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Thermodynamics properties study of diatomic molecules with q-deformed modified Poschl-Teller plus Manning Rosen non-central potential in D dimensions using SUSYQM approach

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Abstract. D-dimensional Dirac equation of q-deformed modified Poschl-Teller plus Manning Rosen non-central potential was solved using supersymmetric quantum mechanics (SUSY QM). The relativistic energy spectra were analyzed by using SUSY QM and shape invariant properties from radial part of D dimensional Dirac equation and the angular quantum numbers were obtained from angular part of D dimensional Dirac equation. The SUSY operators was used to generate the D dimensional relativistic wave functions both for radial and angular parts. In the non-relativistic limit, the relativistic energy equation was reduced to the non-relativistic energy. In the classical limit, the partition function of vibrational, the specific heat of vibrational, and the mean energy of vibrational of some diatomic molecules were calculated from the equation of non-relativistic energy with the help of error function and Mat-lab 2011.

1. Introduction

In some area of physics, relativistic quantum mechanics play important roles. Finding an accurate exact solution of Dirac equation for a certain potential is one of its important roles. Various methods have been applied to solve the Dirac equation for some potentials, central and non-central potentials, with or without tensor coupling potentials, such as NU method, [1-6] SUSY QM method, [7-11] and Romanovski polynomial method. [12-16]

For very limited potentials, three dimensional radial Dirac equations are exactly solved only for s-wave ($l = 0$). However, the three dimensional radial Dirac equations for the spherically symmetric potentials can only be solved approximately for $l \neq 0$ states due to the approximation scheme of the centrifugal term $\sim r^{-2}$. [17-23]

Furthermore, the extension in higher dimensional spaces for some physical problems is very important in some physics area. The D-dimensional non-relativistic and relativistic physical systems have been investigated by many authors, such as ring-shaped pseudoharmonic potential, [21] the isotropic harmonic oscillator and inverse quadratic potential, [22] Pseudoharmonic potential, [23] Kratzer-Fues potential, [24-25] hydrogen atom, [26] modified Poschl-Teller potential, [27] linearly energy dependent quadratic potential, [28] trigonometric scarf potential, [29] ring-shaped Kratzer potential. [30]

The Dirac equation for a charged particle that moves in a field governed by q-deformed Poschl-Teller potential [31] in D dimension is investigated using supersymmetric quantum mechanics (SUSY QM) with idea of shape invariance. SUSY QM method is developed based on Witten proposal [32] and the idea of shape invariant potential is proposed by Gendenshtein [33]. SUSY QM is a powerful tool to determine energy spectrum and wave function of shape invariant potentials for one dimensional Schrodinger equation. Thus the relativistic energy spectrum is obtainable by using the idea of shape invariance and the wave functions are achieved by using lowering and raising SUSY operators. Some of hyperbolic and trigonometric potentials are exactly solvable within the approximation of centrifugal term and their solutions have been reported in the previous papers [10-11]. The q-deformed modified Poschl-Teller plus Manning Rosen non-central potential that govern the diatomic molecules vibration is expressed as

$$V(r, \theta) = t^2 \left(\left(a(a-1)/\sinh_q^2 tr \right) - \left(b(b+1)/\cosh_q^2 tr \right) \right) - \left(1/r^2 \right) \left[\left(p(p+1)/\sin^2 \theta \right) - 2s \cot \theta \right] \quad (1)$$



with $a > 1, b > 0; \alpha > 0; p > 0, s > 0$, a, b, p and s are positive constants that control the depth of the potential, t is positive constant which controls the width of the potential, $0 < r < \infty$, q causes the deformation of the potential shape, $0 < q < 1$. In the non-relativistic limit, the relativistic energy equation reduces the non-relativistic energy equation. In the non relativistic limit is the condition when the energy is subtracted by the mass equals to the non-relativistic energy which is usually obtained from Schrodinger equation solution of the system, and the condition where the sum of the energy and the mass equals to twice of masses. The non-relativistic energy is used in calculation of thermal properties in classical limit. In classical regime, the thermal properties including vibrational partition function Z , mean energy U , and specific heat C [34,35] are determined by applying the non-relativistic energy equation.

This paper is organized as follows. Brief review of SUSY quantum mechanics is presented in section 2, solution of Dirac equations and its application to study thermodynamical properties are presented in section 3 and conclusion is presented in section 4.

2. Review of Supersymmetric Quantum Mechanics Approach Using Operator

2.1. Supersymmetry Quantum Mechanics (SUSY QM)

According to the definition proposed by Witten, in a SUSY QM there are super charge operators Q_i which commute with the Hamiltonian H_{ss} and obey to anti commutation algebra [32]

$$[Q_i, H_{ss}] = 0; \{Q_i, Q_j\} = \delta_{ij} H_{ss} \text{ with, } i = 1, 2, 3, \dots \quad (2)$$

with H_{ss} is called supersymmetric Hamiltonian. Witten proposed that the SUSY QM is the one dimensional model of SUSY field theory and he stated that the simplest SUSY QM system has $N=2$ [32] where the two charge operators are given as

$$Q_1 = (1/\sqrt{2}) \left(\sigma_1 \left(p/\sqrt{2m} \right) + \sigma_2 \phi(x) \right); Q_2 = (1/\sqrt{2}) \left(\sigma_2 \left(p/\sqrt{2m} \right) + \sigma_1 \phi(x) \right) \quad (3)$$

where σ_i are the usual Pauli spin matrices, $p = -i\hbar(\partial/\partial x)$ is the usual one dimensional momentum operator, and $\phi(x)$ is superpotential. By inserting equation (3) into the second equation of equation (2) we get,

$$H_{ss} = \begin{pmatrix} -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}} \frac{d\phi(x)}{dx} + \phi^2(x) & 0 \\ 0 & -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}} \frac{d\phi(x)}{dx} + \phi^2(x) \end{pmatrix}; H_{ss} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \quad (4)$$

From equation (4) we have

$$V_-(x) = \phi^2(x) - \frac{\hbar}{\sqrt{2m}} \phi'(x); \quad V_+(x) = \phi^2(x) + \frac{\hbar}{\sqrt{2m}} \phi'(x) \quad (5)$$

Here H_- and H_+ , are supersymmetry partner of the Hamiltonian, $V_-(x)$ and $V_+(x)$ are the supersymmetry partner potential. To simplify the determination of the energy spectrum and the wave functions, the new operators, raising and lowering operators, are introduced as

$$A^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \text{ and } A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \quad (6)$$

By inserting equation (6) into equation (5) we get the SUSY Hamiltonian as

$$H_-(x) = A^+ A \text{ and } H_+(x) = A A^+ \quad (7)$$

to factorize the usual Hamiltonian as

$$H = H_- + E_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x; a_0) + E_0 \quad (8)$$

By using equation (5) and equation (8) it is obtained that

$$V(x) = V_-(x; a_0) + E_0 = \phi^2(x; a_0) - \frac{\hbar}{\sqrt{2m}} \phi'(x; a_0) + E_0 \quad (9)$$

where $V(x)$ is the effective potential, while $\phi(x)$ is determined hypothetically from equation (9) based on the shape of effective potential from the associated system.

2.2. Shape Invariance

It is observed that the supersymmetry only gives the relationship between the eigenvalues and eigenfunctions between the two Hamiltonian partners but does not provide the actual spectrum. [36]. The energy spectrum is only obtainable by implementing SUSY charged operators properties and the condition of shape invariant proposed by Gendenshtein.[33]. If a pair of potentials $V_{\pm}(x)$ defined in equation (5) are similar in shape but different in the parameters, then they are called to be shape invariant. More specifically, if $V_{\pm}(x, a_0)$ satisfy the condition that

$$V_+(x; a_j) = V_-(x; a_{j+1}) + R(a_{j+1}) \quad (10)$$

$$\text{with } V_+(x; a_j) = \phi^2(x; a_j) + \frac{\hbar}{\sqrt{2m}} \phi'(x; a_j); V_-(x; a_{j+1}) = \phi^2(x; a_{j+1}) - \frac{\hbar}{\sqrt{2m}} \phi'(x; a_{j+1}) \quad (11)$$

where $j = 0, 1, 2, \dots$, and a is a parameter in our original potential, V_- , whose ground state energy is zero, $a_j = f_j(a_0)$ where f_j is a function applied j times, the remainder $R(a_j)$ is a 's dependence but is independent of x , then $V_{\pm}(x, a_0)$ is said to be shaped invariant. The energy eigenvalue of the Hamiltonian H_- is given by [33]

$$E_n^{(-)} = \sum_{k=1}^n R(a_k) \quad (12)$$

and by using equation (8) and equation (12) we get the energy spectra of the system given as,

$$E_n = E_n^{(-)} + E_0 \quad (13)$$

Based on the characteristics of lowering operator, the ground state wave function of H_- , whose ground state energy is zero, is obtained from condition that,

$$H_- \psi_0^{(-)} = 0 \rightarrow A \psi_0^{(-)} = 0 \quad (14)$$

Subsequently, the excited wave function, $\psi_1^{(-)}(x, a_0), \dots, \psi_n^{(-)}(x, a_0)$ of H_- are obtained by using raising operator operated to the lower wave function, [37] given as

$$\psi_n^{(-)}(x; a_0) \approx A^+(x; a_0) A^+(x; a_1) \dots A^+(x; a_{n-1}) \psi_0^{(-)}(x; a_n) \quad (15)$$

In the Dirac equation we have applied the SUSY potential partner and the SUSY operator equations (5-6) as

$$V_-(x) = \phi^2(x) - \phi'(x); V_+(x) = \phi^2(x) + \phi'(x) \quad (16)$$

$$A = \frac{d}{dx} + \phi(x); A^+ = -\frac{d}{dx} + \phi(x) \quad (17)$$

The SUSY QM and the idea of shape invariance potential are suitable to be used in solving one dimensional Schrodinger equation problems. One dimensional Dirac equation reduces to one dimensional Schrodinger type equation by suitable change in parameters which are the coefficient of the potential function and the energy term.

By obtaining the super-potential, the potential partners, $V_+(x, a_0)$ and $V_-(x, a_0)$, and the SUSY operators, A^+ and A are obtained and so the energy spectrum and the wave function.

3. Solution of Dirac Equation for Non-central Potential in D dimension

3.1. Solution of radial part of D dimensional Dirac Equation

The Dirac equation with the scalar potential $S(\vec{r})$ and magnitude of vector potential $V(\vec{r})$ is given as in Hu et al. [38]

$$\{\bar{\alpha} \vec{p} + \beta (M + S(\vec{r}))\} \psi(\vec{r}) = \{E - V(\vec{r})\} \psi(\vec{r}) \quad (18)$$

where M is the relativistic mass of the particle, E is the total relativistic energy, and \vec{p} is the three-dimensional momentum operator, $-i\nabla$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}; \vec{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (19)$$

with $\vec{\sigma}$ are the three-dimensional Pauli matrices and I is the 2×2 identity matrix. The potential in equation (18) is spherically symmetric potential, it does not only depend on the radial coordinate $r = |\vec{r}|$, and we have taken $\hbar = 1$ and $c = 1$. The Dirac equation expressed in equation (18) is invariant under spatial inversion and therefore its eigenstates have definite parity. By writing the spinor in D dimension as

$$\psi(\vec{r}) = \begin{pmatrix} \zeta(\vec{r}) \\ \Omega(\vec{r}) \end{pmatrix} \quad (20)$$

and if we insert equations (19) and (20) into equation (18) and use matrices multiplication, we achieve

$$\vec{\sigma} \cdot \vec{p} \Omega(\vec{r}) = \{-M - S(\vec{r}) + E - V(\vec{r})\} \zeta(\vec{r}) \quad (21)$$

$$\vec{\sigma} \cdot \vec{p} \zeta(\vec{r}) = \{M + S(\vec{r}) + E - V(\vec{r})\} \Omega(\vec{r}) \quad (22)$$

In the exact spin symmetric case, when the scalar potential is equal to the magnitude of vector potential $S(\vec{r}) = V(\vec{r})$, then the upper Dirac spinor obtained from equations (21) and (22) are

$$\vec{\sigma} \cdot \vec{p} \frac{\vec{\sigma} \cdot \vec{p}}{M + E} \zeta(\vec{r}) = \{-M + E - 2V(\vec{r})\} \zeta(\vec{r}) \quad (23)$$

By applying the Pauli matrices, it is simply shown that if $(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) = p^2$, then equation (23) becomes

$$p^2 \zeta(\vec{r}) + 2V(\vec{r})(M + E)\zeta(\vec{r}) = (E^2 - M^2)\zeta(\vec{r}) \quad (24)$$

Since $p^2 = -\Delta_D = -\nabla_D^2$ with the hyperspherical Laplacian ∇_D^2 is given by [39-40]

$$\nabla_D^2 = r^{1-D} \frac{\partial}{\partial r} \left(r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin^{D-2} \theta_{D-1}} \frac{d}{d\theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{d}{d\theta_{D-1}} \right) - \frac{L_{D-2}^2}{\sin^2 \theta_{D-1}} \right] \quad (25)$$

with $L_{D-2}^2 = l_{D-2}(l_{D-2} + D - 3)$.

To reduce equation (24) into Schrodinger like equation we set $V \rightarrow (1/2)V$ and if V is modified Poschl-Teller plus trigonometric Manning-Rosen non-central potential then equation (24) becomes

$$r^{1-D} \frac{\partial}{\partial r} \left\{ r^{D-1} \frac{\partial}{\partial r} (\zeta(\vec{r})) \right\} + \frac{1}{r^2} \left[\frac{1}{\sin^{D-2} \theta_{D-1}} \frac{d}{d\theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{d(\zeta(\vec{r}))}{d\theta_{D-1}} \right) - \frac{L_{D-2}^2}{\sin^2 \theta_{D-1}} (\zeta(\vec{r})) \right] - \left(t^2 \left(\frac{a(a-1)}{\sinh^2 tr} - \frac{b(b+1)}{\cosh^2 tr} \right) - \frac{1}{r^2} \left[\frac{p(p+1)}{\sin^2 \theta} - 2s \cot \theta \right] (M + E) \right) (\zeta(\vec{r})) = -(E^2 - M^2) (\zeta(\vec{r})) \quad (26)$$

The upper component of Dirac spinor in equation (20) is given as

$$\zeta(\vec{r}) = \frac{1}{r^{\frac{D-1}{2}}} F_l(r) Y_{\ell_1 \dots \ell_{D-2}}^\ell(\hat{x} = \theta_1, \theta_2, \dots, \theta_{D-1}) \text{ and } Y_{\ell_1 \dots \ell_{D-2}}^\ell(\hat{x} = \theta_1, \dots, \theta_{D-1}) = \Phi(\theta_1 = \phi) H(\theta_2, \dots, \theta_{D-1}) \quad (27)$$

where \vec{x} is a D dimensional position vector in hyper-spherical Cartesian coordinate [40,41], the unit vector along \vec{x} vector is denoted as $\hat{x} = \vec{x}/r$. The hyper-spherical Cartesian coordinate components x_1, x_2, x_3, \dots are given as

$$x_1 = r \cos \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{D-1}; x_2 = r \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{D-1};$$

$$x_j = r \cos \theta_{j-1} \sin \theta_j \sin \theta_{j+1} \dots \sin \theta_{D-1} \quad 3 \leq j \leq D-1 \quad (28)$$

$$x_{D-1} = r \cos \theta_{D-2} \sin \theta_{D-1}; x_D = r \cos \theta_{D-1} \quad (29)$$

The simultaneous eigenfunctions are given as

$$L_1^2 Y_{\ell_1 \dots \ell_{D-2}}^{(\ell)} = m^2 Y_{\ell_1 \dots \ell_{D-2}}^{(\ell)}; m = \ell_1; L_{D-1}^2 Y_{\ell_1 \dots \ell_{D-2}}^{(\ell)} = \ell_{D-1} (\ell_{D-1} + D - 2) Y_{\ell_1 \dots \ell_{D-2}}^{(\ell)} \quad (30)$$

$$L_j^2 Y_{\ell_1 \dots \ell_{D-2}}^{(\ell)} = \ell_j (\ell_j + j - 1) Y_{\ell_1 \dots \ell_{D-2}}^{(\ell)}; 1 < j < D - 1 \quad (31)$$

$$L_k^2 = \sum_{a=b=2}^{k+1} L_{ab}^2 = - \left[\frac{1}{\sin^{k-1} \theta_k} \frac{d}{d\theta_k} \left(\sin^{k-1} \theta_k \frac{d}{d\theta_k} \right) - \frac{L_{k-1}^2}{\sin^2 \theta_k} \right]; 2 \leq k \leq D - 2 \quad (32)$$

By applying the variable separation method we have radial and angular parts of D dimensional Dirac equation which are given as

$$\frac{d^2 F(r)}{dr^2} - t^2 \left(\frac{a(a-1)(E+M) + \lambda_t + (\frac{D-1}{2})(\frac{D-3}{2})}{\sinh_q^2 tr} - \frac{(E+M)b(b+1)}{\cosh_q^2 tr} \right) F(r) + (E^2 - M^2 - t^2 \left(\lambda_t + (\frac{D-1}{2})(\frac{D-3}{2}) \right) d_0) F(r) = 0 \quad (33)$$

$$\left[\frac{1}{\sin^{D-2} \theta_{D-1}} \frac{d}{d\theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{d}{d\theta_{D-1}} \right) + \lambda_t - \frac{1}{\sin^2 \theta_{D-2}} (L_{D-2}^2) - \left[\frac{P(P+1)}{\sin^2 \theta_{D-1}} - 2s \cot \theta_{D-1} \right] (M + E) \right] H(\theta_{D-1}) = 0 \quad (34)$$

In order to solve the radial Dirac equation in Eq.(33), we use the approximation value for the centrifugal term as in Greene and Aldirch, and in Ikdhair [20-21], $1/r^2 \cong t^2 (d_0 + 1/\sinh_q^2 tr)$, for $tr \ll 1$ and $d_0 = 1/12$. By setting

$$(a(a-1))(E+M) + \lambda_t + (\frac{D-1}{2})(\frac{D-3}{2}) = c(c-1) \quad ; \quad b(b+1)(E+M) = b'(b'+1) \quad (35)$$

$$E^2 - M^2 - t^2 \left(\lambda_t + (\frac{D-1}{2})(\frac{D-3}{2}) \right) d_0 = E' \quad (36)$$

equation (33) becomes

$$-\frac{d^2 F(r)}{dr^2} + t^2 \left(\frac{c(c-1)}{\sinh_q^2 tr} - \frac{b'(b'+1)}{\cosh_q^2 tr} \right) F(r) = E' F(r) \quad (37)$$

Equation (37) is solved using SUSY QM and by introducing the hypothetical super-potential as in [26-27]

$$\phi(r) = tP \coth_q tr + tB \tanh_q tr \quad (38)$$

using equation (9), equation (37) and equation (38) we get

$$t^2 \left(\frac{c(c-1)}{\sinh_q^2 tr} - \frac{b'(b'+1)}{\cosh_q^2 tr} \right) = t^2 (P^2 + P)(q \csc h_q^2 tr) - t^2 (B^2 + B)(q \sec h_q^2 tr) + E'_0 + (2BP + P^2 + B^2) t^2 \quad (39)$$

that gives

$$P = -(1/2) - \sqrt{(c(c-1)/q) + 1/4}; B = -(1/2) + \sqrt{(b'(b'-1)/q) + 1/4}; E'_0 = -(P+B)^2 t^2 \quad (40)$$

The superpotential, super-partner potentials, ground state energy and raising-lowering operators obtained from equations (5), (6), (12), (13), and (38) are

$$\varphi(r) = \left(-\frac{1}{2}t - t \sqrt{\frac{c(c-1)}{q} + \frac{1}{4}} \right) \coth_q tr + \left(-\frac{1}{2}t + t \sqrt{\frac{b'(b'-1)}{q} + \frac{1}{4}} \right) \tanh_q tr \quad (41)$$

$$V_-(r, a_0) = q \{ t^2 P(P+1) \} \csc h_q^2 tr - qt^2 B(B+1) \operatorname{sech}_q^2 tr + t^2 (P+B)^2 \quad (42)$$

$$V_+(r, a_0) = q \{ t^2 P(P-1) \} \csc h_q^2 tr - qt^2 B(B-1) \operatorname{sech}_q^2 tr + t^2 (P+B)^2 \quad (43)$$

$$A^+ = -\frac{d}{dr} + tP \coth_q tr + tB \tanh_q tr; A = \frac{d}{dr} + tP \coth_q tr + tB \tanh_q tr \quad (44)$$

By comparing the super-partner potential $V_-(r, a_0)$ and $V_+(r, a_0)$ we get the translation parameters as

$$a_0 = P, a_1 = P-1, \dots, a_n = P-n, \text{ and } b_0 = B, b_1 = B-1, \dots, b_n = B-n, \quad (45)$$

By implementing equations (10-13) and (42-43) and (45) we get the equation of relativistic energy spectra given as

$$E'_n = -t^2 \left(\left(-\sqrt{c(c-1)/q+1/4} \right) + \left(\sqrt{b'(b'+1)/q+1/4} \right) - 1 - 2n \right)^2 \quad (46)$$

The relativistic energy equation obtained from equations (35-36) and (46) is given as

$$(E^2 - M^2) = \left\{ \left[\lambda_l + \left(\frac{D-1}{2} \right) \left(\frac{D-3}{2} \right) \right] t^2 d_0 \right\} - t^2 \left(\left(-\sqrt{\frac{c(c-1)}{q} + \frac{1}{4}} \right) + \left(\sqrt{\frac{b'(b'+1)}{q} + \frac{1}{4}} \right) - 1 - 2n \right)^2 \quad (47)$$

with $c(c-1), b'(b'+1)$ is expressed in equation (35) and

$$\lambda_l = -[(D-2)/2]^2 + (p'-n)^2 - \frac{s'^2}{(p'-n)^2}; \quad p' = -1/2 + \sqrt{(\ell_{D-2} + (D-3)/2)^2 + (E+M)p(p+1)} \quad (48)$$

which are obtained from angular solution in the next section. From the relativistic energy equation in equation (47), we can obtain the numerical value of relativistic energy by using Mat-Lab in Table 1.

Table 1. Relativistic energy for various dimensions level

$$(t = 0.8, s = 0.3, d_0 = 1./12, a = 6, b = 2, p = 4, s = 2, q = 0.75, M = 5(1/fm), \ell_{D-2} = 2)$$

D	E_0	E_1	E_2	E_3	E_4
3	-5.9455	-7.1363	-7.9716	-9.2079	-9.2079
4	-5.8748	-7.0824	-7.9306	-8.6091	-9.1800
5	-5.8075	-7.0212	-7.8823	-8.5695	-9.1467

By using equations (14-15) and (44) we obtain the un-normalized relativistic ground state and first excited state wave functions for upper Dirac spinor

$$F_0 = C(\sinh_q tr)^{-P}(\cosh_q tr)^{-B}; \quad F_1 = 2(tP \coth_q tr + tB \tanh_q tr)(\sinh_q tr)^{-P}(\cosh_q tr)^{-B} \quad (49)$$

with the values of P and B are expressed in equation (40). By using raising operator in equation (44) we obtain all excited states of wave functions in exact spin symmetric case.

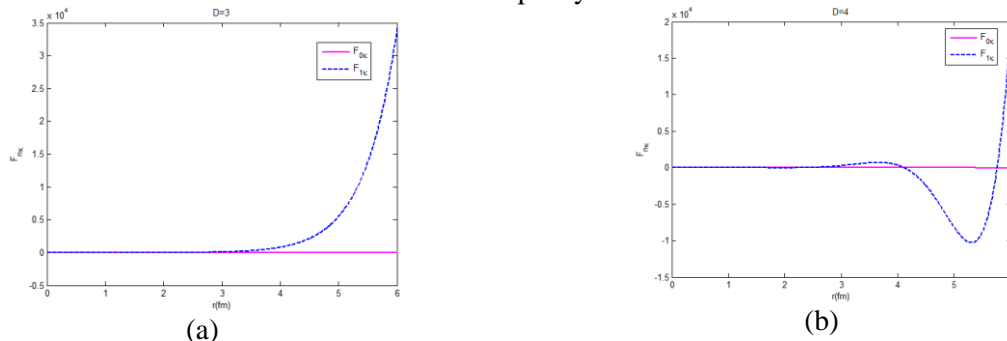


Figure 1. Ground state (solid line) and first excited (dash line) radial wave functions for (a) $D = 3$, (b) $D = 4$

3.2. Solution of Angular part of D dimensional Dirac Equation

The angular part of D dimensional Dirac equation for Manning Rosen angular potential are

$$\left[\frac{1}{\sin^{j-1} \theta_j} \frac{d}{d\theta_j} \left(\sin^{j-1} \theta_j \frac{d}{d\theta_j} \right) + \ell_j(\ell_j + j - 1) - \frac{\ell_{j-1}(\ell_{j-1} + j - 2)}{\sin^2 \theta_j} \right] H(\theta_j) = 0; \quad j \in [2, D-2] \quad (50)$$

$$\left[\frac{1}{\sin^{D-2} \theta_{D-1}} \frac{d}{d\theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{d}{d\theta_{D-1}} \right) + \lambda_\ell - \frac{(L_{D-2}^2)}{\sin^2 \theta_{D-1}} - (E+M) \left(\frac{p(p+1)}{\sin^2 \theta_j} - 2s \cot \theta_j \right) \right] H(\theta_{D-1}) = 0 \quad (51)$$

These two equations are solvable by using SUSY QM approach by reducing it into one dimensional Schrodinger like equation by suitable substitution of wave functions in equations (50) and (51) as

$$H(\theta_j) = Q(\theta_j) / \sin^{(j-1)/2} \theta_j \quad \text{and} \quad H(\theta_{D-1}) = Q(\theta_{D-1}) / \sin^{(D-2)/2} \theta_{D-1} \quad (52)$$

By inserting equation (52) into equations (50) and (51) we obtain

$$\left[d^2 Q(\theta_j) / d\theta_j^2 \right] + \left[\ell_j(\ell_j + j - 1) + ((j-1)/2)^2 - \ell_{j-1}(\ell_{j-1} + j - 2) + [(j-1)/2][(j-3)/2] / \sin^2 \theta_j \right] Q(\theta_j) = 0 \quad (53)$$

and

$$d^2 Q(\theta_{D-1}) / d\theta_{D-1}^2 + \left[\lambda_\ell + [(D-2)/2]^2 - \frac{(L_{D-2}^2) + [(D-2)/2][(D-4)/2]}{\sin^2 \theta_{D-1}} - (E+M) \left(\frac{p(p+1)}{\sin^2 \theta_{D-1}} - 2s \cot \theta_{D-1} \right) \right] Q(\theta_{D-1}) = 0 \quad (54)$$

By setting the parameters in equation (53) as

$$\ell_j(\ell_j + j - 1) + ((j-1)/2)^2 = \varepsilon; \quad \ell_{j-1}(\ell_{j-1} + j - 2) + [(j-1)/2][(j-3)/2] = o(o+1) \quad (55)$$

then we have

$$\left[d^2 Q(\theta_j) / d\theta_j^2 \right] + \left[\varepsilon - o(o+1) / \sin^2 \theta_j \right] Q(\theta_j) = 0; \quad (56)$$

where ε is assumed to stand for the energy. The hypothetical superpotential for equation (56) is set as

$$\phi(\theta_j) = I \cot \theta_j \quad (57)$$

By using equations (9), (56), and (57) we get the values

$$I = -(o+1) \quad \text{and} \quad \varepsilon_0 = (o+1)^2 = \left(-1/2 + (\ell_{j-1} + (j-2)/2) \right)^2 \quad (58)$$

The superpotential, super-partner potentials, and raising-lowering operators obtained from equations (5), (6), (12), (13), and (57) are

$$V_-(a_0, \theta) = o(o+1) / \sin^2 \theta - (o+1)^2; \quad V_+(a_0, \theta) = (o+2)(o+1) / \sin^2 \theta - (o+1)^2 \quad (59)$$

$$A = d/d\theta_j - (o+1) \cot \theta_j; \quad A^+ = -d/d\theta_j - (o+1) \cot \theta_j \quad (60)$$

By shifting $o \rightarrow o+1$ in equation (59) and by applying equations (10-13) with equations (59-60) we obtain

$$\varepsilon = \ell_j(\ell_j + j - 1) + ((j-1)/2)^2 = (o+n+1)^2 \rightarrow \ell_j = (-1 + \ell_{j-1} + n) \quad (61)$$

The ground state and first excited state wave functions are obtained by using equations (14-15), and (60) as

$$Q_0(\theta_j) = C \left(\sin \theta_j \right)^{o+1}; \quad Q_1(\theta_j) = -(o+1) \cos \theta_j \left(\sin \theta_j \right)^o \quad (62)$$

The angular wave function for the highest level of angular component is obtained from the solution of equation (54) by setting

$$\left(L_{D-2}^2 \right) + [(D-2)/2][(D-4)/2] + (E+M)p(p+1) = p'(p'+1) \quad (63)$$

$$(E+M)s = s'; \quad E' = \left\{ \lambda_\ell + [(D-2)/2]^2 \right\} \quad (64)$$

such that equation (54) become

$$d^2 Q(\theta_{D-1}) / d\theta_{D-1}^2 - \left[p'(p'+1) / \sin^2 \theta_{D-1} - 2s' \cot \theta_{D-1} \right] Q(\theta_{D-1}) = -E' Q(\theta_{D-1}) \quad (65)$$

By setting the hypothetical superpotential for equation (65) as

$$\phi(\theta_{D-1}) = N \cot \theta_{D-1} + B/N \quad (66)$$

and by using equations (9) and (65-66) we have

$$N = -1/2 + (p'+1/2) = p'; -s' = B; p'^2 - s'^2/p'^2 = E'_0 \quad (67)$$

By combining equations (5-7) and (65-67) we have the super-potential, super partner potential and SUSY operator as given as

$$\varphi(\theta_{D-1}) = p' \cot \theta_{D-1} - s'/p' \quad (68)$$

$$V_-(\theta_{D-1}, a_0) = \frac{p'(p'+1)}{\sin^2 \theta_{D-1}} - 2s' \cot \theta_{D-1} - p'^2 + \frac{s'^2}{p'^2}; V_+(\theta, a_0) = \frac{p'(p'-1)}{\sin^2 \theta_{D-1}} - 2s' \cot \theta_{D-1} - p'^2 + \frac{s'^2}{p'^2} \quad (69)$$

$$A = d/d\theta_{D-1} + p' \cot \theta_{D-1} - s'/p'; A^+ = -d/d\theta_{D-1} + p' \cot \theta_{D-1} - s'/p' \quad (70)$$

By shifting the parameter $p' \rightarrow p'-1$ in equation (69) and by applying equation (10-13) with equations (68-69) we obtain relativistic energy equation

$$E'_n = (p'-n)^2 - s'^2/(p'-n)^2 \quad (71)$$

with p' in equation (63).

By using equations (63-64), and (71) we get orbital quantum number equation given as

$$\lambda_l = -[(D-2)/2]^2 + (p'-n)^2 - s'^2/(p'-n)^2 \quad (72)$$

By using equations (8), (17), (18) and (89) we get the relativistic angular wavefunctions for ground state and first excited state which are obtained using equations (14-15) and (70) are ,

$$Q_0^{(-)} = (\sin \theta_{D-1})^{-p'} e^{\frac{s'}{p'} \theta_{D-1}}; Q_1 = 2(p' \cot \theta_{D-1} - s'/p') (\sin \theta_{D-1})^{-p'} e^{\frac{s'}{p'} \theta_{D-1}} \quad (73)$$

The total relativistic ground state and first excited state wave function for any D dimension is obtained by combining equations (49), (62), and (73).

3.3. Thermodynamical Properties

In non-relativistic condition, the relativistic energy equation expressed in equation (47) reduces into non-relativistic energy by taking $(M+E) \rightarrow 2\mu$ where μ is the non-relativistic mass, $(E-M) \rightarrow E_{NR}$, E_{NR} is the non-relativistic energy, and if d_0 is small then equation (47)

$$E_{nr} = -(t^2/2\mu) \left(\left(-\sqrt{c_{nr}(c_{nr}-1)/q+1/4} \right) + \left(\sqrt{b_{nr}'(b_{nr}'+1)/q+1/4} \right) - 1 - 2n \right)^2 \quad (74)$$

$$\text{with } (a(a-1))2\mu + \lambda_l + ((D-1)/2)((D-3)/2) = c_{nr}(c_{nr}-1); b(b+1)2\mu = b_{nr}'(b_{nr}'+1) \quad (75)$$

In classical regimes [43], the vibrational partition function, vibrational mean energy, and specific heat are obtained from the non-relativistic energy equation in equation (74). The vibrational partition function is defined as

$$Z(\zeta, \beta) = \sum_{n=0}^{\zeta} e^{-\beta E_{nr}}, \beta = \frac{1}{kT} \quad (76)$$

k is Boltzman constant, E_{nr} is non-relativistic energy spectrum of the system. When the temperature, T , is high enough, then the value of ζ is high, β is small, and if $1/\delta^2 = 2t^2\beta/\mu$ then equation (76) becomes

$$Z(\xi, \beta) = \sum_{n=0}^{\xi} e^{\gamma^2} = \delta \int_{\delta}^{\xi} e^{\gamma^2} d\gamma = \delta \frac{\sqrt{\pi}}{2} \text{erfi} \left(\frac{\xi}{\delta} \right) \quad (77)$$

$$\text{where } E_{NR} = -(2t^2/\mu)(\xi-n)^2; \xi = 1/2 \left\{ \left(-\sqrt{c_{nr}(c_{nr}-1)/q+1/4} \right) + \left(\sqrt{b_{nr}'(b_{nr}'+1)/q+1/4} \right) - 1 \right\} \quad (78)$$

with $y = (n - \xi)/\delta$ and $erfi$ is the imaginary error function [44].

The vibrational mean energy and the vibrational specific heat are defined as

$$U(\beta, \xi) = -(\partial/\partial\beta) \ln Z(\xi, \beta); C = -(\partial/\partial T)U = -k\beta^2 (\partial/\partial\beta)U \quad (79)$$

By using equations (77-79) we obtain the vibrational mean energy and specific heat equation given as

$$U(\beta, \xi) = (-1/\beta) \left((-1/2) + \left((1/\sqrt{\pi}) \left(\xi t \sqrt{2\beta} / \sqrt{\mu} \right) \exp(-\xi^2 t^2 2\beta / \mu) \right) / \operatorname{erf} \left(\xi t \sqrt{2\beta} / \sqrt{\mu} \right) \right) \quad (80)$$

and

$$C(\xi, \beta) = k \left[\frac{1}{2} - \frac{\left(\frac{1}{\sqrt{\pi}} \frac{\xi t \sqrt{2\beta}}{\sqrt{\mu}} \exp\left(-\frac{\xi^2 t^2 2\beta}{\mu}\right) \right)}{\operatorname{erf}\left(\frac{\xi t \sqrt{2\beta}}{\sqrt{\mu}}\right)} \left(\frac{1}{2} + \frac{\xi^2 t^2 2\beta}{\mu} \right) - \frac{\left(\frac{\xi^2 t^2 2\beta}{2\sqrt{\pi}\mu} \exp\left(-2\frac{\xi^2 t^2 2\beta}{\mu}\right) \right)}{\left(\operatorname{erf}\left(\frac{\xi t \sqrt{2\beta}}{\sqrt{\mu}}\right) \right)^2} \right] \quad (82)$$

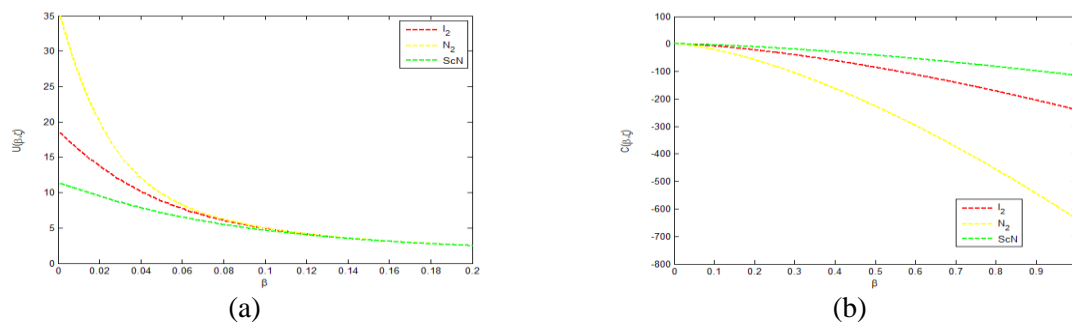


Figure 3. Graph of (a) mean energy U as a function of β , (b) specific heat as a function of β (for $D = 4, n = 5, l = 4, a = 6, b = 3, p = 4, s = 2, q = 0.75$)

The graphs of mean energy and specific heat as a function of β are shown in figure 3 (a and b). From figure 3 (a) and (b) we see that for larger values of β the values of vibrational mean energy and specific heat are constant. The specific heat for system whose is governed by q-deformed modified Poschl-Teller plus trigonometric Manning-Rosen non-central potential are negatives. The negative specific heat may occur for the astronomical objects [45].

4. Conclusion

The Dirac equation in D dimensions of q-deformed modified Poschl-Teller potential plus trigonometric Manning-Rosen non-central potential is solved using SUSY QM. The radial part of D-dimensions of the Dirac equation reduces to one dimensional Schrodinger type equation in centrifugal approximation scheme. There are two solutions of angular Dirac equations, the first one is for hyper-spherical harmonic and the other one is for angular potential functions.

In the exact spin symmetric case, the relativistic energy equation reduces to the non-relativistic energy in the non-relativistic condition. In the classical regime, some thermodynamics properties are derived from the non-relativistic energy equation. The mean energy and specific heat are numerically calculated from non-relativistic energy equation by using Mat-Lab.

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