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A remark on geometric desingularization of a non-hyperbolic point using hyperbolic space

Christian Kuehn

Institute for Analysis and Scientific Computing, Vienna University of Technology, 1040 Vienna, Austria

E-mail: `ck274@cornell.edu`

Abstract. A steady state (or equilibrium point) of a dynamical system is hyperbolic if the Jacobian at the steady state has no eigenvalues with zero real parts. In this case, the linearized system does qualitatively capture the dynamics in a small neighborhood of the hyperbolic steady state. However, one is often forced to consider non-hyperbolic steady states, for example in the context of bifurcation theory. A geometric technique to desingularize non-hyperbolic points is the blow-up method. The classical case of the method is motivated by desingularization techniques arising in algebraic geometry. The idea is to blow up the steady state to a sphere or a cylinder. In the blown-up space, one is then often able to gain additional hyperbolicity at steady states. The method has also turned out to be a key tool to desingularize multiple time scale dynamical systems with singularities. In this paper, we discuss an explicit example of the blow-up method where we replace the sphere in the blow-up by hyperbolic space. It is shown that the calculations work in the hyperbolic space case as for the spherical case. This approach may be even slightly more convenient if one wants to work with directional charts. Hence, it is demonstrated that the sphere should be viewed as an auxiliary object in the blow-up construction. Other smooth manifolds are also natural candidates to be inserted at steady states. Furthermore, we conjecture several problems where replacing the sphere could be particularly useful, i.e., in the context of singularities of geometric flows, for avoiding compactification, and generating 'interior' steady states.

1. Introduction

Consider an ordinary differential equation (ODE) given by

$$\frac{dz}{dt} = z' = f(z), \tag{1}$$

where $z = z(t) \in \mathbb{R}^N$, $N \in \mathbb{N}$, $t \in \mathbb{R}$ and $f : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be sufficiently smooth. Suppose $z^* \in \mathbb{R}^N$ is a steady state (or equilibrium point) of (1), i.e., $f(z^*) = 0$. Using a translation of coordinates, if necessary, we may assume for the following analysis without loss of generality that $z^* = 0 := (0, 0, \ldots, 0) \in \mathbb{R}^N$. The first standard calculation for steady states is to consider the linearized system in a neighborhood of the steady state

$$Z' = [\mathrm{D}f(0)]Z, \tag{2}$$

where $Z \in \mathbb{R}^N$ and $\mathrm{D}f(0) \in \mathbb{R}^{N\times N}$ denotes the total derivative of f evaluated at $z = 0$. It is also common to refer to $\mathrm{D}f(0)$ as the Jacobian matrix or simply the Jacobian. Let λ_n for

$n \in \{1,2,\ldots,N\}$ denote the eigenvalues of $\mathrm{D}f(0)$. If the eigenvalues have no zero real parts, $\mathrm{Re}(\lambda_n) \neq 0$ for all n, then the steady state $z^* = 0$ is called hyperbolic. The Hartman-Grobman Theorem (see e.g. [22, p.120-121]) implies that in a neighborhood of a hyperbolic steady state, the flows generated by (1) and (2) are topologically conjugate. For most practical purposes this implies that we may just use the linear ODE (2) to study the dynamics near $z^* = 0$.

However, non-hyperbolic points are unavoidable if we want to analyze bifurcation points [7, 18]. The linearization approach breaks down and one has to carefully consider the influence of nonlinear terms. One possible technique that can be very successful in this context is geometric desingularization; see e.g. [4, p.67-70] for a particular example or [3] for general planar singularities. We are going to introduce geometric desingularization via the blow-up method in more detail in Section 2.

The main geometric idea of the method arose in algebraic geometry in the context of desingularization of algebraic varieties [9, p.29], where one replaces certain singular points by projective space. The resulting variety either has no singular points anymore or one can try to repeat the blow-up. Under certain conditions one may indeed reach a complete desingularization as stated in the celebrated Hironaka Theorem [10, 11].

In the context of ODEs, the classical strategy involves using a spherical blow-up as one works in real space and not in the context of (complex) projective space. The key difference to the algebraic geometry blow-up is that one also has to keep track of the dynamics on the blown-up space. There has been a tremendous amount of work on using the blow-up technique for planar ODEs [3, 4, 2], canard solutions [6, 12, 14, 24], traveling wave problems [20, 5] and a large variety of other problems in the theory of multiple time scale dynamical systems [19, 8, 13, 15, 17]. For fast-slow dynamical systems of the form

$$\begin{array}{rcl} x' &=& f_1(x,y,\varepsilon), \\ y' &=& \varepsilon g(x,y,\varepsilon), \end{array} \tag{3}$$

with $0 < \varepsilon \ll 1$ and $(x,y) \in \mathbb{R}^{m+n}$, it is crucial to observe [6] that the blow-up is frequently very helpful. Indeed, consider the formal limit $\varepsilon \to 0$ in (3) which yields

$$\begin{array}{rcl} x' &=& f_1(x,y,0), \\ y' &=& 0. \end{array} \tag{4}$$

Linearizing around a steady state, say $(x^*,y^*) = (0,0) =: 0$ of the fast subsystem (4) means computing the total derivative matrix $D_x f(0) \in \mathbb{R}^{m\times m}$, which is generically not hyperbolic due to the presence of the free parameters (or frozen slow variables) $y \in \mathbb{R}^n$. Therefore, one has to deal with non-hyperbolic points for fast-slow systems [6, 16] and re-write them in the form

$$\begin{array}{rcl} x' &=& f_1(x,y,\varepsilon), \\ y' &=& \varepsilon g(x,y,\varepsilon), \\ \varepsilon' &=& 0. \end{array} \tag{5}$$

Setting $z = (x,y,\varepsilon)$, $f_2(x,y,\varepsilon) = \varepsilon g(x,y,\varepsilon)$, $f_3(x,y,\varepsilon) = 0$, and $f := (f_1,f_2,f_3)^\top$ yields precisely a problem of the form (1) with a non-hyperbolic point at the origin.

Using a spherical, or cylindrical, space is currently the standard choice to desingularize non-hyperbolic steady states of ODEs. However, there seems to be no apparent reason why other manifolds could not function equally well, or even better. In this paper, we investigate this idea in more detail and consider a simple example to illustrate the main idea. The spherical case is discussed in Section 2, which is also a fully self-contained introduction to the blow-up method.

In Section 3 we replace the sphere by hyperbolic space, i.e. by using a manifold with constant negative curvature. We emphasize that the word 'hyperbolic' is then used in two distinct ways: (1) for the dynamical type of a steady state and (2) for a smooth manifold which replaces the sphere in the blown-up space. The results in Section 3 confirm the intuition that using a spherical blown-up space is not crucial and hyperbolic space works also for geometric desingularization in the example. This indicates that one should be open-minded about trying to use different manifolds for geometric desingularization. Further directions for this new approach are sketched in Section 4. Three areas are identified, where using manifolds different from spheres could be beneficial.

2. Spherical Blow-Up

In this section a basic test example for the blow-up method is reviewed from [4] and more explicit calculations for this example are provided. The spherical blow-up is constructed in this context, which leads to a geometric desingularization of the problem.

Consider the following planar ODE [4] for $z(t) = (x(t), y(t)) \in \mathbb{R}^2$

$$\begin{array}{lllllll} \frac{dx}{dt} & = & x' & = & ax^2 - 2xy & =: & f_1(x,y), \\ \frac{dy}{dt} & = & y' & = & y^2 - axy & =: & f_2(x,y), \end{array} \tag{6}$$

where $a > 0$ is a positive parameter, we abbreviate $(x, y) = (x(t), y(t))$ and we denote the vector field by $f := (f_1, f_2)^\top$, where $(\cdot)^\top$ denotes the transpose.

Remark: We note that we are not interested here in trying to solve the problem (6) directly but to use it to illustrate how the blow-up method can resolve the fine-scale geometry of the problem.

We may view the vector field f as a smooth section into the tangent bundle $f : \mathbb{R}^2 \to \mathrm{T}\mathbb{R}^2$. If $p \in \mathbb{R}^2$ is a given point, then we shall usually employ the natural identification of the tangent space $\mathrm{T}_p\mathbb{R}^2 \cong \mathbb{R}^2$.

Observe that $(x, y) = (0, 0) =: 0$ is a steady state, i.e. $f_1(0) = 0 = f_2(0)$, for (6). It is straightforward to compute the linearized system $Z = (X, Y) \in \mathbb{R}^2$ at the origin

$$\begin{aligned} \begin{pmatrix} X' \\ Y' \end{pmatrix} = [\mathrm{D}f(0)] \begin{pmatrix} X \\ Y \end{pmatrix} &= \left.\begin{pmatrix} 2ax - 2y & -2x \\ -ay & 2y - ax \end{pmatrix}\right|_{\{x=0=y\}} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \end{aligned}$$

where we shall always employ capital variables $Z = (X, Y) \in \mathbb{R}^2$ to emphasize when we work with a linearized problem. We see that the origin is a non-hyperbolic steady state since $\mathrm{D}f(0)$ has two zero eigenvalues; see also Figure 1(a). Hence, further analysis is required and the blow-up method provides one approach to understand the dynamics.

For planar vector fields, the classical approach of the blow-up method is to use a transformation which replaces the point p with a (unit) circle

$$\mathcal{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{(x, y) \in \mathbb{R}^2 : x = \cos\theta, y = \sin\theta, \theta \in [0, 2\pi)\}.$$

In higher-dimensional cases, one usually uses spheres or cylinders. Formally, we fix $r_0 > 0$, consider the interval $\mathcal{I} := [0, r_0]$ and define the manifold

$$\mathcal{B} := \mathcal{S}^1 \times \mathcal{I}. \tag{7}$$

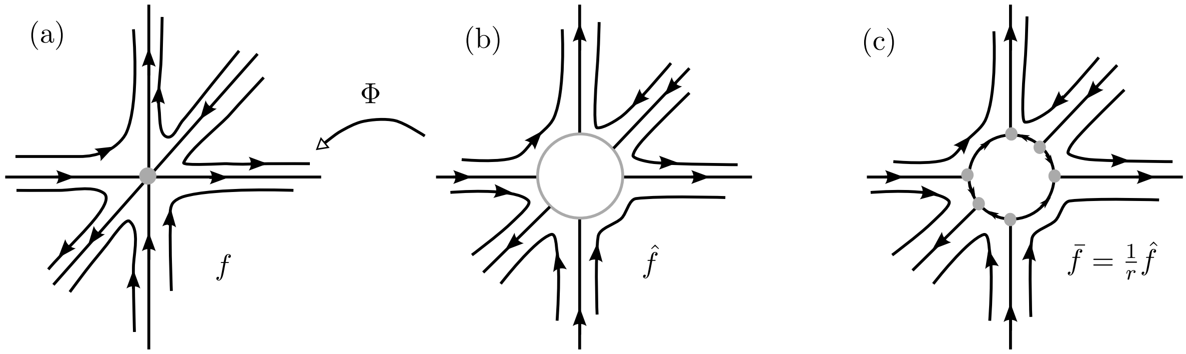

Figure 1. Sketch of the main steps of the (spherical) blow-up method for the example (6). (a) Original vector field f with non-hyperbolic steady state (gray) at the origin. (b) Blown-up vector field $\hat{f}$ on $\mathcal{B}$ with a full circle of steady states (gray) given by $\mathcal{S}^1 \times \{r = 0\}$. (c) Desingularized blown-up vector field $\bar{f}$ with precisely six hyperbolic saddle steady states (gray). The small arrows on $\mathcal{S}^1 \times \{r = 0\}$ indicate the qualitative part of the flow which is different from $\hat{f}$. Observe that the flow directions are compatible with the phase portrait for $\mathcal{S}^1 \times \{r > 0\}$.

Sometimes other choices for $\mathcal{I}$ are convenient such as $\mathcal{I} = \mathbb{R}$, $\mathcal{I} = [-r_0, r_0]$ or $\mathcal{I} = [0, \infty)$ but in our context $\mathcal{I} := [0, r_0]$ will suffice. A spherical blow-up transformation is given by

$$\Phi : \mathcal{B} \to \mathbb{R}^2,$$

where the map Φ will be defined algebraically below. We already note that if Φ is differentiable then the push-forward $\Phi_* : \mathrm{T}\mathcal{B} \to \mathbb{R}^2$ induces a vector field $\hat{f}$ on the blown-up space $\mathcal{B}$ if we require the condition

$$\Phi_* \left(\hat{f}\right) = f.$$

One possibility to define Φ algebraically is to use the weighted polar blow-up. Let $(\theta, r) \in \mathcal{S}^1 \times [0, r_0]$ be coordinates for $\mathcal{B}$ and define

$$\Phi(\theta, r) = (r^\alpha \cos\theta, r^\beta \sin\theta) = (x, y),$$

where $\alpha, \beta \in \mathbb{R}$ are the weights to be chosen below and $\theta \in [0, 2\pi)$. Observe that Φ is a diffeomorphism outside of the circle $\mathcal{S}^1 \times \{r = 0\}$, which corresponds to the steady state $p = (0, 0)$. Hence, the polar blow-up transformation indeed inserts a circle at the non-hyperbolic point and topologically conjugates the dynamics between

$$\mathbb{R}^2 - \{(0,0)\} \qquad \text{and} \qquad \mathcal{B} - \left[\mathcal{S}^1 \times \{r = 0\}\right].$$

To determine good weights α and β one may use quasi-homogeneity of the vector field; recall that f is quasi-homogeneous of type (α, β) and degree $k + 1$ if

$$f(r^\alpha x, r^\beta y) = (r^{\alpha+k} f_1(x, y), r^{\beta+k} f_2(x, y))^\top. \tag{8}$$

Substituting the vector field (6) into (8) yields

$$\begin{array}{rcl} r^{2\alpha} a x^2 - r^{\alpha+\beta} 2xy & = & r^{\alpha+k}(ax^2 - 2xy), \\ r^{2\beta} y^2 - r^{\alpha+\beta} axy & = & r^{\beta+k}(y^2 - axy). \end{array} \tag{9}$$

Therefore, the vector field f is quasi-homogeneous of type $(\alpha, \beta) = (1, 1)$ and degree 2 (with $k = 1$). Then one chooses the blow-up weights as the type of the quasi-homogeneous vector field so that for (6) we just have a polar coordinate change

$$\Phi(\theta, r) = (r \cos\theta, r \sin\theta) = (x, y).$$

Lemma 2.1. *The vector field $\hat{f}$ in polar coordinates is given by*

$$\begin{array}{rcl}\theta' & = & r\left(3\cos\theta\sin^2\theta - 2a\sin\theta\cos^2\theta\right),\\ r' & = & r^2(a\cos\theta - 2\sin\theta - 2a\cos\theta\sin^2\theta + 3\sin^3\theta).\end{array} \tag{10}$$

Proof. One possibility is to note that $\hat{f}(\theta, r) = (\mathrm{D}\Phi)^{-1} f(\Phi(\theta, r))$ and calculate. Alternatively, one may proceed slightly more directly

$$\begin{array}{rcccl} ar^2\cos^2\theta - 2r\cos\theta\sin\theta & = & x' & = & r'\cos\theta - r\theta'\sin\theta,\\ r^2\sin^2\theta - ar^2\sin\theta\cos\theta & = & y' & = & r'\sin\theta + r\theta'\cos\theta,\end{array} \tag{11}$$

and then solve for θ' and r' in (11). □

The ODE (10) has an entire circle of steady states given by $\mathcal{S}^1 \times \{r = 0\}$; see Figure 1(b). However, it is possible to desingularize the vector field $\hat{f}$ by division by $1/r$, i.e. we define

$$\bar{f} := \frac{1}{r}\hat{f}.$$

The division by $1/r$ does not change the qualitative dynamics on the set $\mathcal{S}^1 \times \{r > 0\}$ up to a time rescaling [1, Sec.1.4.1]. However, the $1/r$ scaling does drastically change the dynamics on the circle $\mathcal{S}^1 \times \{r = 0\}$. The desingularized vector field $\bar{f}$ is given by

$$\begin{array}{rcl}\theta' & = & 3\cos\theta\sin^2\theta - 2a\sin\theta\cos^2\theta,\\ r' & = & r(a\cos\theta - 2\sin\theta - 2a\cos\theta\sin^2\theta + 3\sin^3\theta).\end{array} \tag{12}$$

Having computed (12), the dynamics follows by direct calculation of the steady states and linearization.

Proposition 2.2. *For $a > 0$ fixed, There are six steady states for (12) on $\mathcal{S}^1 \times \{r = 0\}$. Four are given by*

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

while the remaining two are defined by the condition $\tan\theta = \frac{2}{3}a$. The six steady states are hyperbolic saddle points as shown in Figure 1(c).

Although the calculations using polar coordinates are easy for our example problem, they become quickly very involved for other problems [6]. In particular, consider the situation when the blow-up has to be used iteratively when new steady states on the sphere associated to $\{r = 0\}$ are also non-hyperbolic.

It is more convenient to use charts for $\mathcal{B}$ in combination with a so-called weighted directional blow-up. Introduce coordinates on $\mathcal{B}$ given by $(\bar{x}, \bar{y}, \bar{r}) \in \mathcal{S}^1 \times [0, r_0]$ with $\bar{x}^2 + \bar{y}^2 = 1$. Then define the weighted directional blow-up map by

$$\Psi : \mathcal{B} \to \mathbb{R}^2, \qquad \Psi(\bar{x}, \bar{y}, \bar{r}) = (\bar{r}\bar{x}, \bar{r}\bar{y}). \tag{13}$$

So how should we define charts $\kappa_i : \mathcal{B} \to \mathbb{R}^2$ to make the calculations as simple as possible? One approach is to require that the induced local coordinate changes

$$\psi_i = \Psi \circ \kappa_i^{-1}$$

are easy to compute and the vector fields $\mathrm{D}\psi_i^{-1} \circ f \circ \psi$ have a tractable algebraic form. Let $x_i, y_i \in \mathbb{R}$, $r_i \in [0, r_0]$ and let (r_1, y_1), (r_2, x_2) be coordinates on $\mathbb{R}^2$. One possibility to design the charts is to consider (13) and try to require

$$\psi_1(r_1, y_1) = (r_1, r_1 y_1) \qquad \text{and} \qquad \psi_2(r_2, x_2) = (r_2 x_2, r_2). \tag{14}$$

The following diagram illustrates the main aspects of the weighted directional blow-up:

$$\begin{tikzcd}
 & & \mathcal{B} = \mathcal{S}^1 \times [0, r_0] \arrow[dll, "\kappa_2"'] \arrow[dl, "\kappa_1"'] \arrow[dr, "\Psi"] & \\
(r_2, x_2) \in \mathbb{R}^2 \arrow[r, shift left, "\kappa_{12}"] & (r_1, y_1) \in \mathbb{R}^2 \arrow[l, shift left, "\kappa_{21}"] \arrow[rr, "\psi_1"] & & (x, y) \in \mathbb{R}^2,
\end{tikzcd}$$

where κ_{12} and κ_{21} denote the transition maps between the two charts κ_1 and κ_2. If (14) holds then this leads to

$$\begin{aligned}
\kappa_1(\bar{x}, \bar{y}, \bar{r}) &= \psi_1^{-1} \circ \Psi(\bar{x}, \bar{y}, \bar{r}) = \psi_1^{-1}(\bar{r}\bar{x}, \bar{r}\bar{y}) = (\bar{r}\bar{x}, \bar{r}\bar{y}/(\bar{r}\bar{x})) = (\bar{r}\bar{x}, \bar{y}/\bar{x}), \\
\kappa_2(\bar{x}, \bar{y}, \bar{r}) &= \psi_2^{-1} \circ \Psi(\bar{x}, \bar{y}, \bar{r}) = \psi_2^{-1}(\bar{r}\bar{x}, \bar{r}\bar{y}) = (\bar{r}\bar{x}/(\bar{r}\bar{y}), \bar{r}\bar{y}) = (\bar{x}/\bar{y}, \bar{r}\bar{y}).
\end{aligned} \tag{15}$$

Hence we may use (15) as definitions of the charts and obtain that the corresponding coordinate changes on $\mathbb{R}^2$ are given by (14).

Lemma 2.3. *The vector fields using the charts $\kappa_{1,2}$ are given by*

$$\left\{ \begin{array}{lcl} r_1' &=& r_1^2(a - 2y_1), \\ y_1' &=& r_1 y_1 (3y_1 - 2a), \end{array} \right. \qquad \left\{ \begin{array}{lcl} r_2' &=& r_2^2(1 - a x_2), \\ x_2' &=& r_2 x_2 (2a r_2 - 3). \end{array} \right. \tag{16}$$

Proof. As before, we may formally carry out the coordinate change. Or one may use direct calculations, for example, we have

$$r_2' = y' = r_2^2 - a r_2^2 x_2, \qquad x' = r_2' x_2 + r_2 x_2' = a r_2^2 x_1^2 - 2 r_2^2 x_2.$$

From these results, the vector field in (r_2, x_2)-coordinates easily follows. The calculation for the κ_1-chart is similar. $\square$

The ODEs (16) are still polynomial vector fields and algebraically a lot simpler to treat in comparison to long expressions using trigonometric functions. As for the polar case, we may again desingularize the problem using a division by $1/r_i$. For the first chart this yields

$$\begin{array}{lcl} r_1' &=& r_1(a - 2y_1), \\ y_1' &=& y_1(3y_1 - 2a). \end{array} \tag{17}$$

We have that (17) is defined in $(r_1, y_1) \in [0, r_0] \times \mathbb{R}$. We may consider this domain as corresponding to covering the right-half plane of $\mathcal{B} \subset \mathbb{R}^2$ outside of the open half-disc $\{x > 0, x^2 + y^2 < 1\}$; see Figure 2.

There are two steady states for (17) given by

$$(r_1, y_1) = (0, 0), \qquad (r_1, y_1) = \left(0, \frac{2}{3}a\right)$$

which correspond to the steady states with angles $\theta = 0$ and the smallest positive zero of $\tan\theta = \frac{2}{3}a$. In the form (17) it is easier to check the eigenvalues of the linearized system

$$\left(\begin{array}{c} R_1' \\ Y_1' \end{array} \right) = \left(\begin{array}{cc} a - 2y_1 & -2r_1 \\ 0 & 6y_1 - 3a \end{array} \right) \left(\begin{array}{c} R_1 \\ Y_1 \end{array} \right)$$

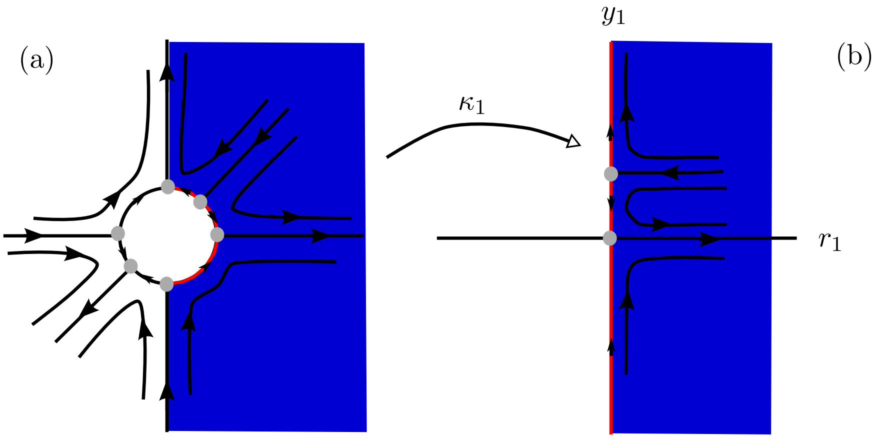

Figure 2. Sketch of the coordinate chart κ_1 associated to the x-directional blow-up. (a) Blown-up space $\mathcal{B}$ with phase portrait (black). (b) Directional coordinates $(r_1, y_1) \in \mathbb{R}^2$; the blue region corresponds to the blue region in (a) using the chart map κ_1, respectively its inverse κ_1^{-1}. Note that the half-circle from (a) is mapped to the vertical y_1-axis.

to conclude that the two steady states are hyperbolic saddle points. The calculations for the second desingularized system

$$\begin{aligned} r_2' &= r_2(1 - ax_2), \\ x_2' &= x_2(2ar_2 - 3), \end{aligned} \tag{18}$$

are similar and we also find two saddle points. The system (17) covers the outside of the open half-disc $\{y > 0, x^2 + y^2 < 1\}$ similar to the case shown in Figure 2 just for the upper half-plane. We can define two more charts, which also cover the left-half plane and the lower half-plane. If we define

$$\begin{aligned} \kappa_3(\bar{x}, \bar{y}, \bar{r}) &= (-\bar{r}\bar{x}, \bar{y}/\bar{x}), \\ \kappa_4(\bar{x}, \bar{y}, \bar{r}) &= (\bar{x}/\bar{y}, -\bar{r}\bar{y}), \end{aligned} \tag{19}$$

then the local coordinate changes are given by

$$\psi_3(r_3, y_3) = (-r_3, r_3 y_3) \qquad \text{and} \qquad \psi_4(r_4, x_4) = (r_4 x_4, -r_4). \tag{20}$$

With the four charts, one checks that there are six hyperbolic saddle points on $\mathcal{B} \times \{r = 0\}$ and one determines the direction of the flow as shown in Figure 1(c).

As a remaining question we consider the relation between the directional and polar blow-up maps. For example, if we would like to change from polar coordinates (θ, r) to Euclidean coordinates (r_1, y_1), we want the following diagram to commute:

$$\begin{tikzcd} & \mathcal{B} = \mathcal{S}^1 \times [0, r_0] \arrow[dl, "\alpha_1"'] \arrow[dr, "\Phi"] & \\ (r_1, y_1) \in \mathbb{R}^2 \arrow[rr, "\psi_1"'] & & (x, y) \in \mathbb{R}^2. \end{tikzcd}$$

In particular, this yields the requirement

$$\Phi(\theta, r) = (r\cos\theta, r\sin\theta) = (x, y) = (r_1, r_1 y_1) = \psi_1(r_1, y_1).$$

Therefore, we must have $r_1 = r\cos\theta$ which implies

$$r_1 y_1 = y_1 r\cos\theta = r\sin\theta \quad \Rightarrow \quad y_1 = \tan\theta.$$

The coordinate change

$$\alpha_1(\theta, r) = (r\cos\theta, \tan\theta) = (r_1, y_1) \tag{21}$$

is not well-defined when $\theta = \pi/2, 3\pi/2$ but it is a diffeomorphism otherwise. Note that this implies the polar blow-up is indeed equivalent to the directional blow-up in the x-direction expect on the vertical y_1-axis. This is geometrically clear as we cannot map the circle diffeomorphically, or even homeomorphically, onto the y_1-axis. In some sense, this fact leads one to the viewpoint that using a spherical blow-up, if one eventually wants to calculate in directional coordinates anyway, is not the only choice for the blown-up space. In fact, there may be manifolds that work more naturally with directional coordinate charts.

3. Hyperbolic Space Blow-Up

In this section we address the question whether it is possible to consider a blown-up space other than the sphere to analyze the dynamics. As we shall show below, the answer to this question is positive. The second question is whether other blow-up spaces are more convenient from a practical and/or theoretical perspective. Again, this question has at least a 'non-negative' answer, i.e. we shall show that for our test example, the calculation for hyperbolic space work equally well; in fact, it may be even more convenient to use hyperbolic space if we have distinguished directions and want to work in charts. Furthermore, having the additional freedom to pick a manifold adapted to the problem could be beneficial as outlined in Section 4.

Instead of the sphere, we shall now work with hyperbolic space [23] via the hyperboloid model and define

$$\mathbb{H}_x := \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}, \qquad \mathbb{H}_y := \{(x, y) \in \mathbb{R}^2 : y^2 - x^2 = 1\}.$$

Furthermore, we define the associated blow-up spaces

$$\mathcal{B}_x := \mathbb{H}_x \times [0, \rho_0], \qquad \mathcal{B}_y := \mathbb{H}_y \times [0, \rho_0]$$

for some fixed $\rho_0 > 0$; note that ρ_0 plays the same role as r_0 for the spherical case. We start with the blow-up using just the space $\mathcal{B}_x$. Note that we can again use a (weighted) blow-up similar to the polar coordinate map Φ if we recall that $\cosh^2(\varphi) - \sinh^2(\varphi) = 1$. Indeed, we may just define the blow-up map by

$$\Xi : \mathcal{B}_x \to \mathbb{R}^2, \qquad \Xi(\varphi, \rho) = (\rho\cosh\varphi, \rho\sinh\varphi)$$

and apply it to our main example (6). As for the spherical polar blow-up, the map Ξ induces a vector field, which we denote by $\hat{h}$, on $\mathcal{B}_x$ by the requirement

$$\Xi_*\left(\hat{h}\right) = f.$$

Lemma 3.1. *The vector field $\hat{h}$ is given by*

$$\begin{array}{rcl} \varphi' & = & \rho(3\sinh^2\varphi\cosh\varphi - 2a\cosh^2\varphi\sinh\varphi), \\ \rho' & = & \rho^2(a\cosh\varphi - 2\sinh\varphi - 3\sinh^3\varphi - 2a\cosh\varphi\sinh^2\varphi). \end{array} \tag{22}$$

The proof of Lemma 3.1 follows the same approach as Lemma 2.1. As before, we may desingularize the vector field and consider

$$\bar{h} := \frac{1}{\rho}\hat{h}.$$

Then we look for steady states on $\mathbb{H}_x \times \{\rho = 0\}$ and we have to solve

$$\sinh^2 \varphi = \frac{2}{3} a \cosh \varphi \sinh \varphi$$

since $\cosh \varphi \geq 1$.

Proposition 3.2. *For the desingularized vector field $\bar{h}$, there is one steady state at $(\varphi, \rho) = (0, 0)$ and a second one at $(\varphi, \rho) = \left(0, \tanh\left(\frac{2}{3}a\right)\right)$. Both points are hyperbolic saddles.*

The result is expected from the previous computations. Next, we observe that the geometry of the problem for the hyperbolic blow-up space $\mathbb{H}_x$ is similar to the directional blow-up in the x-direction; see Figure 3.

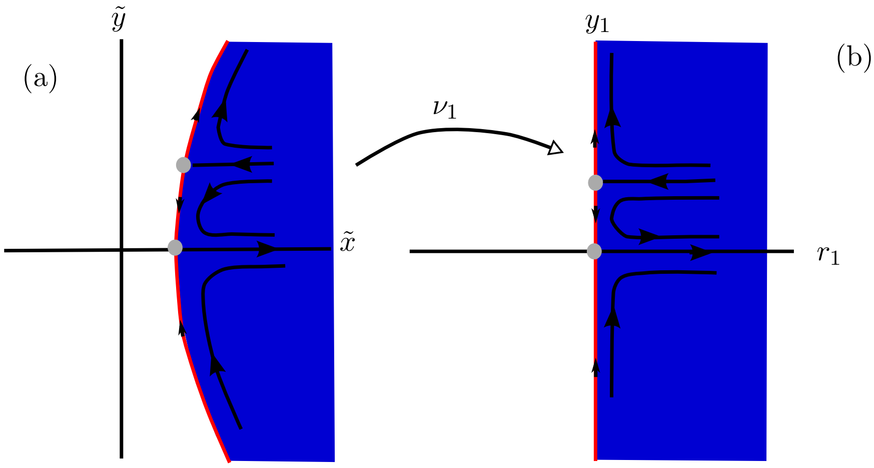

Figure 3. Sketch of the coordinate chart ν_1 associated to the x-directional blow-up. (a) Blown-up space $\mathcal{B}_x = \mathbb{H}_x \times [0, \rho)$ with phase portrait (black). (b) Directional coordinates $(r_1, y_1) \in \mathbb{R}^2$; the blue region corresponds to the blue region in (a) using the chart map ν_1, respectively its inverse ν_1^{-1}. Note that the curve $\{\tilde{x}^2 - \tilde{y}^2 = 1\} \times \{\rho = 0\}$ from (a) is mapped to the vertical y_1-axis.

Next, we check how to define the directional blow-ups based upon $\mathcal{B}_x$. Let $(\tilde{x}, \tilde{y}, \tilde{\rho})$ be coordinates on $\mathcal{B}_x$ with $\tilde{x}^2 - \tilde{y}^2 = 1$ and $\tilde{\rho} \in [0, \rho_0]$. Define the blow-map

$$\Gamma(\tilde{x}, \tilde{y}, \tilde{\rho}) = (\tilde{\rho}\tilde{x}, \tilde{\rho}\tilde{y}).$$

Let $\nu_i : \mathcal{B}_x \to \mathbb{R}^2$ be coordinate charts. As before, we want to construct the charts such that the local coordinate changes are given, as for the spherical case in (14), by

$$\gamma_1(\rho_1, y_1) = (\rho_1, \rho_1 y_1) \qquad \text{and} \qquad \gamma_2(\rho_2, x_2) = (\rho_2 x_2, \rho_2), \tag{23}$$

where $\gamma_i = \Gamma \circ \nu_i^{-1}$. In particular, the following diagram should commute

$$\begin{tikzcd} & \mathcal{B}_x = \mathbb{H}_x \times [0, \rho_0] \arrow[dl, "\nu_2"'] \arrow[d, "\nu_1"] \arrow[dr, "\Gamma"] & \\ (\rho_2, x_2) \in \mathbb{R}^2 \arrow[r, shift left, "\nu_{12}"] & (\rho_1, y_1) \in \mathbb{R}^2 \arrow[l, shift left, "\nu_{21}"] \arrow[r, "\gamma_1"] & (x, y) \in \mathbb{R}^2, \end{tikzcd}$$

where ν_{12}, ν_{21} denote the transition maps. The conditions (23) yield

$$\begin{array}{l}\nu_1(\tilde{x},\tilde{y},\tilde{\rho}) = \gamma_1^{-1}\circ\Gamma(\tilde{x},\tilde{y},\tilde{\rho}) = \gamma_1^{-1}(\tilde{\rho}\tilde{x},\tilde{\rho}\tilde{y}) = (\tilde{\rho}\tilde{x},\tilde{\rho}\tilde{y}/(\tilde{\rho}\tilde{x})) = (\tilde{\rho}\tilde{x},\tilde{y}/\tilde{x}),\\ \nu_2(\tilde{x},\tilde{y},\tilde{\rho}) = \gamma_2^{-1}\circ\Gamma(\tilde{x},\tilde{y},\tilde{\rho}) = \gamma_2^{-1}(\tilde{\rho}\tilde{x},\tilde{\rho}\tilde{y}) = (\tilde{r}\tilde{x}/(\tilde{\rho}\tilde{y}),\tilde{\rho}\tilde{y}) = (\tilde{x}/\tilde{y},\tilde{\rho}\tilde{y}),\end{array} \tag{24}$$

so the calculations are almost exactly the same as for the spherical case. However, there are some subtle differences when we consider the relation between the directional and hyperbolic polar blow-up maps. If we would like to change from the coordinates (φ,ρ) to Euclidean coordinates (ρ_1,y_1) we get the requirement

$$\Gamma(\varphi,\rho) = (\rho\cosh\varphi,\rho\sinh\varphi) = (x,y) = (\rho_1,\rho_1 y_1) = \gamma_1(\rho_1,y_1).$$

Therefore, it follows that $\rho_1 = \rho\cosh\theta$ which implies

$$\rho_1 y_1 = y_1\rho\cosh\varphi = \rho\sinh\varphi \quad\Rightarrow\quad y_1 = \tanh\varphi.$$

The coordinate change $\beta_1 : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\beta_1(\varphi,\rho) = (\rho\cosh\varphi,\tanh\varphi) = (\rho_1,y_1) \tag{25}$$

is analytic and well-defined everywhere. Geometrically, this is expected since we can easily map the domain

$$\{\tilde{x} : \tilde{x} > 0, \tilde{x}^2 - \tilde{y}^2 = 1\} \times [0,\rho_0]$$

diffeomorphically onto a rectangular strip of the form $\{(x,y) : x \in [0,\rho_0]\}$; see Figure 3. For the second chart we get

$$\Gamma(\varphi,\rho) = (\rho\cosh\varphi,\rho\sinh\varphi) = (x,y) = (\rho_2 x_2,\rho_2) = \gamma_2(\rho_2,x_2).$$

Therefore, it follows that $\rho_2 = \rho\sinh\theta$ which implies

$$\rho_2 x_2 = x_2\rho\sinh\varphi = \rho\cosh\varphi \quad\Rightarrow\quad x_2 = \frac{1}{\tanh\varphi}.$$

The coordinate change $\beta_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\beta_2(\varphi,\rho) = \left(\rho\sinh\varphi,\frac{1}{\tanh\varphi}\right) = (\rho_2,x_2) \tag{26}$$

is is not defined at $\varphi = 0$ as $\tanh(0) = 0$. Again, this is expected from the geometry as shown in Figure 3.

4. Conclusion and Outlook

In Section 4.1 a small summary is provided based upon the calculations of the hyperbolic space desingularization example. In Sections 4.2-4.4, three potential applications for blow-up without spheres are suggested, which provide additional motivation for the overall construction proposed here.

4.1. Classical Calculations in Different Coordinates

We may conclude from the example in Section 3 that the space $\mathcal{B}_x$, which is built upon $\mathbb{H}_x$, basically yields immediately a directional blow-up in the x-direction up to the analytic coordinate change β_1. Similarly, one may show that using $\mathcal{B}_y$ corresponds, up to an analytic coordinate change, to a y-direction blow-up. As for the spherical case, we may define charts that also cover the negative half-planes.

In summary, the example demonstrates that the classical choice of a spherical blow-up in $\mathbb{R}^N$ with $\mathcal{S}^{N-1}\times\mathcal{I}$ for some interval $\mathcal{I}\subseteq\mathbb{R}$ is certainly not the only option. In particular, if we already know a certain direction for $z\in\mathbb{R}^N$ where we do not need the directional blow-up, say z_1, then hyperbolic space $\mathbb{H}_{z_1}$ is one good choice as it corresponds via an analytic coordinate change to the respective directional blow-ups. Of course, the intrinsic dynamics of the problem does not change, regardless of the blow-up transformation considered. However, as has been shown in the case of the polar and directional blow-ups, sometimes a certain coordinate system is preferable. Therefore, hyperbolic space, as well as other suitable manifolds, could be viable alternatives to classical methods.

4.2. Geometric Flows and Singularities

Here we briefly outline one viewpoint that motivated the investigation of hyperbolic space blow-up. The basic ODE (1) we started with, as well as the example (6), are vector fields on $\mathbb{R}^N$ given by $f:\mathbb{R}^N\to\mathrm{T}\mathbb{R}^N$. However, what happens if we replace the base manifold $\mathbb{R}^N$ by some more general smooth manifold $\mathcal{M}$? The ODEs under study defined by a vector field $F:\mathcal{M}\to\mathrm{T}\mathcal{M}$ could still have non-hyperbolic points. A natural example is to start considering ODEs arising in differential geometry. Consider the hyperbolic plane $\mathbb{H}$ now viewed in the Poincaré upper half-plane model

$$\mathbb{H}=\{z=x+\mathrm{i}y\in\mathbb{C}:y>0\},\qquad \mathrm{d}s=\frac{\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}}{y},$$

where $\mathrm{d}s$ is the arclength element [23]. In particular, consider a curve $\gamma(t)=(x(t),y(t))\in\mathbb{H}$ between two points $p_1,p_2\in\mathbb{H}$ with $\gamma(0)=p_0$ and $\gamma(1)=p_1$ and define the Lagrangian energy functional

$$L[\gamma]:=\frac{1}{2}\int_0^1\frac{(x')^2+(y')^2}{y^2}\,\mathrm{d}t. \tag{27}$$

Then it is classical to minimize (27) to determine the differential equations for the geodesics on $\mathbb{H}$. The associated Euler-Lagrange equations are given by

$$q'=f(q),\qquad f(q):=\left(u,v,\frac{2uv}{y},\frac{v^2-u^2}{y}\right)^\top$$

with coordinates $q=(x,y,u,v)\in\mathrm{T}\mathbb{H}^2=:\mathrm{T}\mathcal{M}$. As discussed previously for desingularization of dividing factors it seems natural to consider $F:=yf$ so that

$$q'=F(q),\qquad F(q):=\left(uy,vy,2uv,v^2-u^2\right)^\top$$

with a suitable time re-parametrization understood. The vector field $F(q)$ has a codimension two submanifold of equilibria $\{u=0=v\}$. If we formally linearize $\mathrm{D}_qF(x,y,0,0)\in\mathbb{R}^{4\times4}$ has a quadruple zero eigenvalue and there is no (dynamical) hyperbolicity of the submanifold $\{u=0=v\}$. Of course, one could then ask whether the blow-up method can help to desingularize the problem and it seems natural to conjecture that spherical blow-up spaces may not be adequate. Indeed, the entire problem is formulated under the assumption that we work with a hyperbolic metric not with a spherical one. For other geometric flows the situation could be similar since (dynamically) non-hyperbolic points or submanifolds can always appear and we leave this direction as an open problem for future research.

4.3. Point Contraction and Compactness

As outlined in the introduction, a typical situation for the blow-up occurs in the context of fast-slow systems

$$\begin{aligned} x' &= f_1(x,y,\varepsilon), \\ y' &= \varepsilon g(x,y,\varepsilon), \\ \varepsilon' &= 0. \end{aligned} \tag{28}$$

The lowest-dimensional non-trivial case occurs when $z := (x, y, \varepsilon) \in \mathbb{R}^3$. Setting $f_2(x, y, \varepsilon) = \varepsilon g(x, y, \varepsilon)$, $f_3(x, y, \varepsilon) = 0$, and $f := (f_1, f_2, f_3)^\top$ leads to

$$z' = f(z), \qquad z \in \mathbb{R}^3 \tag{29}$$

with a non-hyperbolic steady state at $z = 0$ under the assumption that a non-hyperbolic fast subsystem steady state at the origin exists. Inserting a two-sphere $\mathcal{S}^2$ via a blow-up

$$\Phi : \mathcal{S}^2 \times [0, r_0] \to \mathbb{R}^3 \tag{30}$$

means that all new steady states on $\mathcal{S}^2 \times \{0\}$ can now be analyzed locally. A potential disadvantage is that we now treat all directions of the fast-slow system equally and all orbits on the sphere are compact. To illustrate this problem formally, let us consider the transcritical singularity of the form

$$\begin{aligned} x' &= x(x-y) + \kappa\varepsilon, \\ y' &= \varepsilon, \end{aligned} \tag{31}$$

where $\kappa \in \mathbb{R}$ is a parameter. It is easy to check that $(x, y) = (0, 0)$ is a non-hyperbolic steady state of the fast subsystem. At this transcritical singularity it is well-understood (see e.g. [13, Thm. 2.1]) that depending upon κ it is possible that trajectories starting sufficiently close to the attracting critical manifold $C_0^{a-} := \{(x, y) \in \mathbb{R}^2 : x = 0, y < 0\}$ can jump near the origin towards large negative values of x, the so-called jump case. On the other hand, it is also possible for different values of κ that trajectories stay close to $C_0^r := \{(x, y) \in \mathbb{R}^2 : x = 0, y > 0\}$ (the canard case), or that trajectories stay close to the attracting critical manifold $C_0^{a+} := \{(x, y) \in \mathbb{R}^2 : x = y, y > 0\}$ (the exchange-of-stability case). In those last two cases the motion is slow and bounded on all time scales as $\varepsilon \to 0$. However, for the jump case, the trajectories become unbounded in the singular limit $\varepsilon \to 0$ on the slow time scale. So it seems quite plausible that the x-direction is special in this context as the jumps occur almost parallel to the x-axis. Hence, it could be viewed as more natural to try to represent this fact in the blow-up construction and work with a space such as the hyperboloid

$$\mathbb{H}_x = \{(x, y, \varepsilon) \in \mathbb{R}^3 : y^2 + \varepsilon^2 - x^2 = 1\},$$

or another manifold, which has a distinguished direction. Of course, for the transcritical singularity, this is not necessary as demonstrated in [13]. However, it looks unnecessary to compactify a direction along which one wants to observe unbounded orbits while using a compact blown-up space seems more natural if we want to focus on bounded orbits.

4.4. Tori and Interior Steady States

To illustrate another potential advantage of different manifolds we continue with the fast-slow systems case (28)-(29). Frequently the main problem is to extend a suitable invariant submanifold of phase space near the blown-up sphere [13, 6] from one part of phase space to another part. In this context, a technically challenging problem is that this may involve an ODE without steady states. For example, consider the truncated fold point normal form

$$\begin{aligned} x' &= y - x^2, \\ y' &= \varepsilon. \end{aligned} \tag{32}$$

After a suitable blow-up [13], one obtains the system

$$\begin{aligned} \tilde{x}' &= \tilde{y} - \tilde{x}^2, \\ \tilde{y}' &= 1, \end{aligned} \tag{33}$$

where $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2$ are the coordinates in the so-called classical (or re-scaling) chart. Indeed, one just derives (33) from (32) by applying the scaling

$$x = \varepsilon^{1/3}\tilde{x}, \quad y = \varepsilon^{2/3}\tilde{y}, \quad t = \varepsilon^{-1/3}\tilde{t}$$

where differentiation is with respect to $\tilde{t}$ in (33). To analyze (33) one relies on the classical asymptotic theory of Ricatti equations [21] to find the relevant global solutions of (33). One conjecture is that a fully global analysis could be avoided if there would be additional steady states in $\mathbb{R}^2$ for (33). Instead of using a spherical blow-up of the form (30) a potential choice is

$$\Phi : \mathcal{S}^1 \times \mathcal{S}^1 \times [0, r_0] \to \mathbb{R}^3,$$

where $\mathcal{S}^1 \times \mathcal{S}^1 = \{(x^2 + y^2 + \varepsilon^2 + \mathfrak{R}^2 - \mathfrak{r}^2)^2 = 4\mathfrak{R}^2(x^2 + y^2)\}$ is the usual 2-torus with major and minor radii $\mathfrak{R}$ and $\mathfrak{r}$ that can be used as free parameters. A projection of this torus along the ε-axis yields an annulus in the (x, y)-plane. In particular, if we think about the inner boundary of the annulus in a 'classical chart' then it may potentially contain steady states at which the dynamics could be analyzed locally. Of course, as for the transcritical singularity discussed in the last section, this idea is not necessary for the analysis of the fold point [6, 13] but it may have the potential to avoid difficulties for other singularities.

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