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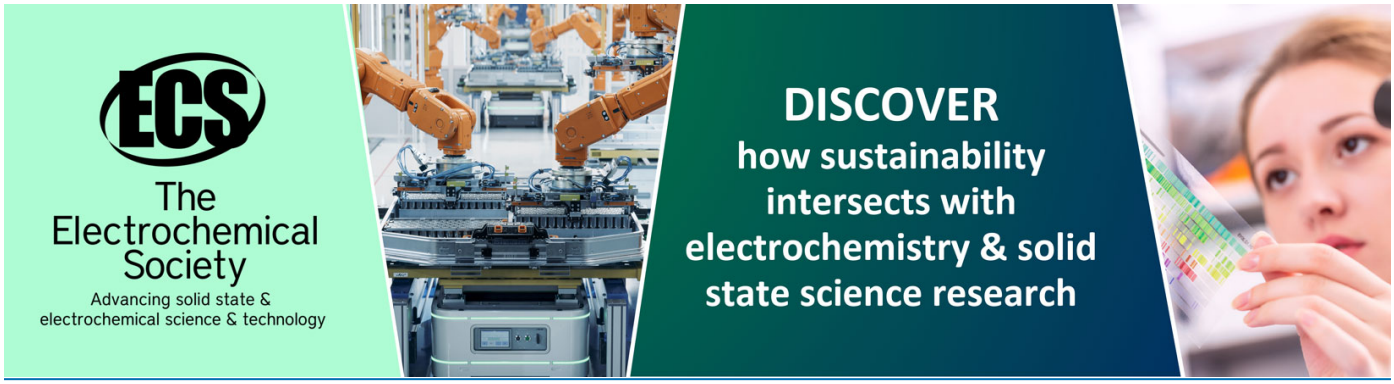

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A new horizon of Few-Body Problems: Exact Coulomb treatment and the energy-momentum translation of the three-body Faddeev equation

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Abstract. We propose an off-shell Coulomb-like amplitude with an on-shell phase shift that is accurate to within 9-10 digits. The full Coulomb amplitude is separated into the leading and the auxiliary amplitudes using "the two-potential theory". The leading amplitude reproduces most of the on- and off-shell parts, while the auxiliary amplitude contributes mainly to the off-shell part and minimally to the on-shell part. We then review the three-body calculation method developed during the last four decades. A reminder to a threshold behavior investigation method is pointed out based on the three-body Faddeev equation. We discuss the Efimov physics and some extensions, which are recovered from our predictions. The method may suggest a promising technique to resolve the existing discrepancies between current experimental and theoretical values.

1. Introduction

The Coulomb phase shift renormalization method has a long history [1, 2, 3, 4, 5, 6, 7]. The most plausible method was presented by Alt et al. [8, 9, 10, 11]. Recently, we checked the reliability of the renormalization method. It was found that the phase shift can be definitely represented by the renormalization method. However, the screening range is unusually long [12]. In order to obtain the proton-proton phase shift $\sigma_0(k) = 96.8$ degrees at $E = 1$ keV by the renormalization method, we need to solve the Lippmann-Schwinger (LS) equation in momentum space with the screening range of 3 nm, 1400 Gauss points, to within 250 significant digits (see figure 1). However, a serious cancelation of significant digits occurs, resulting in converegence to a incorrect, unreliable value. Therefore, we confirmed that the phase shift renormalization is well performed by such a proper significant digits. However, Alt's method cannot treat the nuclear interaction by the usual FORTRAN program.

We presented the following generalized screened Coulomb potential $V^R(r)$ and the renormalization phase $\phi(k, R)$ [13]

$$V^R(r) = \frac{2k\eta(k)}{r} e^{-(r/R)^m} \tag{1}$$

$$\phi(k,R) = \eta(k)\Big(\ln 2kR - \gamma/m\Big). \tag{2}$$

In this case, we obtain a short cut-off potential by increasing m. After that, Deltuva et al. used an effective short-range, screened Coulomb potential, by adopting $m = 5$. In the three-body calculation, such two-body information is required for a wide two-body "sub-energy" region $-\infty < z'' \leq E$. This is because,

$$z'' = E - \frac{q''^2}{2\mu} \tag{3}$$

$$\text{for} \quad 0 \leq q'' < \infty. \tag{4}$$

In order to obtain the two-body off-shell amplitude for the screened Coulomb potential, by the LS equation, one needs zero energy. Therefore, the most remarkable region of the Coulomb off-shell amplitude is the origin i.e. $z = 0$ for the two- and three-body problems. The appropriate ranges in the energy region from 0.1 keV to 100 MeV, are listed in table 1. It is found that the $m = 5$ case yields on the average one order of magnitude shorter range than the case $m = 1$. In order to obtain higher accuracy in the phase shift, still longer ranges are required. If the renormalization method is used, efforts for getting the shorter ranges were shown to be ineffective. Despite of the suggestion of Deltuva et al., it is found that the case $m = 5$ does not essentially improve over $m = 1$ case [14, 15, 16]. Therefore, we suspect that the general use of unsatisfied Coulomb off-shell amplitude could mislead the three-nucleon scattering calculation. This may lead to desparity, considering that the ASZ method may not have been successful for the past four decades.

Table 1. Screening ranges for $m = 1$ and $m = 5$ with respect to the generalized screened Coulomb potential by (1). These ranges are obtained to guarantee the three digit accuracy in the phase shift. Present ranges belongs to the asymptotic region.

Energy [MeV]	$m = 1$	$m = 5$	σ_0 [deg.]
100	11 fm	4 fm	-0.370
50	35 fm	8 fm	-0.523
20	110 fm	17 fm	-0.826
10	300 fm	35 fm	-1.17
1	900 fm	620 fm	-3.66
0.1	1.0×10^4 fm	3.7×10^3 fm	-10.7
0.01	1.0×10^5 fm	7.7×10^4 fm	-16.3
0.001	3.0×10^6 fm	4.5×10^5 fm	96.8
0.0001	3.0×10^7 fm	6.2×10^6 fm	949.9

2. On-shell Coulomb Amplitude

In [12], we introduce an off-shell Coulomb-like amplitude. Before that, we have introduced the so-called "critical range" which satisfies zero renormalization phase [17, 18, 19, 20, 21, 22, 23, 24, 25]. In this paper, we will review our method for the on-shell Coulomb amplitude where the Coulomb phase shift is defined by the screened Coulomb phase shift plus a renormalization phase: $\sigma_0(k) = \delta_0^R(k) + \phi(k,R)$. Therefore, the on-shell scattering amplitude is separated into two parts for the S-wave case by

$$f_0^C(k) = e^{2i\phi(k,R)}\Big(\frac{e^{2i\delta_0^R(k)} - 1}{2ik}\Big) + \Big(\frac{e^{2i\phi(k,R)} - 1}{2ik}\Big). \tag{5}$$

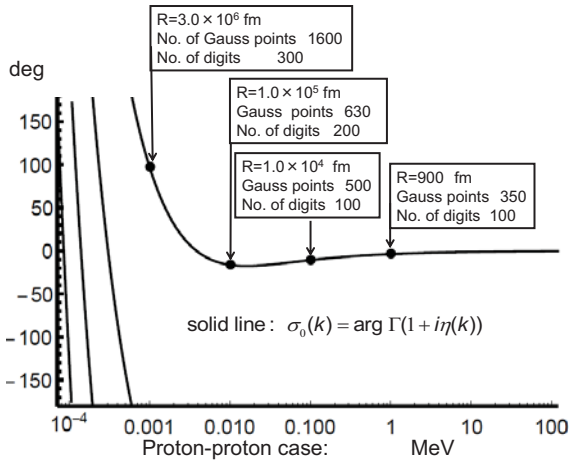

Figure 1. Necessary digits to obtain three digits accurate Coulomb phase shift by the renormalization method with the LS equation for $m = 1$ case.

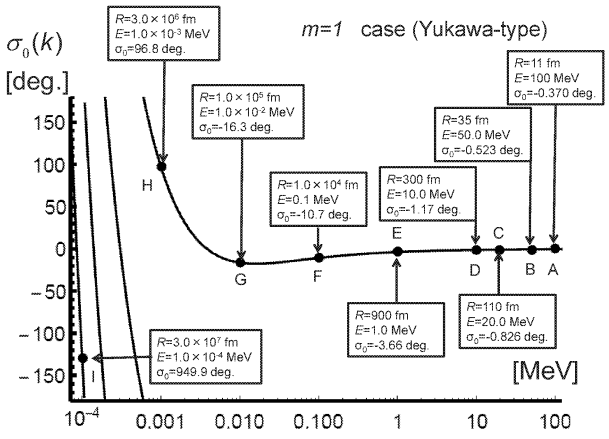

Figure 2. Necessary screening range to represent the three digits accurate Coulomb phase shift by the renormalization method. $m = 1$ case in (1).

However, the "renormalized Coulomb amplitude" is usually defined by the first term of the right hand side of (5), where the second term has been neglected without explanation [9].

In our method, we require $\phi(k, R) = n\pi$ for $n = 0, 1, 2, \cdots$. Using this criterion, a "critical range" is obtained, and the screened Coulomb phase shift becomes the Coulomb phase shift itself: $\delta_0^R(k) = \sigma_0(k)$, and the on-shell amplitude is given by $f_0^C(k) = (e^{2i\delta_0^R(k)} - 1)/2ik$. In other words, the Coulomb phase shift can be obtained directly from the critical range. We pointed out [26] that the screening range is not obtained by increasing a single range but by concatenating many discrete bands. Since Alt's renormalization phase is given by (2) in the asymptotic region [27], we may set:

$$\phi(k, R) \quad = \quad \eta(k)\Big(\ln 2kR - \frac{\gamma}{m}\Big) = n\pi, \tag{6}$$

where $n\pi$ is adopted instead of $2n\pi$, because the difference corresponds to only the sign of the

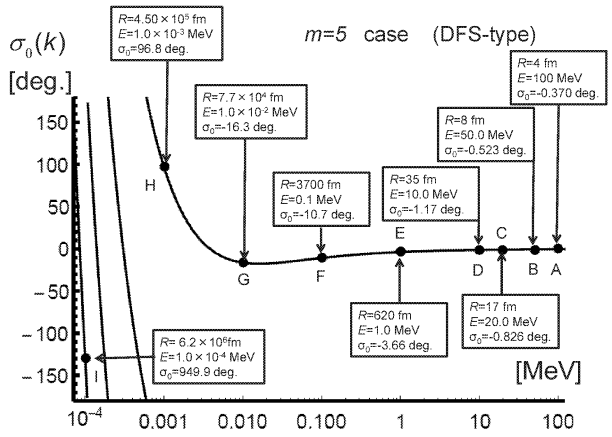

Figure 3. Necessary screening range to represent the three digits accurate Coulomb phase shift by the renormalization method. $m = 5$ case in (1).

wave function. Therefore, our "critical range", hereafter Alt's range, becomes:

$$R = \frac{1}{2k}\exp\Big(\frac{\pi}{\eta(k)}+\frac{\gamma}{m}\Big). \tag{7}$$

We conclude that the screened Coulomb phase shift becomes the Coulomb phase shift by using this critical range. However, a very long range is required to satisfy the asymptotic properties (see figures 2 and 3). Such a long range, due to the cancellation of significant digits, renders the solution of the LS equation quite hard. In order to avoid these difficulties, we obtained the best range parameters, by a fitting method at a region much shorter than the asymptotic domain where the phase shift calculation guarantees nine to ten digits accuracy (see figures 1, 2, 3 and table 1) [27]. Hereafter, the term "critical range" will be used to refer to these best range parameters.

3. A new method for the off-shell Coulomb-like amplitude

We presented an off-shell Coulomb-like amplitude [12]. In this paper, we review it and extend the results. We would like to introduce the two-potential theory in r-space. The Schrödinger equation for the Coulomb wave function may be written in operator form as:

$$(E - H_0 - V^{\mathcal{R}})|\psi^C\rangle = V^{\phi}|\psi^C\rangle = (V^C - V^{\mathcal{R}})|\psi^C\rangle. \tag{8}$$

Now, we assume a relation: $V^C|\psi^C\rangle \approx V^{\mathcal{R}'}|\psi^{\mathcal{R}'}\rangle$, where $V^{\mathcal{R}'}$ is a screened Coulomb potential of range $\mathcal{R}'$ with proper asymptotic properties and $|\psi^{\mathcal{R}'}\rangle$ the corresponding wave function. This assumption makes sense in the frame of the above discussion of Alt's Coulomb renormalization method. Therefore, the asymptotic behavior for such a long range region leads to $V^{\phi}|\psi^C\rangle \approx (V^{\mathcal{R}'} - V^{\mathcal{R}})|\psi^{\mathcal{R}'}\rangle \equiv V^{\phi'}|\psi^{\mathcal{R}'}\rangle \to 0$ (for $r \to \infty$). With this condition, we can define a general solution $|\psi^{\mathcal{R}}\rangle$ for the homogeneous equation in (8). Finally, the general solution of (8) is given by

$$|\psi^C\rangle = |\psi^{\mathcal{R}}\rangle + G^{\mathcal{R}}V^{\phi}|\psi^C\rangle \tag{9}$$

$$G^{\mathcal{R}} = [E - H_0 - V^{\mathcal{R}}]^{-1} = G_0 + G_0T^{\mathcal{R}}G_0 \equiv G_0\overline{\omega}^{\mathcal{R}}, \tag{10}$$

$$\overline{\omega}^{\mathcal{R}} = 1 + T^{\mathcal{R}}G_0, \tag{11}$$

$$\omega^{\mathcal{R}} = 1 + G_0T^{\mathcal{R}}, \tag{12}$$

where $G^{\mathcal{R}}$ is Green's function with the screened Coulomb potential $V^{\mathcal{R}}$.

By multiplying (9) from the left with V^{ϕ}, an amplitude $t^{\phi\mathcal{R}}$ may be defined using $|\psi^{\mathcal{R}}\rangle$ and the plane wave $|\psi_0\rangle$, as:

$$V^{\phi}|\psi^{C}\rangle \equiv t^{\phi\mathcal{R}}|\psi^{\mathcal{R}}\rangle = t^{\phi\mathcal{R}}\omega^{\mathcal{R}}|\psi_0\rangle\,. \tag{13}$$

Therefore, $t^{\phi\mathcal{R}}$ introduces all the long range properties in the Coulomb wave function, and also merits for the proper asymptotic behavior mentioned above. Therefore, (9) is rewritten by using (10), and (13),

$$\begin{aligned} |\psi^{C}\rangle &= |\psi^{\mathcal{R}}\rangle + G^{\mathcal{R}}t^{\phi\mathcal{R}}|\psi^{\mathcal{R}}\rangle = |\psi^{\mathcal{R}}\rangle + G_0\overline{\omega}^{\mathcal{R}}t^{\phi\mathcal{R}}\omega^{\mathcal{R}}|\psi_0\rangle & (14)\\ &= |\psi^{\mathcal{R}}\rangle + G^{\mathcal{R}}V^{\phi}|\psi^{C}\rangle = |\psi^{\mathcal{R}}\rangle + G_0\overline{\omega}^{\mathcal{R}}V^{\phi}|\psi^{C}\rangle. & (15)\end{aligned}$$

Multiplying (14) from the left with V^{ϕ}, and using (13), yields:

$$V^{\phi}|\psi^{C}\rangle = t^{\phi\mathcal{R}}|\psi^{\mathcal{R}}\rangle \;=\; V^{\phi}|\psi^{\mathcal{R}}\rangle + V^{\phi}G^{\mathcal{R}}t^{\phi\mathcal{R}}|\psi^{\mathcal{R}}\rangle. \tag{16}$$

Here, we define a Coulomb-like off-shell T-matrix $\tilde{T}^{C}$ by

$$\begin{aligned} G_0^{-1}|\psi^{C}\rangle &= V^{C}|\psi^{C}\rangle \equiv \tilde{T}^{C}|\psi_0\rangle, & (17)\\ G_0^{-1}|\psi^{\mathcal{R}}\rangle &= V^{\mathcal{R}}|\psi^{\mathcal{R}}\rangle = T^{\mathcal{R}}|\psi_0\rangle, & (18)\end{aligned}$$

where $\tilde{T}^{C}$ has all the properties of the long range Coulomb wave function and the potential. In addition it satisfies the Coulomb asymptotic behavior and our requirement of (6) which is equivalent to the *Lemma* stated in our recent article [26].

Multiplying (14) from the left with V^{C} and using (17), (18), we obtain

$$\begin{aligned} \tilde{T}^{C}|\psi_0\rangle &= T^{\mathcal{R}}|\psi_0\rangle + \overline{\omega}^{\mathcal{R}}t^{\phi\mathcal{R}}\omega^{\mathcal{R}}|\psi_0\rangle & (19)\\ &= T^{\mathcal{R}}|\psi_0\rangle + \overline{\omega}^{\mathcal{R}}V^{\phi}|\psi^{C}\rangle. & (20)\end{aligned}$$

Therefore, (19), (16), and (10) lead to the following operator relations:

$$\begin{aligned} \tilde{T}^{C} &= T^{\mathcal{R}} + T^{\phi\mathcal{R}} = T^{\mathcal{R}} + \overline{\omega}^{\mathcal{R}}t^{\phi\mathcal{R}}\omega^{\mathcal{R}}, & (21)\\ t^{\phi\mathcal{R}} &= V^{\phi} + V^{\phi}G^{\mathcal{R}}t^{\phi\mathcal{R}} & (22)\\ T^{\mathcal{R}} &= V^{\mathcal{R}} + V^{\mathcal{R}}G_0T^{\mathcal{R}}, & (23)\end{aligned}$$

where (21) is known as the amplitude by the two-potential theory. Multiplying from the left (20) with $\langle\psi_0|$ and using $\langle\psi_0|\overline{\omega}^{\mathcal{R}} = \langle\psi^{\mathcal{R}}|$, we obtain

$$\begin{aligned} \langle\psi_0|\tilde{T}^{C}|\psi_0\rangle &= \langle\psi_0|T^{\mathcal{R}}|\psi_0\rangle + \langle\psi_0|\overline{\omega}^{\mathcal{R}}V^{\phi}|\psi^{C}\rangle\\ &= \langle\psi_0|T^{\mathcal{R}}|\psi_0\rangle + \langle\psi^{\mathcal{R}}|V^{\phi}|\psi^{C}\rangle. & (24)\end{aligned}$$

(24) is equivalent to (21), and the amplitude $T^{\phi\mathcal{R}}$ is obtained by using the partial wave functions $|\psi_L^{\mathcal{R}}\rangle$ and $|\psi_L^{C}\rangle$.

$$T_L^{\phi\mathcal{R}}(p,p';E) \;\equiv\; \langle\psi_L^{\mathcal{R}}|V^{\phi}|\psi_L^{C}\rangle = \int_0^{\infty}\psi_L^{\mathcal{R}*}(p,r)V^{\phi}(r,k)\psi_L^{C}(p',r)r^2dr. \tag{25}$$

This result guarantees the existence of the off-shell Coulomb-like T-matrix by (21) and (25), which was not clearly stated in [28]. The off-shell value given by (25), is calculated numerically, where the integrand is highly oscillatory and vanishes for very large values of r. Therefore, (25) contributes to off-shell, and half-on-shell values.

In our method, the on-shell value of $T^{\mathcal{R}}$ in (21) gives 9–10 digits of accuracy [27]. Therefore, it is obvious that the on-shell value of $T^{\phi\mathcal{R}}$ contains only the next order correction need to obtain an accuracy of 9–10 digits in the phase shift. However, the numerical calculation of the on-shell value of (25) is very sensitive to the convergence condition. Therefore, in order to calculate the on-shell value of (25), we can adopt a screened Coulomb potential with a very large range ($\mathcal{R} \ll$)$\mathcal{R}'$ which appears in the asymptotic region and is introduced to obtain the solution of (8). For such an $\mathcal{R}'$, we can accomplish the following relation:

$$T_L^{\phi\mathcal{R}}(k,k;E) = \int_0^\infty \psi_L^{\mathcal{R}*}(k,r)V^{\phi}(r,k)\psi_L^{C}(k,r)r^2dr \equiv \Delta_L \tag{26}$$

$$\rightarrow \int_0^\infty \psi_L^{\mathcal{R}*}(k,r)V^{\phi'}(r,k)\psi_L^{\mathcal{R}'}(k,r)r^2dr \cong -\frac{|T_L^{\mathcal{R}}|^2}{\mathcal{I}m(T_L^{\mathcal{R}})} \times \phi'_L(k) \tag{27}$$

with

$$\phi'_L(k) \equiv \sigma_L(k) - \delta_L^{\mathcal{R}}(k) = \tan^{-1}\frac{\mathcal{I}m(T_L^{\mathcal{R}}+\Delta_L)}{\mathcal{R}e(T_L^{\mathcal{R}}+\Delta_L)} - \tan^{-1}\frac{\mathcal{I}m(T_L^{\mathcal{R}})}{\mathcal{R}e(T_L^{\mathcal{R}})}, \tag{28}$$

where the last term in (27) is an auxiliary amplitude, $\phi'_L(k)$ is the discrepancy between the analytic Coulomb phase shift and our direct phase shift. We can find an adequate $\mathcal{R}'$ which satisfies (27). In [12], we mentioned that the value of (26) vanishes. However, such a condition is satisfied only for the case where the eigen-values of $V^C|\psi^C\rangle$ and $V^{\mathcal{R}}|\psi^{\mathcal{R}}\rangle$ are identical. Our fitting procedure albeit not exact, reproduces the Coulomb phase shift with an accuracy of 9–10 digits. Therefore, we can conclude that the first term $T^{\mathcal{R}}$ represents almost all on- and off-shell Coulomb amplitude, with a compensation coming from the off-shell part of the second term.

4. Review of the three-hadron problem

Half a century has passed since the Faddeev equation was presented [29]. In the beginning, it seemed that the obtained results fitted the experimental data well, in contrast to the results of other methods. After that, improved results have been mainly due to the rapid growth of the computer capacity as to how many partial waves can be incorporated, and what kind of realistic two-body and three-body forces can be treated. These efforts have been done somewhat with a technical professionalism. However, the ideas of the theoretical and methodological approach of the Faddeev equation were investigated only in the beginning, that is, in 1960s and 1970s [30, 31, 32, 33, 34]. It seems that today, is a good time to start a "deep critical rethinking" of the results obtained so far by the usual Faddeev calculations, in order to understand whether the fundamental problems of the Faddeev equation still exist or not.

First of all, the quantitative agreements for A_y are not yet satisfied [35]. Recently, the intensive discussion about the disagreements for the cross section minimum and A_y maximum were pointed out by Sagara [36]. The so-called "Sagara discrepancy" was illustrated by a ratio between theoretical and experimental data in the cross section minimum and the maximum of A_y: $S_{\rm disc}$ ≡(calculation-experiment)/experiment [37, 38, 39]. Such discrepancies have an energy dependence which is commonly inversely proportional to the energy. It was claimed that the cross section minimum problem was solved by using a three-body force named 2π 3NF [40]. However, it seems that the A_y puzzle has not yet been solved by such a systematic way with 2π 3NF and some other intensive 3NFs [41]. We are concerned about the energy dependence of the Sagara discrepancy not only for the A_y puzzle, but also the cross section minimum. If such a 3NF can fundamentally reproduce the cross section minimum, the A_y-puzzle should also be solved by the 3NF at the same time. However, the puzzle is not solved yet. The concern arises from whether the 3NF is fundamental or an ad hoc remedy in the three-nucleon problem.

Another problem discussed but remained unsolved is the break-up problem in the case of "the Space star anomaly", although 3NF and many different so-called realistic potentials were used. Also an incomplete differential cross section could not yet be represented by the Faddeev calculation [30]. On the other hand, the discrepancy in the triton binding energy between the experimental data is compensated by a three-body force [42]. It was found that the doublet n-d scattering length ($a_{\rm nd}^2$) is closely related to the three-body force [43]. The closed relation between $a_{\rm nd}^2$ and 3NF is reminiscent of the relation between the "n-d threshold" and the 3NF. Many attempts are carried out by using realistic potentials and corresponding three-body forces, relativistic effects, the Coulomb interaction, new degrees of freedom etc. However, those methods could not satisfactorily explain the above discrepancies. The three-body force is mainly an isotropic interaction. Furthermore, the renormalization phase of the Coulomb amplitude is also isotropic. Such isotropic interactions can not affect the P-wave dominant A_y value, and the effect of the isotropic interaction could work somewhat negatively. Therefore, we suspect that the energy dependence in the Sagara discrepancy could be closely related to an anisotropic energy-momentum usage or kinematics in the Faddeev equation.

The second is the three-body bound state which is obtained by solving the two-variable eigen-equation as the three-body Faddeev equation. However, the two-variable eigen-equation is "not well defined" mathematically, although the one variable eigen-equation is well defined [44]. Therefore, we should separate the two-variable three-body Faddeev equation into one-variable two-body LS equations, and a one-variable quasi-two-body multi-channel LS equation. Each of them gives eigen-values $\epsilon^{(2)}$ for $(bc) \to b+c$ and ζ by solving one-variable eigen-equations, where we obtain a separation energy ζ for the reaction $(abc) \to (bc) + a$ by the quasi-two-body multi-channel LS equation. Therefore, the three-body binding energy is obtained by $\epsilon^{(3)} = \zeta + \epsilon^{(2)}$ [45]. This method coincides with the Jacobi-coordinate anti-symmetrized molecular dynamics, where bound colonies are grown up sequentially from many free-particle groups, by cooling the system in a timely manner, and not instantaneously as in the usual Faddeev method [46, 47, 48, 49, 50].

4.1. Three-body kinematics

Now, let us start with the well-known three-body kinematics for the scattering with a incident particle and a target. The incident particle has a kinetic energy E_{Lab} with the momentum q_L,

$$E_{Lab} = \frac{q_L^2}{2m} = \frac{q_L^2}{2(m+M_t)} + E_{\rm cm} = \Big(\frac{m}{m+M_t}\Big) E_{Lab} + E_{\rm cm} \tag{29}$$

$$E = E_{\rm cm} - \epsilon_B = \frac{q^2}{2\mu} + \frac{p^2}{2\nu}, \qquad \text{(target binding energy} = \epsilon_B) \tag{30}$$

$$E_{\rm cm} = \frac{\overline{q}^2}{2\mu} + \frac{\overline{p}^2}{2\nu}, \qquad \text{(target unbound system)}, \tag{31}$$

where m and M_t are the masses of the incident particle and the target. μ denotes the reduced mass between a pair and the spectator particle, and ν is the reduced mass in the pair, respectively. Therefore, q is the relative momentum between the pair and the spectator, and p is the relative momentum in the pair after three-body break-up. $E_{\rm cm}$ is the three-body incident center of mass energy, and ϵ_B is the target binding energy. Therefore E is the three-body free energy after the deuteron break-up. If there are no two-body sub-bound states in the intermediate state, then the three-body total energy $E_{\rm cm}$ should be used, where one of the three particles, at least, is virtual.

Usually, the Faddeev equation and the AGS equation commonly use the three-body free energy E in the Green's function: $G_0(E)$ where real-three-particles are transferred in between two different three-body channels above the break up threshold. However, if the initial and final channels are deuteron bound states in the Born term, the spectator particles gain the

deuteron binding energy. Therefore, the initial and the final state's kinetic energies are given by $E_{\rm cm} = \overline{q}^2/2\mu = \overline{q}'^2/2\mu$. In the Born term of AGS equation, the Green's function could be written by $G_0(\overline{q}, \overline{q}'; E)$. Therefore, a transferred momentum in the Green's function could be partly virtual, because all the particle-lines in the Feynman diagram are "outer-lines" where the energy-momentum should be conserved. While, if we choose $E_{\rm cm}$ instead of E in the Green's function, the three-particle momenta are all real by (31) and (36) with $G_0(\overline{q}, \overline{q}'; E_{\rm cm})$. $E_{\rm cm} = 0$ indicates the quasi-two-body threshold (Q2T) in the three-body scattering which causes a divergence of the two-body propagator $\tau(z)$. The n-d scattering length is calculated with such analytic properties as to $E_{\rm cm} = 0$.

On the other hand, in the intermediate three-body lines as the "inner lines" of the Feynman diagram, we are concerned about two-type of three-body energies: the three-body energy $E_{\rm cm}$ before the break up, and the three-body energy E after the break up. It is found that by an energy-momentum translation:

$$E \to E_{\rm cm} = E + \epsilon_B \tag{32}$$
$$q^2/2\mu \to \overline{q}^2/2\mu = q^2/2\mu + \epsilon_B, \tag{33}$$
$$\overline{p}^2/2\nu = p^2/2\nu, \tag{34}$$

we obtain, for the usual Born term in the three-identical particle system,

$$Z_{\alpha,\beta}(q, q'; E) = \frac{g_\alpha(p) g_\beta(p')(1-\delta_{\alpha\beta})}{\Big(E + \epsilon_B\Big) - \Big(q^2/2\nu_\beta + q'^2/2\nu_\alpha + qq'x/m + \epsilon_B\Big)} \tag{35}$$
$$= \frac{g_\alpha(\overline{p}) g_\beta(\overline{p}')(1-\delta_{\alpha\beta})}{E_{\rm cm} - \overline{q}^2/2\nu_\beta - \overline{q}'^2/2\nu_\alpha - \overline{qq}'\overline{x}/m} = Z_{\alpha,\beta}(\overline{q}, \overline{q}'; E_{\rm cm}). \tag{36}$$

where $x = \mathbf{qq}'/qq'$ and $\overline{x} = \overline{\mathbf{qq}}'/\overline{qq}'$. The propagator becomes,

$$\tau_\gamma(E - q'^2/2\mu) = \tau_\gamma([E + \epsilon_B] - [q'^2/2\mu + \epsilon_B]) = \tau_\gamma(E_{\rm cm} - \overline{q}'^2/2\mu). \tag{37}$$

Therefore, the energy-momentum translation does not affect the function itself, however the integral may be modified due to the prolonged range: $0 \le \overline{q}^2/2\mu \le \epsilon_B$ or $-\epsilon_B \le q^2/2\mu \le 0$.

4.2. Non-Efimov but Efimov-like states

One author (S.O.) pointed out that $E_{\rm cm} = 0$ in (36) for NNπ system, generates a singularity-coincidence between the Born term and the propagator $\tau_\gamma(E_{\rm cm} - \overline{q}'^2/2\mu)$ [51]. This coincidence is very similar to the Coulomb scattering in the LS equation where the singularity of the Coulomb potential and of the propagator occurs simultaneously. Since our potential singularity in the present case arises at $E_{\rm cm} = 0$, then we carried out an energy average around $E_{\rm cm} = 0$ after the Fourier transformation of (36). Finally, we obtained $V(r) = V_0 a^{2\gamma+2}/[r(r/2+a)^{2\gamma+2}]$-type potentials with parameters a and γ and a depth constant $V_0(<0)$ for the lowest order term in the local potential expansion of the Born term. Therefore, we obtained $V_0(2a)/r^2$ potential for $\gamma = -1/2$, and Van der Waals potentials $V_0(2a)^5/r^6$ for $\gamma = 3/2$, $V_0(2a)^6/r^7$ for $\gamma = 2$ potentials in the long range region for $a \ll r$. While in the shorter range region $r \ll a$, they become Yukawa-type potentials $V_0 e^{-r/2a}/r$, $V_0 e^{-5r/2a}/r$, and $V_0 e^{-6r/2a}/r$, respectively (see table 2). Beside the lowest order, the higher order terms affect the shorter range region adding to the Yukawa-type potential. Therefore, the shorter range region could be strongly modified from the Yukawa-type potential near the core region, however our longer range predicted potential can not be changed by the higher order terms.

These potentials are closely related to the mass ratio between a transfer particle and the parent particles, which can be easily understood by the reason that if the massless particle is

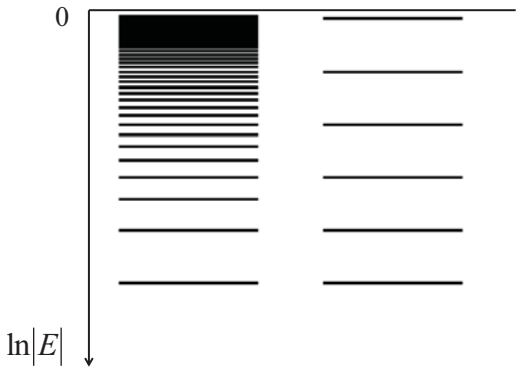

Figure 4. Bound state energy levels for the potentials $1/r$ (left) and $1/r^2$ (right) where the ground state energies are fitted. The vertical unit is arbitrary but the logarithmic unit of the absolute value is taken.

transferred, then the potential will be $1/r$ which is given by $\gamma = -1$. It is interesting that $\gamma = -1$ or a massless particle creates a unique potential V_0/r where two-type of potentials for $a \ll r$ and $r \ll a$ cross each other (see table 2). The above discussion, that two types of potentials, a long range and a short range, arise from a particle-transfer mechanism, seems to be a universal structure in physics. If there is a counterargument about our "energy average" method, we can directly solve the three-body equation with (36), where we need a special technique to treat it in the neighborhood of the singular region. Finally, we can also obtain bound states near the Q2T or the energy dependent two-body quasi (E2Q)-threshold.

Table 2. An effective potential $V_0\, a^{2\gamma+2}/[r(r/2+a)^{2\gamma+2}]$ is illustrated, which is given by an energy average around the Q2T (or $E_{\rm cm} = 0$) with two-parameters a and γ. The potential properties for the longer and shorter ranges are shown with respect to the parameter γ. $V_0(<0)$ is the potential depth given analytically by the Born term of the form (36).

γ	short range potential $r \ll a$	potential	long range potential $a \ll r$
–1	V_0/r	V_0/r	V_0/r
–1/2	$V_0 e^{-r/2a}/r$	$V_0(2a)/[r(r+2a)]$	$V_0(2a)/r^2$
0	$V_0 e^{-2r/2a}/r$	$V_0(2a)^2/[r(r+2a)^2]$	$V_0(2a)^2/r^3$
1/2	$V_0 e^{-3r/2a}/r$	$V_0(2a)^3/[r(r+2a)^3]$	$V_0(2a)^3/r^4$
1	$V_0 e^{-4r/2a}/r$	$V_0(2a)^4/[r(r+2a)^4]$	$V_0(2a)^4/r^5$
3/2	$V_0 e^{-5r/2a}/r$	$V_0(2a)^5/[r(r+2a)^5]$	$V_0(2a)^5/r^6$
2	$V_0 e^{-6r/2a}/r$	$V_0(2a)^6/[r(r+2a)^6]$	$V_0(2a)^6/r^7$
.	...	...	...
.	...	...	...

It is well known that $1/r^2$ potential generates the Efimov-like states, although our case does not require the infinite number of the two-body scattering length: $a_{\rm scatt} \neq \pm\infty$. However, when

the two-body binding energy becomes zero then $0 = E_{\rm cm} = E + \epsilon_B \to E$ which is the case of Efimov states [52, 53]. This is also the case that the energy-momentum translation is denied. In this sense, our prediction covers the Efimov states. We can say that our prediction at the Q2T is generalized to other thresholds such as three-body, four-body, five-body, ..., thresholds [54]. This fact suggests that "observed and unobserved energy levels" in any many-body system give rise to a long range potential near the threshold. We would like to suggest that these levels could be a key which can "control the nuclear reactions" where the long range attractive nuclear interaction could interfere with the Coulomb repulsive force in the pico-size region [51].

5. Conclusion and Discussion

The reliability of the Coulomb renormalization method is confirmed. However, its application to the three-body problem is almost impossible because the screening range is too long. When Alt-type Coulomb renormalization is used, a serious cancelation of significant digits occurs in the two-body LS and in the three-body Faddeev equations. This cancelation causes the method to converge, without any warning, to an erroneous value, hence rendering the calculation unreliable. We present a shorter ranged screened Coulomb potential which can reproduce the Coulomb phase shift to within 9–10 digits accuracy. Furthermore, the off-shell Coulomb-like amplitude is proposed by the two-potential method in r-space representation where the off-shell amplitude can be calculated numerically. By our theory, the Coulomb treatment in the three-body problem could be enormously improved for the first time in a period of half a century. Finally, we would like to emphasize that the method is very general, and it can treat all kinds of charged particle systems, from the electron-electron to the heavy ion-heavy ion, using the universal variables.

We reviewed the discrepancy between the experimental data and the usual Faddeev calculated results. Sagara discrepancy is the most important admonition for this problem. We also added some other problems that seemed solvable by incorporating the three-body force. We pointed out that the Q2T is important for improving the Faddeev treatment. One of the examples was the case of the NNπ system, and we presented a possible existence of a local long-range NN$'$ potential around Q2T, and NN$'$ scattering length and πD one [54]. This potential results to some interesting bound states, and to the existence of long-range potentials such as the $1/r^2$ type and the Van der Waals potential. The $1/r^2$ potential gives infinite number of bound states below the Q2T, without Efimov's requirement for infinite scattering length of two-body sub-system. Therefore, our theory predicts, in addition to the Efimov states, some other interesting aspects of nuclear physics.

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