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D I Sinelshchikov and N A Kudryashov

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Moscow, Russia

E-mail: nakudr@gmail.com, disine@gmail.com

Abstract. We consider a family of nonlinear diffusion equations with nonlinear sources. We assume that all nonlinearities are polynomials with respect to a dependent variable. The traveling wave reduction of this family of equations is an equation of the Lienard–type. Applying recently obtained criteria for integrability of Lienard–type equations we find some new integrable families of traveling wave reductions of nonlinear diffusion equations as well as their general analytical solutions.

1. Introduction

In this work we consider the following family of equations

$$u_t = (u^m u_x)_x + a_1 u^k + a_2 u^l + a_3 u^p, \tag{1}$$

where x, t and $u(x,t)$ are independent and dependent variables correspondingly, a_i, $i = 1,2,3$, $a_1^2 + a_2^2 + a_3^2 \neq 0$ and m, k, l, p, $m^2 + k^2 + l^2 + p^2 \neq 0$ are arbitrary real parameters.

Equation (1) can be considered as a nonlinear diffusion or nonlinear heat equation with polynomial sources, which are both frequent in many applications (see, e.g. [1]). For example, the generalized Fisher equation, the Newell–Whitehead equation and Zeldovich equation belong to family (1) [1–3]. It is known (see, e.g. [1,2,4] and references therein) that traveling wave solutions of equations like (1) are very important for understanding of both underlying phenomenon and solutions' behaviour. Therefore, a problem of constructing general analytical traveling wave solutions of equations from family (1) is worth studying.

Assuming that $u(x,t) = y(z)$, $z = x - C_0 t$ in (1) we obtain its traveling wave reduction

$$y_{zz} + \frac{m}{y} y_z^2 + C_0 y^{-m} y_z + a_1 y^{k-m} + a_2 y^{l-m} + a_3 y^{p-m} = 0, \tag{2}$$

where $C_0 \neq 0$ is traveling waves speed. Equation (2) is a nonlinear ordinary differential equation of the Lienard type. Some approaches for finding integrable cases of such equations have recently been proposed (see, e.g. [5–14] and references therein). In work [14] several new integrability conditions for the Lienard–type equations were found. In this work we apply these conditions for finding integrable cases of (2). As a result, we find four subcases of (2) for which the closed–from general analytical solution can be constructed. We believe that these families of integrable traveling wave reductions of (1) have not been reported previously.

2. Integrable traveling wave reductions of nonlinear diffusion equation (1)

In this section we apply integrability criteria obtained in [14] for finding integrable cases of (2). The approach of work [14] is based on using of the following nonlocal transformations

$$w = F(y), \quad d\zeta = G(y)dz, \quad F_y G \neq 0, \tag{3}$$

where w and ζ are new dependent and independent variables. Connections between the Lienard–type equations and Painleve–Gambier equations (see, e.g. [15])of type III via (3) were studied in [14] and four integrability criteria were found. Below we apply these integrability criteria for finding general analytical solutions for some members of equations family (2). Let us remark that throughout this work we assume that the parameter κ is equal to 0 in all criteria from work [14]. Note also that below we use Ince's notation for the Painleve–Gambier equations [15].

We start with the criterion for integrability of (2) that connects Lienard–type equations with Painleve–Gambier equation XXIII (see Theorem 1, [14]). It can be shown that equation (2) satisfies this criterion when $k = 5 - m$, $l = 1 - m$, $a_2 = 3C_0^2/16$ and $a_3 = 0$. As a result, we obtain that the following integrable equation from family (2)

$$y_{zz} + \frac{m}{y} y_z^2 + C_0 y^{-m} y_z + a_1 y^{5-2m} + \frac{3C_0^2}{16} y^{1-2m} = 0, \tag{4}$$

can be transformed into Painleve–Gambier equation XXIII

$$w_{\zeta\zeta} - \frac{3}{4w} w_\zeta^2 + 2\alpha w_\zeta + w^2 + 3\alpha^2 w = 0, \tag{5}$$

via transformations (3) with $F = 16\alpha^2 a_1 C_0^{-2} y^4$ and $G = (C_0/2\alpha) y^{-m}$. Here $\alpha \neq 0$ is an arbitrary parameter.

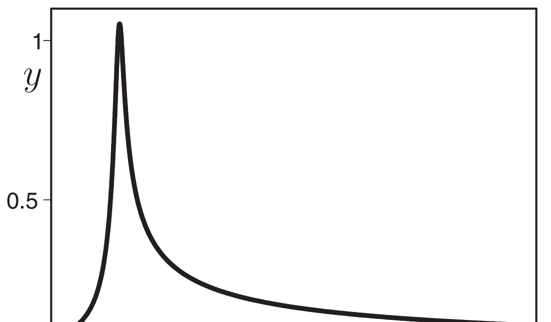

Figure 1. Solution (6) at $m = -2$, $\zeta_0 = 1.1$ and $\alpha = C_0 = g_3 = a_1 = 1$.

Taking into account the general solution of equation (5) we find the general solution of (4)

$$y = \left[\frac{3C_0 g_3}{16 a_1} e^{-2\alpha(\zeta - \zeta_0)} \wp^{-2}\{e^{-\alpha(\zeta-\zeta_0)}, 0, g_3\} \right]^{1/4}, \quad z = \frac{2\alpha}{C_0} \int y^m d\zeta, \tag{6}$$

where $\wp$ is the Weierstrass elliptic function, $g_3 \neq 0$ and ζ_0 are arbitrary constants. We demonstrate a plot of solution (6) for certain values of the parameters in Fig.1. One can see that in this case solution (6) describes a pulse–type structure.

Now we proceed to the next criterion form [14] that connects Lienard–type equations with equation XXV of the Painleve–Gambier type (see Theorem 2, [14]). This criterion holds for equation (2) if $k = -m - 3/5$, $l = -m + 1$, $p = -m - 7/5$ and $a_2 = 5C_0^2/36$. Consequently, the following equation from family (2) is integrable:

$$y_{zz} + \frac{m}{y} y_z^2 + C_0 y^{-m} y_z + a_1 y^{-2m-3/5} + \frac{5C_0^2}{36} y^{1-2m} + a_3 y^{-2m-7/5} = 0. \tag{7}$$

This equation can be transformed into Painleve–Gambier equation XXV

$$w_{\zeta\zeta} - \frac{3}{4w} w_\zeta^2 + \frac{3}{2} w w_\zeta + \frac{1}{4} w^3 + \frac{9a_1}{5} w + \frac{18C_0 a_3}{10} = 0, \tag{8}$$

with the help of transformations (3) with $F = C_0 y^{4/5}$ and $G = (2/3) y^{-4/5-m}$,

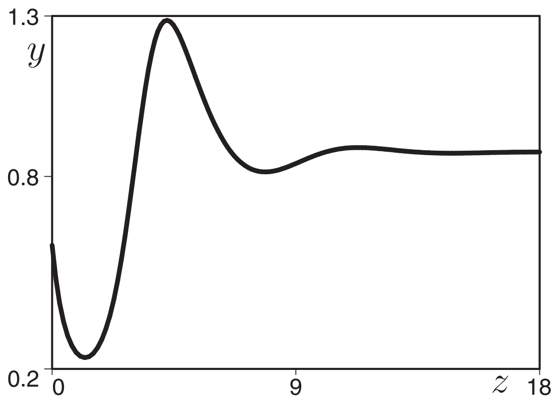

Figure 2. Solution (9) at $m = a_3 = -1$ and $C_0 = 1$.

One can obtain the general solution of (7) by virtue of (3) and the general solution of (8) as follows

$$y = \left(\frac{w}{C_0}\right)^{5/4}, \quad z = \frac{3}{2}\int y^{m+4/5} d\zeta, \quad w = -\frac{9C_0 a_3}{10 v_\zeta + 5v^2 + 9a_1}, \quad v = \frac{\Psi_\zeta}{\psi}, \tag{9}$$

where Ψ is the general solution of the equation

$$\Psi_{\zeta\zeta\zeta} + \frac{9}{5} a_1 \Psi_\zeta + \frac{9C_0}{10} a_3 \Psi = 0. \tag{10}$$

A plot of solution (9) for certain values of the parameters is shown in Fig.2. We see that solution (9) can describe pulse–type structures with oscillations.

Let us consider criterion of equivalence between Lienard–type equations and Painleve–Gambier equation XXVIII (see Theorem 3 work [14]). Equation (2) satisfies this criterion at $a_1 = (n-1)\epsilon n^2/(n-2)^2$, $a_3 = 0$, $a_2 = -(n-1)C_0^2/(n-2)^2$, $k = -m - (n+1)/(n-1)$ and $l = 1 - m$ and in this case it has the form

$$y_{zz} + \frac{m}{y} y_z^2 + C_0 y^{-m} + \frac{(n-1)\epsilon n^2}{(n-2)^2} y^{-2m-\frac{n+1}{n-1}} - \frac{(n-1)C_0^2}{(n-2)^2} y^{1-2m} = 0. \tag{11}$$

Equation (11) can be transformed into

$$w_{\zeta\zeta} - \frac{n-1}{n}\frac{w_\zeta^2}{w} + \frac{n-2}{n}\frac{w_\zeta}{w} + \frac{1}{nw} - n\epsilon w = 0, \tag{12}$$

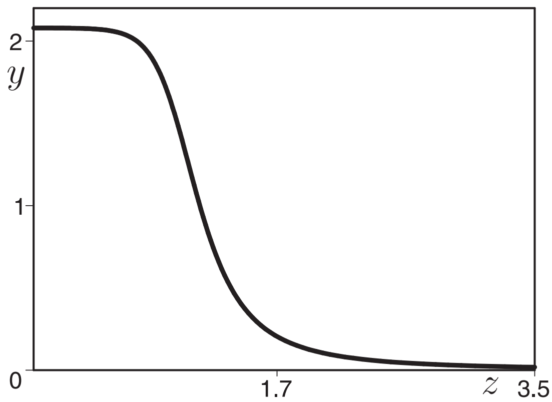

Figure 3. Solution (13) at $n = 3$, $m = -2$, $C_0 = 1$, $C_1 = 0$, $C_2 = 1$ and $C_3 = 10000$.

where $\epsilon \neq 0$ is an arbitrary parameter, $n \neq 0, \pm 1, \pm 2$ is an integer number, with the help of transformations (3) with $F = y^{-\frac{n}{n-1}}/C_0$, $G = ny^{-m-\frac{n}{n-1}}/(n-2)$.

Therefore, we find the general solution of equation (12)

$$y = (C_0 w)^{-\frac{n-1}{n}}, \quad z = \frac{n-2}{n} \int y^{m+\frac{n}{n-1}} d\zeta, \tag{13}$$

where w is defined by the relation

$$w = \psi^n \left(C_1 + \int \psi^{-n} d\zeta \right), \quad \psi = C_2 e^{\sqrt{\epsilon}\zeta} + C_3 e^{-\sqrt{\epsilon}\zeta}, \tag{14}$$

where C_1, C_2 and C_3 are arbitrary constants. We demonstrate a plot of solution (13) for particular values of the parameters in Fig. 3. We see that this solution can describe kink–type structures.

Finally, we consider connections via (3) between equation (2) and Painleve–Gambier XVIII equation. Using Theorem 4 from [14] we find that equation (2) satisfies this criterion in the case of $a_3 = 0$, $a_1 = 108\beta/C_0^2$, $a_2 = -3C_0^2/4$, $k = -m - 5/3$ and $l = -m - 1$. Consequently, we get the corresponding equation from family (2)

$$y_{zz} + \frac{m}{y} y_z^2 + C_0 y^{-m} y_z + \frac{108\beta}{C_0^2} y^{-2m-5/3} - \frac{3C_0^2}{4} y^{-2m+1} = 0, \tag{15}$$

that can be transformed into

$$w_{\zeta\zeta} - \frac{1}{2w} w_\zeta^2 + w w_\zeta - \frac{1}{2} w^3 + \frac{72\beta}{w} = 0 \tag{16}$$

by virtue of with (3) $F = C_0 y^{2/3}$ and $G = y^{-2/3-m}$. Here $\beta \neq 0$ is an arbitrary parameter. The general solution of (15) can be found via the general solution of (16) and corresponding transformations (3)

$$y = \left[\frac{6(\wp^2\{\zeta - \zeta_0, 12\beta, g_3\} - \beta)}{C_0 \wp_z(\zeta - \zeta_0, 12\beta, g_3)} \right]^{3/2}, \quad z = \int y^{m+2/3} d\zeta, \tag{17}$$

where g_3 and ζ_0 are arbitrary constants. Solution (17) as long as $\beta \neq 0$ and $g_3 \neq 0$ is periodic, although it may be singular on the real line.

3. Conclusion

In this work we have considered a family of nonlinear diffusion equations with polynomial sources. Traveling wave reductions of these equations are nonlinear ordinary differential equations of Lienard type. Applying recently obtained integrability conditions for the Lienard–type equations, we have found four new integrable families of traveling wave reductions of nonlinear diffusion equations with polynomial sources. We have also constructed the corresponding general solutions in the explicit form.

4. Acknowledgments

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