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A Short Introduction to Hilbert Space Theory

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A Short Introduction to Hilbert Space Theory

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Abstract. We present a short introduction to Hilbert spaces and linear operators defined on them.

1. Introduction

Hilbert spaces are the closest generalization to infinite dimensional spaces of the Euclidean spaces. These notes were written for students wishing a basic introduction to Hilbert space theory but who have no knowledge of Banach spaces. First, we consider a normed space and we see that if the space is finite dimensional all the norms defined on it generate the same topology, so that the convergence of sequences does not depend on the norm that is used. Later we consider linear transformations defined in a normed space and we see that all of them are continuous if the space is finite dimensional. Continuous linear transformations are called bounded operators and, by introducing a norm in the space of all bounded operators, we convert it into a Banach space under certain conditions. Once we have clear the concepts of convergence of a sequence and completeness of the space, we define Hilbert spaces and consider some of their properties. Mainly we focus on the Pythagorean theorem, the Cauchy-Schwarz inequality, the Parallelogram Identity and in the introduction of the concept of basis for a Hilbert space. Furthermore, we show that every Hilbert space of dimension n is isomorphic to $\mathbb{C}^n$ and that every separable Hilbert space is isomorphic to ℓ^2, the space of all square summable sequences.

One of the main theorems related to Hilbert spaces is the Riesz Representation Theorem, which characterizes the continuous linear transformations defined in a Hilbert space with values in $\mathbb{C}$. This theorem allows us to define the adjoint operator of a bounded operator. Finally we study the spectrum of an operator and we see some of the properties of the operator in terms of its spectrum.

The results included here are classical and can be found in many books of functional analysis.

2. Metric spaces and continuous functions

Let X be a set, *a metric (or distance)* d, defined in X, is a function $d : X \times X \to \mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$ such that

(i) $d(x, y) = d(y, x)$,

(ii) $d(x, z) \leq d(x, y) + d(y, z)$,

(iii) $d(x, y) = 0$ if and only if $x = y$.

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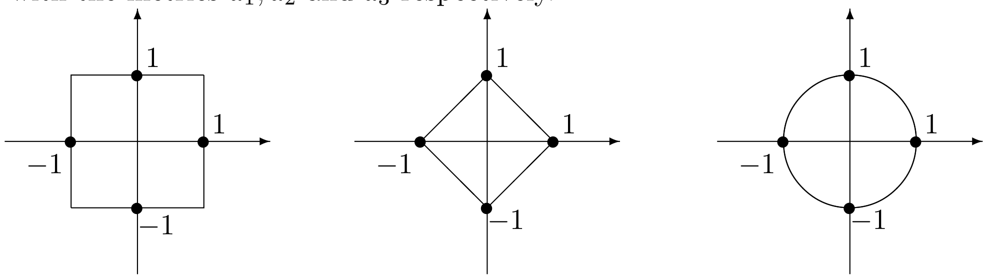
The classical examples of metrics defined in $\mathbb{R}^n$ are the following: for $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$

$$d_1(a,b) = \max\{|a_i - b_i|, i = 1, \ldots, n\},$$
$$d_2(a,b) = \sum_{k=1}^{n} |a_i - b_i|,$$
$$d_3(a,b) = \left(\sum_{k=1}^{n} (a_i - b_i)^2\right)^{1/2}.$$

Every set is a metric space with the *discrete metric*:

$$d(x,y) = \begin{cases} 1 & x \neq y, \\ 0 & x = y. \end{cases}$$

Let (X, d) be a metric space, $x \in X$ and $r > 0$. The *open ball* centered at the point x and with radius r is the set $B_r(x) = B(x,r) = \{y \in X : d(x,y) < r\}$. The following figure shows $B_1(0)$ in $\mathbb{R}^2$ with the metrics d_1, d_2 and d_3 respectively.



Open sets in a metric space (X, d) are defined as follows. A set $U \subset X$ is called open if, for each $x \in U$ there exists a radius $r > 0$, depending on x, such that $B_r(x) \subset U$. Observe that

(i) The union of open sets is an open set.
(ii) The intersection of a finite collection of open sets is an open set.
(iii) X and $\emptyset$ are open sets.

A family τ of subsets of X with properties i), ii) and iii) is called a *topology* for X.

For example, if X has the discrete metric then, any subset of X is open since

$$B_r(x) = \begin{cases} \{x\} \text{ if } r < 1, \\ X \text{ if } r \geq 1. \end{cases}$$

Different metrics defined on a set can produce the same topology, for example, in $\mathbb{R}^2$, the metrics d_1, d_2 and d_3 generate the same topology.

Let (x_n) be a sequence of points in a metric space (X, d). We say that the sequence (x_n) *converges* to x, and write $x_n \to x$, if given $\epsilon > 0$ there exists $N > 0$ such that for $n \geq N$, $d(x_n, x) < \epsilon$. A function $f : (X, d_X) \to (Y, d_Y)$ is called *continuous at the point* $x \in X$ if given any sequence (x_n) such that $x_n \to x$ then $f(x_n) \to f(x)$.

A sequence (x_n) in a metric space (X, d) is called a *Cauchy sequence* if given $\epsilon > 0$ there exists N such that for $n, m > N, d(x_n, x_m) < \epsilon$. Any convergent sequence is a Cauchy sequence and since the elements of a Cauchy sequence are getting closer and closer when n tends to ∞, it is natural to think that every Cauchy sequence is convergent. However, the following

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example shows that this is not the case. Let P be the set of polynomials in $[0,1]$ with the metric $d(p,q) = \sup_{x\in[0,1]} |p(x) - q(x)|$ and consider the sequence $p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Note that $p_n(x) \to e^x$, which is not a polynomial anymore. Another classical example of a Cauchy sequence that is not convergent is obtained by considering the vector space $\mathbb{Q}$, consisting of all rational numbers and an irrational point $x \in \mathbb{R} \setminus \mathbb{Q}$. The decimal expansion of x, $x = a.a_1a_2a_3...$ is the limit of the sequence of rational numbers $b_n = a.a_1 \dots a_n$. Thus, (b_n) is a Cauchy sequence that does not converge in $\mathbb{Q}$. Metric spaces such that every Cauchy sequence is convergent are called *complete spaces.*

Among all metric spaces those that have the structure of a vector space are quite important. Most of the examples given above are vector spaces. Let X be a vector space over $\mathbb{C}$ or $\mathbb{R}$. A norm $\| \, \|$ defined in X is a function $\| \, \| : X \to \mathbb{R}_+$, such that

(i) $\|x\| \geq 0, \|x\| = 0$ if and only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda| \|x\|$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

A normed space has a direct relation between its algebraic structure and its topological structure. A normed space which is a complete metric space with the distance $d(x,y) = \|x - y\|$ is called a *Banach space*. The most natural examples of Banach spaces are $\mathbb{C}^n, \mathbb{R}^n$ with the Euclidean norm. Actually, they are Banach spaces with any norm defined in them. This property is not exclusive of these spaces, every finite dimensional normed space is a Banach space (see [3], Theorem 2.4-2, for details).

As an example of an infinite dimensional Banach space we have $C([0,1])$, the set of all complex-valued continuous functions defined on $[0,1]$ with the uniform norm, $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

Consider now the set ℓ^2, consisting of all complex valued sequences (a_n) such that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, with the norm

$$\|(a_n)\| = \sqrt{\sum_{n=1}^{\infty} |a_n|^2}.$$

This space is a vector space with the following operations: if $a = (a_n), b = (b_n) \in \ell^2$ and $\lambda \in \mathbb{C}$ then

$$a + b = (a_n + b_n),$$
$$\lambda a = (\lambda a_n).$$

The space ℓ^2 is a Banach space that will play a very important role in the characterization of separable Hilbert spaces, in the sense we define later on.

Let (x_n) be a sequence in a normed space X, the series $\sum_{n=1}^{\infty} x_n$ is said to be *convergent* if the sequence (s_n), where $s_n = x_1 + x_2 + \cdots + x_n$, is convergent. The series $\sum_{n=1}^{\infty} x_n$ is said to be *absolutely convergent* if $\sum_{n=1}^{\infty} \|x_n\|$ converges. The following theorem, which proof follows easily, characterizes Banach spaces.

Theorem 2.1 *A normed space is a Banach space if and only if every absolutely convergent sequence is convergent.*

3. Hilbert spaces

For simplicity from now on we only consider vector spaces over $\mathbb{C}$. Let V be a vector space, an *inner product* is a function $\langle , \rangle : V \times V \to \mathbb{C}$ such that for $u, v, w \in V$ and $\lambda \in \mathbb{C}$,

(i) $\langle u, v\rangle = \overline{\langle v, u\rangle}$;
(ii) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$;
(iii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$;
(iv) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$.

One of the most important inequalities in an inner product space is the *Cauchy-Schwarz inequality*. It states the following, for any u, v in an inner product space

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Using last inequality it is easy to prove that $\|u\| = \sqrt{\langle u, u\rangle}$ is a norm in V.

A *Hilbert space* is a vector space with inner product $\langle\, ,\, \rangle$ such that it is a Banach space with the norm induced by the inner product.

The simplest example of a Hilbert space is $\mathbb{C}^n$ with the inner product:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The metric induced by this inner product is the usual Euclidean metric in $\mathbb{C}^n$.

The space ℓ^2 is an infinite dimensional vector space that is a Hilbert space with the product:

$$\langle a, b\rangle = \sum_n a_n \overline{b_n}, \text{ where } a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots).$$

It is easy to prove that if $\|\;\|$ is a norm induced by an inner product then,

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2.$$

The latter equation is called the *Parallelogram Identity* and is an important tool to determine if a Banach space is a Hilbert space or not. For example, the space of all continuous functions defined on $[0, 1]$ is not a Hilbert space since it does not hold the Parallelogram Identity. Indeed, let $f(x) = x$, $g(x) = 1$, then $\|f + g\| = 2, \|f - g\| = 1, \|f\| = 1, \|g\| = 1$.

If a norm holds the Parallelogram Identity this norm defines an inner product in the space by the *polarization identity*

$$\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2).$$

Two points x and y in an inner product space are *orthogonal* if $\langle x, y\rangle = 0$. A set of vectors $\{x_\alpha\}_{\alpha\in I}$ is *orthonormal* if $\|x_\alpha\| = 1$ for each $\alpha \in I$, and $\langle x_\alpha, x_\beta\rangle = 0$ if $\alpha \neq \beta$. Observe that, if x, y are orthogonal then,

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2,$$

which is the Pythagorean Theorem.

An orthonormal set in a Hilbert space $\mathcal{H}$ is called a *basis* of $\mathcal{H}$ if the set of all linear combinations of its elements is dense in $\mathcal{H}$. It means that we can approximate each element of $\mathcal{H}$ by linear combinations of the elements of its basis. If $\mathcal{H}$ is not a finite dimensional vector space, a basis in the sense given above is never a basis of $\mathcal{H}$ as a vector space or Hamel basis (see [1] exercise 16, Chapter 1, Section 4). Such as in the case of vector spaces a Hilbert space always has a basis.

The space $\mathbb{C}^n$ is a finite dimensional Hilbert space. A basis of $\mathbb{C}^n$ is $\{e_m\}_{m=1}^n$ where

$$e_m = (0, \dots, 0, 1, 0, \dots, 0).$$

In every vector space one can write an element of the space as a linear combination of the elements of the basis. If $\mathcal{H}$ is a Hilbert space which basis is not finite, we can write each element in terms of the elements of the basis.

In the following theorem, which proof can be found in [6], Theorems 4.14 and 4.15, the series $\sum_{n=1}^{\infty} a_n e_n$ has the usual meaning

$$\sum_{n=1}^{\infty} a_n e_n = \lim_{n\to\infty} \sum_{n=1}^{m} a_n e_n.$$

We consider a Hilbert space that has a numerable basis.

Theorem 3.1 *If $\{e_n\}_{n\geq 1}$ is a basis of a Hilbert space $\mathcal{H}$ then,*

(i) Each element $a \in \mathcal{H}$ can be written uniquely as

$$a = \sum_{n=1}^{\infty} a_n e_n, \; a_n = \langle a, e_n \rangle.$$

(ii) The inner product of two elements $a = \sum_{n=1}^{\infty} a_n e_n, b = \sum_{n=1}^{\infty} b_n e_n$, is given by

$$\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$$

and

$$\|a\|^2 = \sum_{n=1}^{\infty} |a_n|^2$$

For simplicity, in last theorem we considered a Hilbert space with a numerable basis. The general version of this theorem can be found in [1], Chapter 1, Lemma 4.13.

A Hilbert space with numerable or finite basis is called *separable*. On the other hand, a metric space is called separable if it has a numerable dense set. Since any Hilbert space is a metric space we could get confused with these two definitions of separability. However, a Hilbert space is separable if and only if, it is separable as metric space. It is easy to see that the set of all linear combinations of elements of the basis of $\mathcal{H}$, whose coefficients are complex numbers with rational components, is a numerable dense set in $\mathcal{H}$.

Let $(\mathcal{H}, \langle\, ,\, \rangle_{\mathcal{H}})$ and $(\mathcal{K}, \langle\, ,\, \rangle_{\mathcal{K}})$ be two Hilbert spaces. A linear operator $T : \mathcal{H}$ onto $\mathcal{K}$ is called an isomorphism if, for each $x, y \in \mathcal{H}$

$$\langle x, y \rangle_{\mathcal{H}} = \langle T(x), T(y) \rangle_{\mathcal{H}}.$$

Theorem 3.2 *Let $\mathcal{H}$ be a separable Hilbert space. Then,*

(i) If the dimension of the space is n then, $\mathcal{H}$ is isomorphic to $\mathbb{C}^n$.

(ii) If the space has a numerable basis, then $\mathcal{H}$ is isomorphic to ℓ^2.

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Proof. We only consider the infinite dimensional case. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Define $T:\mathcal{H}\to\ell^2$ by

$$T(y)=(\langle y,e_n\rangle)_{n\in\mathbb{N}},\ \text{if}\ y=\sum_{n=1}^{\infty}\langle y,e_n\rangle e_n.$$

Then,

$$\begin{aligned}\langle T(x),T(y)\rangle &= \sum_{n=1}^{\infty}\langle x,e_n\rangle\overline{\langle y,e_n\rangle}=\sum_{n=1}^{\infty}\langle x,\langle y,e_n\rangle e_n\rangle\\ &= \langle x,\sum_{n=1}^{\infty}\langle y,e_n\rangle e_n\rangle=\langle x,y\rangle.\end{aligned}$$

Thus, $\mathcal{H}$ is isomorphic to ℓ^2. $\square$

The latter theorem is very important concerning the Hilbert structure of the space. However, most of the Hilbert spaces used in applications have additional properties related to the elements of the space. For example, the Hilbert space could be the Bergman space, the Harmonic space, the Hardy space, etc. In any case the nature of the elements in the Hilbert space is very important.

4. Orthogonal projections

A set A in a metric (in general in a topological space) is *closed* if its complement is an open set. For the special case of metric spaces we can prove that a set A is closed using sequences of points in A. Indeed, A is closed if and only if for any convergent sequence (x_n) of points in A the limit of the sequence is also a point in A.

For infinite dimensional Banach spaces we can find non closed subspaces. For example, the set of all polynomials in $C[0,1]$ is a non closed subspace.

Theorem 4.1 *Any finite dimensional subspace of a Banach space is closed.*

Let $\mathcal{H}$ be a Hilbert space and $M\subset\mathcal{H}$. For a point $x\in\mathcal{H}$ the *distance* from x to M is defined by

$$d(x,M)=\inf\{d(x,m)|m\in M\}.$$

Theorem 4.2 *Let $\mathcal{H}$ be a Hilbert space and M a closed subspace of $\mathcal{H}$. Let $x\in\mathcal{H}\setminus M$, then there exists a unique point $z\in M$ such that*

$$\|x-z\|=\inf\{d(h,m)|m\in M\}=d(x,M):=d.$$

Proof. Let (y_n) be a sequence of points in M such that

$$\lim_{n\to\infty}\|x-y_m\|=d.$$

Then, using the Parallelogram Identity, we have that

$$\|y_n+y_m-2x\|^2+\|y_n-y_m\|^2=2\|y_n-x\|^2+2\|y_m-x\|^2.$$

This fact implies that

$$\|y_n-y_m\|^2=2\|y_n-x\|^2+2\|y_m-x\|^2-4\|\frac{y_n+y_m}{2}-x\|^2.$$

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Then,

$$\|y_n - y_m\|^2 \leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4d^2 \to 0 \text{ when } n, m \to \infty.$$

Thus, there exists $z \in M$ such that $y_n \to z$. Then,

$$d(x, M) = \lim_{n\to\infty} \|x - y_n\| = \|x - \lim_{n\to\infty} y_n\| = \|x - z\|.$$

Assume that there exists $z_1 \in M$ such that $d = \|x - z_1\|$. Then, the sequence $z, z_1, z, z_1, z, z_1, \ldots$ is such that $d = \lim_{n\to\infty} \|x - w_n\|$ where,

$$w_n = \begin{cases} z & \text{if } n \text{ is even,} \\ z_1 & \text{if } n \text{ is odd.} \end{cases}$$

Since $(x - w_n)$ converges, it is a Cauchy sequence which implies that (w_n) is a Cauchy sequence, then $z = z_1$. □

The existence of the element z in last theorem looks quite natural for normed spaces, this fact is not true for general Banach examples. The reader can find an example in [6], exercise 3.9. The reason our intuition fails is because we think of distances as Euclidean distances.

Let $M \subset \mathcal{H}$, the *orthogonal complement* of M is the set

$$M^\perp = \{h \in \mathcal{H} | \langle h, m\rangle = 0 \, \forall \, m \in M\}.$$

The orthogonal complement of M holds the following properties:

(i) $M^\perp$ is a closed subspace of $\mathcal{H}$.

(ii) If M is a closed subspace of $\mathcal{H}$ then, $(M^\perp)^\perp = M$.

(iii) $M \cap M^\perp = \{0\}$.

Theorem 4.3 *Let $\mathcal{H}$ be a Hilbert space and M a closed subspace of $\mathcal{H}$. Then, every element $x \in \mathcal{H}$ is written uniquely as*

$$x = z + w, \; z \in M \text{ and } w \in M^\perp.$$

That is, $\mathcal{H} = M \oplus M^\perp$.

Proof. Let $x \in \mathcal{H}$ and $z \in M$ such that

$$\|x - z\| = d(x, z) = d(x, M).$$

Let $w = x - z$, then using [6], Lemma 4.23, $w \in M^\perp$.

If $x = z_1 + w_1$, $z_1 \in M$ and $w_1 \in M^\perp$. Then,

$$z + w = z_1 + w_1,$$

Then

$$w - w_1 = z_1 - z \in M \cap M^\perp = \{0\}.$$

□

Using last decomposition of $\mathcal{H}$ as a direct sum of M and $M^\perp$, the *orthogonal projection* P_M from $\mathcal{H}$ onto M is defined as

$$P_M(x) = z.$$

Observe that $\|x\|^2 = \|z+w\|^2 = \|z\|^2+\|w\|^2 \geq \|P_M(x)\|^2$, which implies that P_M is a continuous linear operator.

5. Continuous linear functionals and the Riesz representation theorem

Let $\mathcal{H}$ be a Hilbert space, a linear function $\varphi : \mathcal{H} \to \mathbb{C}$ is called a functional (or linear functional). For example, let $\mathbb{C}^n, n_0 \in \{1, 2, \ldots, n\}$. Define

$$\varphi(z_1, z_2, \ldots, z_n) = z_{n_0}.$$

Then

$$\varphi(z) = \langle z, e_{n_0} \rangle,$$

where

$$e_{n_0} = (0, \ldots, 0, 1, 0, \ldots, 0), \text{ where the } n_0 \text{ component of } e_{n_0} \text{ equals } 1.$$

The functional φ is continuous.

Theorem 5.1 *Let $\mathcal{H}$ be a Hilbert space and $\varphi : \mathcal{H} \to \mathbb{C}$ a continuous functional. Then,*

$$\dim N(\varphi)^{\perp} = 1,$$

where $N(\varphi) = \{h \in \mathcal{H} | \varphi(h) = 0\}$.

Proof. It is easy to see that $N(\varphi)$ is a closed subspace of $\mathcal{H}$. Since $\varphi \neq 0, N(\varphi) \neq \mathcal{H}$. Then, $N(\varphi)^{\perp} \neq \{0\}$ (recall that $\mathcal{H} = N(\varphi) \oplus N(\varphi)^{\perp}$). Take $x_1 \neq 0, x_2 \neq 0$ in $N(\varphi)^{\perp}$. Then, there exists a complex number $a \neq 0$ such that

$$\varphi(x_1) + a\varphi(x_2) = 0.$$

Then $\varphi(x_1 + ax_2) = 0$, which implies that $x_1 + ax_2 \in N(\varphi) \cap N(\varphi)^{\perp} = \{0\}$. Thus, $\dim N(\varphi)^{\perp} = 1$. □

The following theorem is called the Riesz Representation Theorem.

Theorem 5.2 *Let $\mathcal{H}$ be a Hilbert space and $\varphi : \mathcal{H} \to \mathbb{C}$ a continuous linear functional. Then, there exists a unique $z \in \mathcal{H}$ such that*

$$\varphi(x) = \langle x, z \rangle \, \forall x \in \mathcal{H}.$$

Proof. Assume that $\varphi \neq 0$ then, $N(\varphi)^{\perp}$ has dimension 1. Take z_0 in $N(\varphi)^{\perp}$ with $\|z_0\| = 1$. Let $x \in \mathcal{H}$ then,

$$x = x - \langle x, z_0 \rangle z_0 + \langle x, z_0 \rangle z_0.$$

Observe that $\langle x - \langle x, z_0 \rangle z_0, z_0 \rangle = \langle x, z_0 \rangle - \langle x, z_0 \rangle \langle z_0, z_0 \rangle = 0$. Thus

$$x - \langle x, z_0 \rangle z_0 \in (N(\varphi)^{\perp})^{\perp} = N(\varphi).$$

Then,

$$\varphi(x) = \varphi(\langle x, z_0 \rangle z_0) = \langle x, z_0 \rangle \varphi(z_0) = \langle x, \overline{\varphi(z_0)} z_0 \rangle.$$

Let $z = \overline{\varphi(z_0)} z_0$. If $\varphi(x) = \langle x, z_1 \rangle$ then

$$\langle x, z - z_1 \rangle = 0 \, \forall \, x \in \mathcal{H}.$$

In particular $\langle z - z_1, z - z_1 \rangle = 0$ which implies that $z = z_1$. □

6. Bounded operators

Let $(X, \| \|_X), (Y, \| \|_Y)$ be two normed spaces. A linear transformation $T : X \to Y$ is called *bounded* if there exists $C > 0$ such that

$$\|T(x)\|_Y \leq C\|x\|_X, \qquad \forall x \in X. \tag{6.1}$$

For example,

(i) the identity operator $I : X \to X$ is bounded.

(ii) Let $x_0 \in [a, b]$, the function $T : C[a, b] \to \mathbb{C}$ given by $T(f) = f(x_0)$ is a bounded operator.

It is easy to prove that a linear operator T is bounded if and only if $T(M)$ is bounded for any bounded set M. The following theorem which proof can be found in any book of functional analysis, characterizes bounded operators.

Theorem 6.1 *Let $(X, \| \|_X), (Y, \| \|_Y)$ be two normed spaces and $T : X \to Y$ be a linear operator. The following sentences are equivalent.*

(i) A linear operator T is bounded.

(ii) The operator T is continuous.

(iii) The operator T is continuous at the point 0.

(iv) The operator T is continuous at some point x_0.

The set $\mathcal{B}(X, Y)$ consisting of all bounded operators from X to Y has a natural vector space structure. Furthermore, if $X = Y$, $\mathcal{B}(X) := \mathcal{B}(X, X)$ is an algebra, with the product of two elements given by its composition. Besides, the norm in this space is defined as

$$\|T\| = \inf\{C : C \text{ holds } (6.1)\}.$$

The latter norm can be calculated in many forms.

Theorem 6.2 .

(i) $\|T\| = \sup_{\|x\|\leq 1} \|T(x)\|$.

(ii) $\|T\| = \sup_{\|x\|=1} \|T(x)\|$.

(iii) $\|T\| = \sup_{x\neq 0} \frac{\|T(x)\|}{\|x\|}$.

The space $\mathcal{B}(X, Y)$ is a Banach space if and only if Y is a Banach space. As we mentioned above, $\mathcal{B}(X)$ is an algebra. Besides, if $S, T \in \mathcal{B}(X)$ then

$$\|ST\| \leq \|S\|\|T\|,$$

which makes $\mathcal{B}(X)$ a Banach algebra if X is a Banach space. For a detailed introduction to Banach algebras see [7].

As an example of a bounded operator, consider the set $M_{n,m}(\mathbb{C})$ consisting of all $n \times m$ complex valued matrices. Each linear transformation $T_A : \mathbb{C}^m \to \mathbb{C}^n$ is given by

$$T_A(x) = Ax,$$

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where A is the matrix of the linear transformation in the standard basis for $\mathbb{C}^n$ and $\mathbb{C}^m$:

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{pmatrix}. \tag{6.2}$$

It is easy to see that T_A is a bounded operator. Thus, every linear transformation $T : \mathbb{C}^m \to \mathbb{C}^n$ is bounded. We mentioned above that any two norms defined on a finite dimensional vector space generate the same family of open sets. The following theorem is a consequence of this fact.

Theorem 6.3 *If X is a finite dimensional normed space then, any linear operator T defined in X, is bounded.*

The latter theorem is not valid for infinite dimensional normed spaces. For example, let $P[0,1]$ be the set of all polynomials, defined on the interval $[0,1]$, with the norm

$$\|p\| = \sup_{x\in[0,1]} |p(x)|.$$

And define $T : P[0,1] \to P[0,1]$ by $T(f) = f'$.

For each $n \in \mathbb{N}$, take $f_n(x) = x^n$. Observe that $\|f_n\| = 1$ but $\|T(f_n)\| = \|nx^{n-1}\| = n$, which shows that T is not bounded.

Two important examples of bounded operators defined in ℓ^2 are the shift operators:

$$S(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots), \tag{6.3}$$
$$T(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots). \tag{6.4}$$

Theorem 6.4 *Let $(\mathcal{H}, \langle , \rangle)$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then, there exists a unique bounded operator $T^* : \mathcal{H} \to \mathcal{H}$ such that for $x, y \in \mathcal{H}$,*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle.$$

Proof. Take $y \in \mathcal{H}$ and define

$$L(x) = \langle T(x), y\rangle.$$

Then,

$$\|L(x)\| \le \|T(x)\|\|y\| \le \|T\|\|x\|\|y\|.$$

Using the Riesz Representation Theorem, there exists a unique $h \in \mathcal{H}$ such that

$$L(x) = \langle x, h\rangle.$$

Define $T^*(y) = h$. A detailed proof of the boundedness of T^* can be found in [5], Ch. 12. □

For example, consider the shift operators S, T, defined in (6.3) and (6.4) respectively. Then,

$$\langle S(a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots)\rangle = \sum_{n=1}^{\infty} a_n \overline{b_{n+1}} = \langle (a_1, a_2, a_3, \dots), (b_2, b_3, b_4, \dots)\rangle.$$

Thus $S^* = T$. Similar calculations show that $T^* = S$, which means $(T^*)^* = T$.

Actually the operation $T \mapsto T^*$ is an involution in $\mathcal{B}(\mathcal{H})$. That means, for $S, T \in \mathcal{B}(\mathcal{H})$

(i) $(T^*)^* = T$,
(ii) $(ST)^* = T^*S^*$,
(iii) $(S+T)^* = S^* + T^*$,
(iv) $(\lambda T)^* = \overline{\lambda}T^*$.

Besides, it is easy to see that $\|T^*\| = \|T\|$ and that $\|T^*T\| = \|T\|^2$. The last properties of the adjoint operator and the fact that $\mathcal{B}(\mathcal{H})$ is a Banach algebra imply that $\mathcal{B}(\mathcal{H})$ is a C*-algebra, see [4] for details.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be

(i) *self adjoint or hermitian* if $T = T^*$,
(ii) *normal* if $TT^* = T^*T$,
(iii) *unitary* if $T^*T = TT^* = I$.

7. The spectrum of a linear operator

Let X be a Banach space the operator $T \in \mathcal{B}(X)$ is called *invertible* if there exists $S \in \mathcal{B}(X)$ such that

$$ST = I \text{ and } TS = I,$$

where I denotes the identity operator in X. If X is a finite dimensional normed space and $T : X \to X$ is a linear operator, the following statements are equivalent

(i) T is invertible,
(ii) T is injective,
(iii) T is surjective,
(iv) there exists $S \in \mathcal{B}(X)$ such that $ST = I$,
(v) there exists $S \in \mathcal{B}(X)$ such that $TS = I$.

The last sentences are not equivalent if X is not a finite dimensional normed space. For example, the operators S and T defined in (6.3) and (6.4) respectively satisfy $ST \neq I$ and $TS = I$.

Let $T \in \mathcal{B}(X)$, the *spectrum* of T is defined as the set

$$\sigma(T) = \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not invertible}\}.$$

For example, if we have the linear transformation $T_A : \mathbb{C}^n \to \mathbb{C}^n$, given in (6.2), the spectrum of this operator is the set of all eigenvalues of the matrix A.

The following theorem generalizes the formula

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \text{ if } |z| < 1.$$

Theorem 7.1 *If $T \in \mathcal{B}(X)$ is such that $\|T\| < 1$ then, $I - T \in \mathrm{Inv}(\mathcal{B}(X))$ and*

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Proof. Since $\|T\| < 1$ and $\|T^n\| \leq \|T\|^n$ the series $\sum_{n=0}^{\infty} \|T^n\|$ is convergent which implies that $\sum_{n=0}^{\infty} T^n$ converges (see Theorem 2.1). Define

$$S = \sum_{n=0}^{\infty} T^n = \lim_{N\to\infty} \sum_{n=0}^{N} T^n.$$

Thus,

$$(I-T)S = S - TS = \sum_{n=0}^{\infty} T^n - \sum_{n=0}^{\infty} T^{n+1} = I.$$

Similarly $S(I-T) = I$. Thus,

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

□

Denote by $\mathrm{Inv}(\mathcal{B}(X))$ the set of all invertible operators. Last theorem has two very important consequences.

(i) $\mathrm{Inv}(\mathcal{B}(X))$ is an open set in $\mathcal{B}(X)$.

(ii) The inversion $T \mapsto T^{-1}$ is a continuous function on $\mathrm{Inv}(\mathcal{B}(X))$.

Theorem 7.2 *The spectrum $\sigma(T)$ of an operator T, is a compact set of the complex plane and it is contained in the closed disk $\{z \in \mathbb{C} | |z| \leq \|T\|\}$.*

Proof. $\mathbb{C} - \sigma(T)$ is open because $\mathrm{Inv}(\mathcal{B}(X))$ is open and the inversion is continuous. If $\lambda \in \mathbb{C}$ is such that $|\lambda| > \|T\|$ then, $\lambda I - T = \lambda(I - \frac{T}{\lambda})$ is an invertible operator, which implies that $\lambda \notin \sigma(T)$. □

One can prove that, the spectrum of any element $T \in \mathcal{B}(\mathcal{H})$ is not empty. In [7] can be found the proof of this result in the context of Banach algebras. This proof is based in the Hahn-Banach Theorem and the Liouville Theorem.

The spectrum of an operator plays a very important role in the study of the properties of this operator. The following theorem shows this fact.

Theorem 7.3 ([5]) *A normal operator $T \in \mathcal{B}(\mathcal{H})$ is*

(i) self adjoint if and only if $\sigma(T)$ lies in the real axis,

(ii) unitary if and only if $\sigma(T)$ lies in the unit circle.

Many properties of an operator such as compactness are closed related to the properties of the spectrum of the operator. Fur further reading on this topic we refer the reader to the references.

References

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