbf{\Phi}$ from the space $\mathbf{C}_0^\infty(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}})$. Putting $\mathbf{F}=\mathbf{\Phi}$ in (5.4) and integrating by parts the first term, we obtain

$$\varepsilon_n\int\limits_{\Omega}\mathbf{E}_{\varepsilon_n}:\Delta(\Delta\mathbf{\Phi})\,\mathbf{dx}+\int\limits_{\Omega}\mathbf{E}_{\varepsilon_n}:\mathbf{\Phi}\,\mathbf{dx}-\lambda\sum_{i=1}^{3}\int\limits_{\Omega}u_{\varepsilon_n i}\mathbf{E}_{\varepsilon_n}:\frac{\partial\mathbf{\Phi}}{\partial x_i}\,\mathbf{dx}=\omega\mu\int\limits_{\Omega}\mathbf{D}(\mathbf{u}_{\varepsilon_n}):\mathbf{\Phi}\,\mathbf{dx}.$$

Taking into account (5.8)–(5.10), we pass to the limit $n\to\infty$ in the last equality and obtain

$$\int\limits_{\Omega}\mathbf{E}_*:\mathbf{\Phi}\,\mathbf{dx}-\lambda\sum_{i=1}^{3}\int\limits_{\Omega}u_{*i}\mathbf{E}_*:\frac{\partial\mathbf{\Phi}}{\partial x_i}\,\mathbf{dx}=\omega\mu\int\limits_{\Omega}\mathbf{D}(\mathbf{u}_*):\mathbf{\Phi}\,\mathbf{dx}.$$

Since the set $\mathbf{C}_0^\infty(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}})$ is dense in the space $\mathbf{H}_0^2(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}})$, the last equality remains valid if we replace $\mathbf{\Phi}$ with an arbitrary vector function $\mathbf{F}\in\mathbf{H}_0^2(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}})$. Thus, we have

$$\int\limits_{\Omega}\mathbf{E}_*:\mathbf{F}\,\mathbf{dx}-\lambda\sum_{i=1}^{3}\int\limits_{\Omega}u_{*i}\mathbf{E}_*:\frac{\partial\mathbf{F}}{\partial x_i}\,\mathbf{dx}=\omega\mu\int\limits_{\Omega}\mathbf{D}(\mathbf{u}_*):\mathbf{F}\,\mathbf{dx}\quad\forall\,\mathbf{F}\in\mathbf{H}_0^2(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}}). \tag{5.16}$$

Let us define $\mathbf{S}_*$ by the formula

$$\mathbf{S}_*=\mathbf{E}_*+(1-\omega)\mu\mathbf{D}(\mathbf{u}_*). \tag{5.17}$$

Then relations (5.15) and (5.16) together with (5.17) mean that the triplet $(\mathbf{u}_*,\mathbf{S}_*,\mathbf{E}_*)$ is a weak solution to problem (1.1)–(1.8).

In addition, setting $\mathbf{w}=\mathbf{0}$, from (5.12) we derive

$$(1-\omega)\mu\int\limits_{\Omega}|\mathbf{D}(\mathbf{u}_{\varepsilon_n})|^2\,\mathbf{dx}+\frac{1}{\omega\mu}\int\limits_{\Omega}|\mathbf{E}_{\varepsilon_n}|^2\,\mathbf{dx}+\int\limits_{\partial\Omega}q|\mathbf{u}_{\varepsilon_n}|\,ds\leq\rho\int\limits_{\Omega}\mathbf{f}\cdot\mathbf{u}_{\varepsilon_n}\,\mathbf{dx}.$$

Employing (5.10), (5.11), (5.13), and (5.14), we can pass to the lower limit in the last inequality as $n\to\infty$ and obtain the energy estimate (4.1) with $\mathbf{u}=\mathbf{u}_*$ and $\mathbf{E}=\mathbf{E}_*$. Thus, statement (i) is proved.

Further, using the passage-to-limit procedure as above, we establish that the set of weak solutions to problem (1.1)–(1.8) is sequentially weakly closed in the space

$$\mathbf{X}(\Omega,\mathbf{R}^3)\times\mathbf{L}^2(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}})\times\mathbf{L}^2(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}}).$$

Now we show that equality (4.2) holds. By definition, we have

$$\int\limits_{\Omega}\mathbf{E}_0:\mathbf{F}\,\mathbf{dx}-\lambda\sum_{i=1}^{3}\int\limits_{\Omega}u_{0i}\mathbf{E}_0:\frac{\partial\mathbf{F}}{\partial x_i}\,\mathbf{dx}=\omega\mu\int\limits_{\Omega}\mathbf{D}(\mathbf{u}_0):\mathbf{F}\,\mathbf{dx}$$

for any $\mathbf{F}\in\mathbf{H}_0^2(\Omega,\mathbf{R}^{3\times3}_{\mathrm{sym}})$. After integrating the second term by parts, we get

$$\int\limits_{\Omega}\mathbf{E}_0:\mathbf{F}\,\mathbf{dx}+\lambda\sum_{i=1}^{3}\int\limits_{\Omega}u_{0i}\frac{\partial\mathbf{E}_0}{\partial x_i}:\mathbf{F}\,\mathbf{dx}=\omega\mu\int\limits_{\Omega}\mathbf{D}(\mathbf{u}_0):\mathbf{F}\,\mathbf{dx}.$$

Since $\mathbf{H}_0^2(\Omega, \mathbf{R}_{\text{sym}}^{3\times 3})$ is dense in the space $\mathbf{L}^2(\Omega, \mathbf{R}_{\text{sym}}^{3\times 3})$, it follows that the last equality is valid for any vector function $\mathbf{F} \in \mathbf{L}^2(\Omega, \mathbf{R}_{\text{sym}}^{3\times 3})$. Setting $\mathbf{F} = \mathbf{E}_0$, we find

$$\int_\Omega |\mathbf{E}_0|^2 \, \mathbf{dx} = \omega\mu \int_\Omega \mathbf{D}(\mathbf{u}_0) : \mathbf{E}_0 \, \mathbf{dx}. \tag{5.18}$$

Putting $\mathbf{u} = \mathbf{u}_0$, $\mathbf{E} = \mathbf{E}_0$, $\mathbf{w} = 2\mathbf{u}$ in the inequality (3.2) and taking into account (5.18), we derive the following inequality

$$(1-\omega)\mu \int_\Omega |\mathbf{D}(\mathbf{u}_0)|^2 \, \mathbf{dx} + \frac{1}{\omega\mu} \int_\Omega |\mathbf{E}_0|^2 \, \mathbf{dx} + \int_{\partial\Omega} q|\mathbf{u}_0| \, ds \geq \rho \int_\Omega \mathbf{f} \cdot \mathbf{u}_0 \, \mathbf{dx}. \tag{5.19}$$

On the other hand, substituting $\mathbf{u} = \mathbf{u}_0$, $\mathbf{E} = \mathbf{E}_0$, $\mathbf{w} = \mathbf{0}$ in (3.2), we obtain

$$(1-\omega)\mu \int_\Omega |\mathbf{D}(\mathbf{u}_0)|^2 \, \mathbf{dx} + \frac{1}{\omega\mu} \int_\Omega |\mathbf{E}_0|^2 \, \mathbf{dx} + \int_{\partial\Omega} q|\mathbf{u}_0| \, ds \leq \rho \int_\Omega \mathbf{f} \cdot \mathbf{u}_0 \, \mathbf{dx}.$$

This inequality together with (5.19) give (4.2).

Finally, if there exists a function $h \in \mathbf{H}^1(\Omega, \mathbf{R})$ such that $\mathbf{f} = \nabla h$, then

$$\int_\Omega \mathbf{f} \cdot \mathbf{u}_0 \, \mathbf{dx} = \int_\Omega \nabla h \cdot \mathbf{u}_0 \, \mathbf{dx} = \int_{\partial\Omega} h \, \mathbf{u}_0 \cdot \mathbf{n} \, ds - \int_\Omega h \, \mathrm{div}\, \mathbf{u}_0 \, \mathbf{dx} = 0.$$

Combining this with (4.2) and applying Korn's inequality, we get $(\mathbf{u}_0, \mathbf{S}_0, \mathbf{E}_0) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

The theorem is completely proved.

6. Concluding remarks

In this paper, we studied the threshold-slip problem for an incompressible viscoelastic fluid of Oldroyd type, assuming that the flow domain Ω is not rotationally symmetric. This condition plays an important role in the proof of the existence of weak solutions, because for such domains Korn's inequality ensures the coercivity of the relevant operators. Using Proposition 2 (b) (see the Appendix), one can obtain similar results for the case of a flow in a hollow body of revolution under the additional boundary condition $\mathbf{u}|_\Sigma = \mathbf{0}$, where Σ is a part of $\partial\Omega$.

7. Appendix

For the convenience of readers, we provide two statements used in Section 5.

Proposition 1 (see [25]). *Let* $\mathbf{V}$ *be a reflexive Banach space,* $\mathbf{V}^*$ *its the dual space,* $\mathbf{A}\colon \mathbf{V} \to \mathbf{V}^*$ *a pseudo-monotone operator, and* $J\colon \mathbf{V} \to \mathbf{R}$ *a lower semi-continuous convex functional. Suppose also that*

$$\frac{\langle \mathbf{A}(\mathbf{v}), \mathbf{v}\rangle + J(\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{V}}} \to +\infty$$

as $\|\mathbf{v}\|_{\mathbf{V}} \to +\infty$*. Then, for an arbitrary* $\mathbf{z} \in \mathbf{V}^*$*, there exists an element* $\mathbf{u}_z \in \mathbf{V}$ *such that*

$$\langle \mathbf{A}(\mathbf{u}_z) - \mathbf{z}, \mathbf{w} - \mathbf{u}_z\rangle + J(\mathbf{w}) - J(\mathbf{u}_z) \geq 0 \qquad \forall\, \mathbf{w} \in \mathbf{V}.$$

Proposition 2 (Korn's inequalities; see, e.g., [26]). *Assume that* Ω *is a bounded domain in space* $\mathbf{R}^3$ *and* $\partial\Omega \in C^{0,1}$*. Under these conditions, we have*

(a) *if Ω is not a body of revolution, then there is a constant $C_1 = C_1(\Omega) > 0$ such that*

$$\|\mathbf{D}(\mathbf{v})\|^2_{\mathbf{L}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}})} \geq C_1(\Omega)\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega,\mathbf{R}^3)} \quad \text{for all} \ \ \mathbf{v} \in \mathbf{H}^1(\Omega,\mathbf{R}^3) \ \ \text{such that} \ \ \mathbf{v}|_{\partial\Omega}\cdot\mathbf{n} = 0;$$

(b) *if $\Sigma \subset \partial\Omega$ and the surface measure of Σ is positive, then there is a constant $C_2 = C_2(\Omega) > 0$ such that*

$$\|\mathbf{D}(\mathbf{v})\|^2_{\mathbf{L}^2(\Omega,\mathbf{R}^{3\times 3}_{\mathrm{sym}})} \geq C_2(\Omega)\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega,\mathbf{R}^3)} \quad \text{for all} \ \ \mathbf{v} \in \mathbf{H}^1(\Omega,\mathbf{R}^3) \ \ \text{such that} \ \ \mathbf{v}|_{\Sigma} = \mathbf{0}.$$

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