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The energy method. Application to PDEs of hydrodynamics with nonstandard growth

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Abstract. We present a method for the study of localization properties of solutions of nonlinear equations and systems. An example of application of the method is furnished by results of analysis of a mathematical model for a class of non-Newtonian fluids.

1. Introduction

Numerous mathematical models of continuum mechanics involve combined systems of nonlinear PDEs where different components of the solution, such as velocity, density, pressure, temperature, satisfy equations of different type. These equations may degenerate or become singular at certain values of the solutions or their derivatives which causes localization of solutions in space or time. The typical localization effects are the finite speed of propagation of disturbances from the initial and boundary data, the waiting time phenomenon, extinction or blow-up of solutions in a finite time.

The standard approach to the study of localization properties consists in comparison with suitable explicit solutions or sub/super solutions of the same equation, see, e.g., [1]. However, this method ceases to be applicable if the explicit solutions are not available, or when the maximum principle fails. An alternative approach is based on analysis of the local energy functions which satisfy nonlinear ordinary differential inequalities stemming from the PDE or the system of PDEs being studied. We refer to the monographs [2, 3] for an insight into the method and to papers [4]-[12] for its applications to the study of localization properties of solutions of the mathematical models of hydrodynamics, including the flows of Non-Newtonian viscous fluids.

In this note, we make an emphasis on the possibility of application of the method to the models which include the so-called PDEs with nonstandard growth, i.e., the PDEs with variable nonlinearity. For such models, the method of local energies has turned out to be the most effective in the study of the localization properties of solutions - see [3, 13]. The method is illustrated by the proof of extinction in a finite time of solutions of the system describing the flows of incompressible non-homogeneous non-Newtonian fluids.

2. Flows of non-homogeneous non-Newtonian fluids

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 2$, and $\partial\Omega \in \mathrm{Lip}$. We consider the incompressible flow governed by the laws of balance of mass and momentum, which lead to the following system of

equations posed in the cylinder $Q_T = \Omega \times (0,T)$:

$$D_t\rho \equiv \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho = 0 \qquad \text{(the transport equation, hyperbolic)}, \tag{2.1}$$
$$\operatorname{div}\mathbf{v} = 0 \qquad \text{(the incompressibility equation, elliptic)}, \tag{2.2}$$
$$\rho D_t\mathbf{v} = -\nabla p + \operatorname{div}\mathbf{S} + \rho\mathbf{f} \qquad \text{(the balance of momentum, parabolic)}. \tag{2.3}$$

System (2.1)–(2.3) is completed by the initial and boundary conditions

$$\rho(x,0)\mathbf{v}(x,0) = \rho_0(x)\mathbf{v}_0(x), \quad \rho(x,0) = \rho_0(x) \text{ as } x \in \Omega, \tag{2.4}$$
$$\mathbf{v}(x,t) = 0 \text{ for } (x,t) \in \Gamma_T = \partial\Omega \times (0,T). \tag{2.5}$$

Here $\mathbf{v}(x,t)$, $\rho(x,t)$ and $p(x,t)$ stand for velocity, density and pressure in the fluid, $\mathbf{S}$ is the deviatoric part of the Cauchy stress tensor, $\mathbf{D}$ is the strain rate tensor and $\mathbf{f}(x,t)$ is the prescribed mass force. The tensor $\mathbf{S}$ is a (nonlinear) tensor function of the strain rate tensor $\mathbf{D}$ and the thermodynamic characteristics of the flow, for example, temperature θ, density ρ, or entropy. The tensor $\mathbf{S}$ is usually defined by the constitutive relation

$$\mathbf{S} = \mathbf{F}(\mathbf{D},\rho,\theta) \equiv \nu(|\mathbf{D}|,\rho,\theta)\mathbf{D}, \tag{2.6}$$

see [14, 15] for further details and justifications. A few examples of fluids with different constitutive laws (2.6):

- *viscoplastic fluids* (Eu. Bingham, 1922)

$$\nu(|\mathbf{D}|) := \begin{cases} \infty \Rightarrow \mathbf{D} = \mathbf{0}, & |\mathbf{S}| \le |\mathbf{S}_0|, \quad \mathbf{S}_0 \text{ is a yield stress} \\ \nu_0 + |\mathbf{S}_0||\mathbf{D}|^{-1}, & |\mathbf{S}| > |\mathbf{S}_0|, \quad \nu_0 = const. > 0; \end{cases}$$

- *generalized Newtonian fluids* (A. de Waele, 1923; W. Ostwald, 1925)

$$\nu(|\mathbf{D}|) = \nu_0|\mathbf{D}|^{q-2}, \qquad \nu_0 = Const. > 0,$$

  which include viscoplastic fluids ($q = 1$), pseudoplastic or shear-thinning fluids ($1 < q < 2$), Newtonian fluids ($q = 2$), dilatant or shear-thickening fluids ($q > 2$);
- *a mixture of a Newtonian fluid with thickening agents* (A. W. Sisko, 1958)

$$\nu(|\mathbf{D}|) = \nu_\infty + \nu_0|\mathbf{D}|^{q-2}, \quad \nu_\infty,\ \nu_0 = Const. > 0.$$

A special class of fluids is constituted by *electrorheological fluids*, which consist of solid particles dispersed in an insulating liquid. Their characteristic feature is the possibility of variation of the rheological properties under the influence of an exterior electric field. The mathematical models of motion of such fluids involve the constitutive equation (2.6) with variable exponents of nonlinearity, which depend on the electric field: $\nu \sim \nu_0 + \nu_1|\mathbf{D}|^{q(\mathbf{E})}$ with constant ν_0, ν_1 - see [16]. The flow is described by a modified system of Navier-Stokes equations and the system of Maxwell equations for $\mathbf{E}$. Similar constitutive relations with $q \equiv q(\theta)$ are used in modelling of *thermorheological fluids* whose rheology changes together with the variation of the temperature $\theta(x,t)$ - see [17, 18] for an analysis of the pertinent mathematical models.

The study of well-posedness of the mathematical model and the analysis of the qualitative properties of solutions are often disconnected. For this reason in what follows we assume that the arguments of the function ν in (2.6) (the thermodynamic characteristics ρ, θ of the flow or the electric field $\mathbf{E}$) are known functions of (x,t). We will also assume that $\mathbf{F} = \mathbf{F}(\mathbf{D},x,t)$ satisfies the coercivity condition: for every symmetric tensor $\mathbf{D} \in \mathbb{R}^{N\times N}$

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$$\forall (x,t) \in Q_T \quad \begin{array}{l} \delta |\mathbf{D}|^{q(x,t)} \leq \mathbf{F}(\mathbf{D}) : \mathbf{D} = F^{ij} D^{ij}, \quad \text{where } |\mathbf{D}|^2 = D^{ij} D^{ij}, \\ 1 < \underline{q} \leq q^-(t) \leq q(x,t) \leq q^+(t) \leq \overline{q} < \infty, \quad 0 < \delta \equiv \delta(\rho) < \infty, \end{array} \tag{2.7}$$

with given functions $q(x,t)$, $q^{\pm}(t)$, $\delta(\rho)$ and constants $\underline{q}$, $\overline{q}$.

System (2.1)-(2.6) is a system of combined type. Indeed: if we assume that in (2.1) the velocity $\mathbf{v}(x,t)$ is given, then (2.1) can be regarded as a hyperbolic equation for the density $\rho(x,t)$, and if in (2.3) the density $\rho(x,t)$ is known, then system (2.2)-(2.3) becomes a pseudo-parabolic system of equations for the velocity $\mathbf{v}$ and pressure p. The specific type of system (2.1)-(2.3) is unimportant for the applicability of the method.

3. Energy solutions

It is assumed that the initial data satisfy the conditions

$$E(0) = \frac{1}{2} \int_{\Omega} \rho(x,0) |\mathbf{v}(x,0)|^2 dx < \infty, \qquad \frac{1}{M} \leq \rho_0 \leq M \equiv const. \tag{3.1}$$

The solution $\{\mathbf{v}, \rho\}$ of problem (2.1)-(2.5) is understood in the weak sense, i.e., as functions satisfying the integral identities which appear after multiplying equations (2.1), (2.3) by a smooth solenoidal test-function and integrating by parts. In the result, the pressure p is excluded from the weak formulation.

There exists an abundant literature devoted to the issues of existence and uniqueness of solutions of problem (2.1)-(2.4). In the incompressible homogeneous fluid ρ is constant and equation (2.1) fulfills automatically. For this class of fluids, solvability of the classical Navier-Stokes system with constant ρ and relation (2.7) with the parameters $\delta = const$ and $q = 2$ was studied in [19]-[21]. The flows of incompressible homogeneous trembling electrorheological fluids were considered in [10]. The authors of [10] studied solvability of the problem and the asymptotic behavior of solutions as $t \to \infty$. Global in time existence of weak solutions to problem (2.1)-(2.5) for nonhomogeneous fluids with $\rho(x,t) \not\equiv const$ was proved in [14, 15],[22]-[30] for various constitutive relations (2.7). In particular, for the fluids with the constitutive relation $\mathbf{S} = 2\mu\mathbf{D}$, $\mu = const$, the existence of weak solutions was proved in [14] in the class of functions

$$\{\mathbf{v}, \rho\} \in W_q = \left\{ \begin{array}{l} \mathbf{v} \in L^{\infty}(0,T; L^2(\Omega)) \cap L^q \left(0, T; W_0^{1,q}(\Omega)\right) \\ \mathbf{D}(\mathbf{v}) \in L^q(Q_T), \ \operatorname{div} \mathbf{v} = 0, \ \dfrac{1}{M} \leq \rho \leq M \end{array} \right\}, \qquad q = const > 1.$$

Let us introduce *the energy function*

$$E(t) = \frac{1}{2} (\rho \mathbf{v}, \mathbf{v})_{\Omega} = \frac{1}{2} \int_{\Omega} \rho(x,t) |\mathbf{v}(x,t)|^2 \, dx$$

and consider the *energy solutions* which satisfy the relations

$$E'(t) + (\mathbf{F} : \mathbf{S}, 1)_{\Omega} = (\rho \mathbf{f}, \mathbf{v})_{\Omega}, \qquad E'(t) = \frac{dE}{dt}(t). \tag{3.2}$$

Equality (3.2) can be derived from the integral identity for equation (2.3) with the test-function $\mathbf{v}$, equations (2.1)-(2.2) and the formulas of integration by parts (see, e.g., [10]):

$$\left(\rho \frac{d\mathbf{v}}{dt}, \mathbf{v}\right)_{\Omega} = \frac{1}{2} \frac{d}{dt} (\rho \mathbf{v}, \mathbf{v})_{\Omega} \equiv \frac{dE}{dt}(t),$$

$$(-\nabla p + \operatorname{div} \mathbf{S}, \mathbf{v})_{\Omega} = (p, \operatorname{div} \mathbf{v})_{\Omega} - (\mathbf{F} : \mathbf{S}, 1)_{\Omega} = -(\mathbf{F} : \mathbf{S}, 1)_{\Omega}.$$

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4. Variable Lebesgue and Sobolev spaces

The analysis of the problem with variable nonlinearity in (2.6) is performed in the context of Lebesgue and Sobolev spaces with variable exponents [31]. Given a measurable function $p : \Omega \mapsto (1, \infty)$, we denote by $L^{p(\cdot)}(\Omega)$ the set of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < \infty, \qquad p(x) \in [p^-, p^+] \subset (1, \infty) \tag{4.1}$$

with some constants $p^\pm$. The set $L^{p(x)}(\Omega)$ endowed with the Luxemburg norm,

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 :\ A_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}, \tag{4.2}$$

becomes a Banach space. For every $f \in L^{p(\cdot)}(\Omega)$ the relations between the norm and the modular $A_p(f)$ is given by the inequalities

$$\min\left\{A_{p(\cdot)}^{\frac{1}{p^-}}(f), A_{p(\cdot)}^{\frac{1}{p^+}}(f)\right\} \leq \|f\|_{p(\cdot),\Omega} \leq \max\left\{A_{p(\cdot)}^{\frac{1}{p^-}}(f), A_{p(\cdot)}^{\frac{1}{p^+}}(f)\right\} \tag{4.3}$$

with constants $p^\pm$ from condition (4.1). We assume that for all $(x, y) \in \overline{\Omega}$ and every $t \in [0, T]$

$$1 < \underline{q} \leq q^-(t) \leq q(x,t) \leq q^+(t) \leq \overline{q} < \infty, \qquad |q(x,t) - q(y,t)| \leq \omega(|x-y|), \tag{4.4}$$

where the modulus of continuity ω satisfies the condition $\overline{\lim}_{\tau\to 0^+}\omega(\tau)\ln\frac{1}{\tau} = C_\omega < \infty$. Let $W_0^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega),\ |\nabla u|^{p(x)} \in L^1(\Omega)\right\}$ and

$$\mathcal{W}_{q(\cdot)}(Q_T) = \left\{\mathbf{v} : Q_T \mapsto \mathbb{R}^n \;\middle|\; \begin{array}{l} \mathbf{v} \in \left(L^\infty(0,T; L^2(\Omega) \cap L^1(0,T; W_0^1(\Omega))\right)^n; \\ \operatorname{div}\mathbf{v} = 0,\ \displaystyle\int_0^T\int_\Omega |\mathbf{D}(\mathbf{v})|^{q(x,t)}\, dxdt < \infty \end{array}\right\}.$$

The key technical tool used in the analysis is Korn's inequality for $\mathbf{v}(x,t) \in \mathcal{W}_{q(\cdot)}(Q_T)$.

Lemma 4.1. *If $q(x,t)$ satisfies (4.4), $\mathbf{v} = (v_1, \ldots, v_n)$ with $v_i(\cdot,t) \in W_0^{1,q(\cdot,t)}(\Omega)$ for a.e. $t \in [0,T]$, then*

$$\frac{1}{C}\|\nabla\mathbf{v}(\cdot,t)\|_{q(\cdot,t),\Omega} \leq \|\mathbf{D}(\mathbf{v}(\cdot,t))\|_{q(\cdot,t),\Omega}, \qquad K\|\mathbf{v}(\cdot,t)\|_{r(\cdot,t),\Omega} \leq \|\mathbf{D}(\mathbf{v}(\cdot,t))\|_{q(\cdot,t),\Omega} \tag{4.5}$$

where $q < N$, $r \leq \dfrac{qN}{N-q} < \infty$, $K = K\left(C_\omega, \underline{q}, \overline{q}, N, \Omega\right)$, $C = C\left(C_\omega, \underline{q}, \overline{q}, N, \Omega\right)$.

We refer to [31, Th.14.3.21] for the proof of (4.5) with $\mathbf{v}$, q depending only on x and q satisfying the log-continuity condition (4.4) in Ω. For $q = const$ the proof of Korn's inequality can be found in [32]. Using (4.5) with $r = 2$, applying the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ and inequalities (4.3) we derive the following chain of inequalities: for $\underline{q} \geq \frac{2N}{N+2}$

$$\frac{1}{K}\|\mathbf{v}(\cdot,t)\|_{2,\Omega} \leq \|\mathbf{D}(\mathbf{v}(\cdot,t))\|_{q(\cdot),\Omega} \leq \max\left\{A_{q(\cdot,t)}^{\frac{1}{q^-(t)}}(\mathbf{D}(\mathbf{v}(\cdot,t))), A_{q(\cdot,t)}^{\frac{1}{q^+(t)}}(\mathbf{D}(\mathbf{v}(\cdot,t)))\right\}. \tag{4.6}$$

5. Pseudoplastic fluids. Localization in time via analysis of the energy function

Let us consider the energy solutions of system (2.1)-(2.3) which satisfy identity (3.2). Assume that

$$\sup_{t\in(0,T)} E(t) \le \overline{E} < \infty, \qquad \int_0^T \int_\Omega |\mathbf{D}|^{q(x,t)}\, dxdt < \infty, \qquad \frac{1}{M} \le \rho(x,t) \le M < \infty \tag{5.1}$$

with some constants $\overline{E}$ and $M > 0$. The last two inequalities are byproducts of the maximum principle for the solutions of equation (2.1) and restriction (3.1) on $\rho(x,0)$. Without loss of generality we may assume that $\overline{E} \le 1$. It follows from (4.6), (5.1) that

$$C_0 E^{\frac{\overline{q}}{2}}(t) \le C_0 E^{\frac{q^+(t)}{2}}(t) \le \delta \int_\Omega |\mathbf{D}(\mathbf{v}(x,t))|^{q(x,t)}\, dx \le (\mathbf{F}:\mathbf{S},1)_\Omega\,, \quad C_0 = C_0(K,M,\overline{E}). \tag{5.2}$$

Theorem 5.1. *Let* $\{\mathbf{v},\rho,p\}$, $\mathbf{v}\in \mathcal{W}_{q(\cdot)}(Q_T)$, *be a weak solution of problem* (2.1)–(2.5). *Assume that condition* (3.1) *is fulfilled and, additionally,*

$$(2.7) \textit{ holds with } \delta = \delta_0 = const \textit{ and } q(x,t) \in \left(\frac{2N}{N+2}, 2\right). \tag{5.3}$$

(1) If $\mathbf{f}\equiv 0$, *then*

$$E(t) = \frac{1}{2}\int_\Omega \rho(x,t)|\mathbf{v}(x,t)|^2 dx = 0 \qquad \textit{for } t \ge t^* = \frac{2E^{(2-\overline{q})/2}(0)}{C_0(2-\overline{q})}.$$

In particular, $\mathbf{v}\,(x,t) \equiv 0$ *in* $Q_T \cap \{t \ge t^*\}$.

(2) Let us denote $u_+ = \max\{u,0\}$. *Assume that* $\mathbf{f}\not\equiv 0$, $\overline{q}<2$ *and*

$$\|\mathbf{f}(\cdot,t)\|_{2,\Omega} \le \epsilon\,(1-t/t_f)_+^{\overline{q}/(2-\overline{q})} \tag{5.4}$$

with $\varepsilon = const > 0$, $t_f > t^*$. *If the data satisfy the condition*

$$\frac{\overline{q}E(0)}{(2-\overline{q})t_f} + C_0 E^{\frac{\overline{q}}{2}}(0) = \varepsilon\sqrt{2M}, \tag{5.5}$$

then there exists a constant C *such that* $E(t) \le C\,[1-t/t_f]_+^{\overline{q}/(2-\overline{q})}$, *that is,* $E(t) = 0$ *for* $t \ge t_f$ *and* $\mathbf{v}(x,t)\equiv 0$ *in* $Q\cap\{t\ge t_f\}$.

The mechanical meaning of Theorem 5.1 is that if the flow of a nonhomogeneous non–Newtonian pseudoplastic fluid is driven by the initial data, then the fluid becomes immobile in a finite time. If the flow is stirred by the source term $\mathbf{f}\ne 0$ that vanishes at the instant t_f, then the fluid is at rest for all $t \ge t_f$, provided that the intensity of the source is appropriately small.

Sketch of the proof. Applying (2.7), (3.2) and (5.2) we derive the nonlinear integral and differential inequalities

$$\begin{aligned} E(t) + C_0\int_0^t E^{\frac{\overline{q}}{2}}(\tau)d\tau &\le E(t) + C_0\int_0^t E^{\frac{q^+(\tau)}{2}}(\tau)d\tau \\ &\le E(t) + \int_0^t (\mathbf{F}:\mathbf{S},1)_\Omega d\tau = E(0) + \int_0^t (\rho\mathbf{f},\mathbf{v})_\Omega d\tau, \end{aligned}$$

$$E'(t) + C_0 E^{\frac{\bar{q}}{2}}(t) \le E'(t) + C_0 E^{\frac{q^+(\tau)}{2}}(t) \le E'(t) + \int_0^t (\mathbf{F} : \mathbf{S}, 1)_\Omega d\tau = (\rho \mathbf{f}, \mathbf{v})_\Omega. \tag{5.6}$$

If $\mathbf{f} \equiv 0$, we obtain the ordinary differential inequality $E'(t) + C_0 E^{\frac{\bar{q}}{2}}(t) \le 0$ and after a straightforward integration arrive at the estimate $E(t) \le \left[E^{\frac{2-\bar{q}}{2}}(0) - tC_0 \frac{2-\bar{q}}{2}\right]$. Since $E(t) \ge 0$ by definition, it is necessary that $E(t) \equiv 0$ for all $t \ge t^*$. Under the assumptions of item (2), from (5.4), (5.6) we derive the following inequality:

$$E'(t) + C_0 E^{\frac{\bar{q}}{2}}(t) \le \sqrt{2M}(1 - t/t_f)_+^{\bar{q}/(2-\bar{q})}, \ (\bar{E} \le 1).$$

Let $W(t)$ be a solution of the problem

$$W'(t) + C_0 W^{\frac{\bar{q}}{2}}(t) = \sqrt{2M}(1 - t/t_f)_+^{\bar{q}/(2-\bar{q})}, \quad W(0) = E(0).$$

Under condition (5.5) the problem for W has a unique solution $W(t) = E(0)\,[1 - t/t_f]_+^{\bar{q}/(2-\bar{q})}$, which is a majorant for $E(t)$, whence $E(t) \le W(t)$. $\square$

The borderline case $q^+(t) \nearrow 2$ as $t \to \infty$ was considered in [10]. The analysis of behavior of the fluid as $t \to \infty$ reduces to the study of the ordinary differential inequality with the variable exponent $E'(t) + C_0 E^{\frac{q^+(t)}{2}}(t) \le (\rho \mathbf{f}, \mathbf{v})_\Omega$. Properties of the functions satisfying the inequalities of this type were studied in [3, Ch.6]. Analogous localization results were proved for pseudoplastic fluid with vanishing or unbounded initial density $\rho(x, 0) = \rho_0(x)$ under the assumptions

$$\|1/\rho_0(x)\|_{L^m(\Omega)} \le C_1, \quad \|\rho_0(x)\|_{L^M(\Omega)} \le C_2, \quad \min\{m, M\} > 1, \tag{5.7}$$

$$\sqrt{\rho_0}\mathbf{v}_0 \in L^2(\Omega), \quad E(0) = \frac{1}{2}\int_\Omega \rho_0 |\mathbf{v}_0|^2 dx < \infty; \quad \delta |\mathbf{D}(\mathbf{r})|^q \le \mathbf{F}(\mathbf{r}) : \mathbf{D}(\mathbf{r}) \ \ \forall \mathbf{r} \in \mathbb{R}^N$$

with $\delta = const > 0$, $q \in \left(\frac{2MN}{N(M-1)+2M}, 2\right)$, $M > \frac{N}{2}$. Condition (5.7) allows the initial density ρ_0 to vanish or become infinite on any set of zero measure.

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