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The Connected and Disjoint Union of Semi Jahangir Graphs Admit a Cycle-Super (a,d)-Atimagic Total Labeling

Dafik[1,2], **I.H. Agustin**[1,3], **D. Hardiyantik**[3]

[1]CGANT - University of Jember
[2]Mathematics Education Department - University of Jember
[3]Mathematics Department - University of Jember

E-mail: `d.dafik@unej.ac.id; hestyarin@gmail.com`

Abstract. We assume that all graphs in this paper are finite, undirected and no loop and multiple edges. Given a graph G of order p and size q. Let H', H be subgraphs of G. By H'-covering, we mean every edge in $E(G)$ belongs to at least one subgraph of G isomorphic to a given graph H. A graph G is said to be an (a,d)-H-antimagic total labeling if there exist a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that for all subgraphs H' isomorphic to H, the total H-weights $w(H) = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$ form an arithmetic sequence $\{a, a+d, a+2d, ..., a+(s-1)d\}$, where a and d are positive integers and s is the number of all subgraphs H' isomorphic to H. Such a labeling is called super if $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$. In this paper, we will discuss a cycle-super (a,d)-atimagicness of a connected and disjoint union of semi jahangir graphs. The results show that those graphs admit a cycle-super (a,d)-atimagic total labeling for some feasible $d \in \{0, 1, 2, 4, 6, 7, 10, 13, 14\}$.

We use a handbook of graph theory written by Gross *et. al* [4] to define all basic definitions of graph in this paper. For p and q are respectively the order and size of graph, by a labeling of a graph, we mean any mapping that sends some set of graph elements to a set of positive integers. The labelings are called vertex labelings or edge labelings If the domain is respectively a vertex-set $V(G)$ or a edge-set $E(G)$. Moreover, the labelings are called *total* labelings if the domain is $V(G) \cup E(G)$. Simanjuntak *et al.* in [13] introduced an (a,d)-*edge-antimagic total labeling* of G of order p and size q. It is a one-to-one mapping f taking the vertices and edges of G onto $\{1, 2, \ldots, p+q\}$ such that the edge-weights $W_f(uv) = f(u) + f(v) + f(uv), uv \in E(G)$ form an arithmetic sequence $\{a, a+d, \ldots, a+(q-1)d\}$, where the first term a is $a > 0$ and the common difference d is $d \geq 0$. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

Gutiérrez, and Lladó in [3, 8] expanded the edge-magic total labeling into a magic total covering. They defined that a graph G admits an H'-magic covering, where H' is subgraph of G isomorphic to a given graph H, if the total H-weights $w(H) = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e) = \lambda(H)$ is a constant magic sum and $\lambda(H)$ is a constant supermagic sum of H if $f : V(G) \to \{1, 2, \ldots, p\}$. Some relevant results can be found in [7, 9, 10, 12]. Recently Feňovčiková *et. al* [2] proved that wheels are cycle antimagic.

Motivated by these two previous labelings, Inayah *et al.* [5] introduced the $(a,d) - H$- antimagic total labeling. A graph G is said to be an (a,d)-H-antimagic total labeling if there exist a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that for all subgraphs H' isomorphic to H, the total H-weights $w(H) = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$ form an arithmetic sequence

$\{a, a+d, a+2d, ..., a+(s-1)d\}$, where a and d are positive integers and s is the number of all subgraphs H' isomorphic to H. Similarly, such a labeling is called super if $f: V(G) \rightarrow \{1, 2, \ldots, p\}$. Inayah *et. al* [6] proved that, $shack(H, k)$ which contains exactly k subgraphs isomorphic to H is H-super antimagic, for H is a non-trivial connected graph and $k \geq 2$ is an integer.

We will discuss the existence of a cycle-super (a,d)-atimagicness of a connected and disjoint union of semi jahangir graphs. For H-supermagic graphs, we have found some results. For example Rizvi, *et.al.* [11] proved the disjoint union of isomorphic copies of fans, triangular ladders, ladders, wheels, and graphs obtained by joining a star $K_{1,n}$ with K_1, and also disjoint union of non-isomorphic copies of ladders and fans are cycle-supermagic labelings, but for super antimagic labelings, it remains widely open to explore.

The Results

Prior to present the main results, we repropose a lemma proved by Dafik *et.al* in [1], it will be useful to find the existence of H-super antimagic graphs. This lemma showed a least upper bound for feasible value of d for a graph to be super (a, d)-H- antimagic total labeling.

Lemma 1. *[1] Let G be a simple graph of order p and size q. If G is super (a, d)-H- antimagic total labeling then $d \leq \frac{(p_G - p_{H'})p_{H'} + (q_G - q_{H'})q_{H'}}{s-1}$, for H'_j are subgraphs isomorphic to H, $p_G = |V(G)|$, $q_G = |E(G)|$, $p_{H'} = |V(H')|$, $q_{H'} = |E(H')|$, and $s = |H'_j|$.*

Proof: Assume that a (p, q)-graph has a super (a, d)-H- antimagic total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p_G + q_G\}$ and the total H-weights $w(H) = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = \{a, a+d, a+2d, ..., a+(s-1)d\}$. The minimum possible total H-weight in the labeling f is at least $1 + 2 + \ldots + p_{H'} + (p_G + 1) + (p_G + 2) + \ldots + (p_G + q_{H'}) = \frac{p_{H'}}{2} + \frac{p_{H'}^2}{2} + q_{H'}p_G + \frac{q_{H'}}{2} + \frac{q_{H'}^2}{2}$. Thus, $a \geq \frac{p_{H'}}{2} + \frac{p_{H'}^2}{2} + q_{H'}p_G + \frac{q_{H'}}{2} + \frac{q_{H'}^2}{2}$. On the other hand, the maximum possible total H-weight is at most $p_G + p_G - 1 + p_G - 2 + ... + (p_G - (p_{H'} - 1)) + (p_G + q_G) + (p_G + q_G - 1) + (p_G + q_G - 2) + ... + (p_G + q_G - (q_{H'} - 1)) = p_{H'}p_G - \frac{p_{H'} - 1}{2}(p_{H'}) + q_{H'}p_G + q_{H'}q_G - \frac{q_{H'} - 1}{2}(q_{H'})$. So we obtain $a + (s-1)d \leq p_{H'}p_G - \frac{p_{H'} - 1}{2}(p_{H'}) + q_{H'}p_G + q_{H'}q_G - \frac{q_{H'} - 1}{2}(q_{H'})$. Simplifying the inequality then we will have the desired upper bound of d. $\square$

From now on we will introduce our terminology of connected semi Jahangir and disjoint union of semi Jahangir graphs.

A semi Jahangir graph, denoted by SJ_n, is a connected graph with vertex set $V(SJ_n) = \{p, x_i, y_k; \text{for } 1 \leq i \leq n+1,\ 1 \leq k \leq n\}$ and edge set $E(SJ_n) = \{px_i; 1 \leq i \leq n+1\} \cup \{x_iy_i; 1 \leq i \leq n\} \cup \{y_ix_{i+1}; 1 \leq i \leq n\}$. Since we study a super (a, d)-H- antimagic total labeling for $H' = C_4$ isomorphic to H, thus $p_G = |V(SJ_n)| = 2n + 2$, $q_G = |E(SJ_n)| = 3n + 1$, $p_{H'} = |V(C_4)| = 4$, $q_{H'} = |E(C_4)| = 4$, $s = |H'_j| = |C_4| = n$. If semi Jahangir graph SJ_n has a super (a, d)-C_4-antimagic total labeling then it follows from Lemma 1 the upper bound of $d \leq 20$.

A disjoint union of semi Jahangir graph, denoted by mSJ_n, is a disconnected graph with vertex set $V(mSJ_n) = \{p^j, x_i^j, y_k^j; \text{for } 1 \leq i \leq n+1,\ 1 \leq k \leq n,\ 1 \leq j \leq m\}$ and edge set $E(mSJ_n) = \{p^jx_i^j; 1 \leq i \leq n+1,\ 1 \leq j \leq m\} \cup \{x_i^jy_i^j; 1 \leq i \leq n,\ 1 \leq j \leq m\} \cup \{y_i^jx_{i+1}^j; 1 \leq i \leq n,\ 1 \leq j \leq m\}$. Since we study a super (a, d)-H- antimagic total labeling for $H' = C_4$ isomorphic to H, thus $p_G = |V(mSJ_n)| = 2mn + 2m$, $q_G = |E(mSJ_n)| = 3mn + m$, $p_{H'} = |V(C_4)| = 4$, $q_{H'} = |E(C_4)| = 4$, $s = |H'_j| = |C_4| = nm$. If disjoint union of semi Jahangir graph mSJ_n has a super (a, d)-F_n-antimagic total labeling then it follows from Lemma 1 the upper bound of $d \leq 25$.

Now we start to describe the result of the super (a, d)-C_4-antimagic total labeling of semi Jahangir graph, denoted by SJ_n, in the following theorems.

Theorem 1. *For $n \geq 2$, the graph SJ_n admits a super $(15n + 21, 1) - C_4$ antimagic total labeling.*

Proof. Define the vertex and edge labeling f_1 as follows

$$\begin{aligned}
f_1(p) &= 1\\
f_1(x_i) &= i+1, \text{for } 1 \ \leq i \leq n+1\\
f_1(y_i) &= n+i+2, \text{for } 1 \ \leq i \leq n\\
f_1(px_i) &= 2n+i+2, \text{for } 1 \leq i \leq n+1\\
f_1(x_iy_i) &= 5n-2i+4, \text{for } 1 \leq i \leq n\\
f_1(y_ix_{i+1}) &= 5n-2i+5, \text{for } 1 \leq i \leq n
\end{aligned}$$

The vertex and edge labelings f_1 are a bijective function $f_1 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \ldots, 5n+3\}$. The H-weights of SJ_n, for $1 \leq i \leq n$ under the labeling f_1, constitute the following sets $w_{f_1} = f_1(p)+f_1(x_i)+f_1(x_{i+1})+f_1(y_i) = (1)+(i+1)+(i+1+1)+(n+i+2) = n+3i+6$, and the total H-weights of SJ_n constitute the following sets $W_{f_1} = w_{f_1}+f_1(px_i)+f_1(px_{i+1})+f_1(x_iy_i)+f_1(y_ix_{i+1}) = (n+3i+6)+(2n+i+2)+(2n+i+1+2)+(5n-2i+4)+(5n-2i+5) = 15n+i+20$. It is easy to see that the set $W_{f_1} = \{15n+21, 15n+22, \ldots, 16n+20\}$. Therefore, the graph SJ_n admits a super $(15n+21, 1)$-C_4 antimagic total labeling, for $n \geq 2$. □

Theorem 2. *For $n \geq 2$, the graph SJ_n admits a super $(14n+22, 7) - C_4$ antimagic total labeling.*

Proof. Define the vertex labeling f_2 as $f_2(p) = f_1(p), f_2(x_i) = f_1(x_i), f_2(y_i) = f_1(y_i)$ and edge labeling f_2 as follows

$$\begin{aligned}
f_2(px_i) &= 4n+i+2, \text{for} 1 \leq i \leq n+1\\
f_2(x_iy_i) &= 2n+i+2, \text{for} 1 \leq i \leq n\\
f_2(y_ix_{i+1}) &= 3n+i+2, \text{for} 1 \leq i \leq n
\end{aligned}$$

The vertex and edge labelings f_2 are a bijective function $f_2 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \ldots, 5n+3\}$. The H-weights of SJ_n, for $1 \leq i \leq n$ under the labeling f_2, constitute the following sets $w_{f_2} = w_{f_1}$, and the total H-weights of $SJ_n)$ constitute the following sets $W_{f_2} = w_{f_2}+f_2(px_i)+f_2(px_{i+1})+f_2(x_iy_i)+f_2(y_ix_{i+1}) = (n+3i+6)+(4n+i+2)+(4n+i+1+2)+(2n+i+2)+(3n+i+2) = 14n+7i+15$. It is easy to see that the set $W_{f_2} = \{14n+22, 14n+29, \ldots, 21n+15\}$. Therefore, the graph SJ_n admits a super $(14n+22, 7)$-C_4 antimagic total labeling, for $n \geq 2$. □

Theorem 3. *For $n \geq 2$, the graph SJ_n admits a super $(13n+23, 10) - C_4$ antimagic total labeling.*

Proof. Define the vertex and edge labeling f_3 as follows

$$\begin{aligned}
f_3(p) &= 1\\
f_3(x_i) &= 2i, \text{for } 1 \leq i \leq n+1\\
f_3(y_i) &= 2i+1, \text{for } 1 \leq i \leq n\\
f_3(px_i) &= f_2(px_i)\\
f_3(x_iy_i) &= f_2(x_iy_i)\\
f_3(y_ix_{i+1}) &= f_2(y_ix_{i+1}
\end{aligned}$$

The vertex and edge labelings f_3 are a bijective function $f_3 : V(SJ_n) \cup E(SJ_n) \rightarrow \{1, 2, 3, \ldots, 5n+3\}$. The H-weights of SJ_n, for $1 \leq i \leq n$ under the labeling f_3, constitute the following sets $w_{f_3} = f_3(p)+f_3(x_i)+f_3(x_{i+1})+f_3(y_i) = (1)+(2i)+(2(i+1))+(2i+1) = 6i+4$, and the total H-weights of SJ_n constitute the following sets $W_{f_3} = w_{f_3}+f_3(px_i)+f_3(px_{i+1})+f_3(x_iy_i)+f_3(y_ix_{i+1}) = (6i+4)+(4n+i+2)+(4n+i+1+2)+(2n+i+2)+(3n+i+2) = 13n+10i+13$. It is easy to see that the set $W_{f_3} = \{13n+23, 13n+33, \ldots, 23n+13\}$. Therefore, the graph SJ_n admits a super $(13n+23, 10)$-C_4 antimagic total labeling, for $n \geq 2$. □

Theorem 4. *For $n \geq 2$, the graph SJ_n admits a super $(11n+25, 13) - C_4$ antimagic total labeling.*

Proof. Define the vertex and edge labeling f_4 as follows

$$\begin{aligned}
f_4(p) &= 1\\
f_4(x_i) &= n+i+1, \text{for } 1 \leq i \leq n+1\\
f_4(y_i) &= n-i+2, \text{for } 1 \leq i \leq n\\
f_4(px_i) &= 2n+3i, \text{for } 1 \leq i \leq n+1\\
f_4(x_iy_i) &= 2n+3i+1, \text{for } 1 \leq i \leq n\\
f_4(y_ix_{i+1}) &= 2n+3i+2, \text{for } 1 \leq i \leq n
\end{aligned}$$

The vertex and edge labelings f_4 are a bijective function $f_4 : V(SJ_n) \cup E(SJ_n) \to \{1, 2, 3, \ldots, 5n+3\}$. The H-weights of SJ_n, for $1 \leq i \leq n$ under the labeling f_4, constitute the following sets $w_{f_4} = f_4(p) + f_4(x_i) + f_4(x_{i+1}) + f_4(y_i) = (1) + (n+i+1) + (n+i+1+1) + (n-i+2) = 3n+i+6$, and the total H-weights of SJ_n constitute the following sets $W_{f_4} = w_{f_4} + f_4(px_i) + f_4(px_{i+1}) + f_4(x_iy_i) + f_4(y_ix_{i+1}) = (3n+i+6) + (2n+3i) + (2n+3(i+1)) + (2n+3i+1) + (2n+3i+2) = 11n+13i+12$. It is easy to see that the set $W_{f_4} = \{11n+25, 11n+38, \ldots, 24n+12\}$. Therefore, the graph SJ_n admits a super $(11n+25, 13)$-C_4 antimagic total labeling, for $n \geq 2$. □

Theorem 5. *For $n \geq 2$, the graph SJ_n admits a super $(\frac{19n+54}{2}, 14) - C_4$ antimagic total labeling for n is even, and for $n \geq 2$, the graph SJ_n admits a super $(\frac{19n+53}{2}, 14) - C_4$ antimagic total labeling for n is odd.*

Proof. Define the vertex and edge labeling f_5 as follows

$$\begin{aligned}
f_5(p) &= 1\\
f_5(x_i) &= \begin{cases} \frac{i+3}{2}, & \text{for } 1 \leq i \leq n+1 \text{ ; } i \text{ is odd}\\ \frac{n+i+4}{2}, & \text{for } 1 < i < n+1 \text{ ; } i \text{ is even}, n \text{ is even}\\ \frac{n+i+3}{2}, & \text{for } 1 < i \leq n+1 \text{ ; } i \text{ is even}, n \text{ is odd}\end{cases}\\
f_5(y_i) &= n+i+2, \text{for } 1 \leq i \leq n\\
f_5(px_i) &= f_4(px_i)\\
f_5(x_iy_i) &= f_4(x_iy_i)\\
f_5(y_ix_{i+1}) &= f_4(y_ix_{i+1})
\end{aligned}$$

The vertex and edge labelings f_4 are a bijective function $f_5 : V(SJ_n) \cup E(SJ_n) \to \{1, 2, 3, \ldots, 5n+3\}$. The H-weights of SJ_n, for $1 \leq i \leq n$ under the labeling f_5, constitute the following sets $w_{f_5} = f_5(p) + f_5(x_i) + f_5(x_{i+1}) + f_5(y_i) = 1 + (\frac{i+3}{2}) + (\frac{n+i+1+4}{2}) + (2n+2i+4) = \frac{3n+4i+14}{2}$ for even n, $w_{f_5} = f_5(p) + f_5(x_i) + f_5(x_{i+1}) + f_5(y_i) = 1 + (\frac{i+3}{2}) + (\frac{n+i+1+3}{2}) + (2n+2i+4) = \frac{3n+4i+14}{2}$ for odd n and the total H-weights of SJ_n constitute the following sets $W_{f_5} = w_{f_5} + f_5(px_i) + f_5(px_{i+1}) + f_5(x_iy_i) + f_5(y_ix_{i+1}) = (\frac{3n+4i+14}{2}) + (2n+3i) + (2n+3(i+1)) + (2n+3i+1) + (2n+3i+2) = \frac{19n+28i+26}{2}$ for even n and $W_{f_5} = w_{f_5} + f_5(px_i) + f_5(px_{i+1}) + f_5(x_iy_i) + f_5(y_ix_{i+1}) = (\frac{3n+4i+13}{2}) + (2n+3i) + (2n+3(i+1)) + (2n+3i+1) + (2n+3i+2) = \frac{19n+28i+25}{2}$ for odd n. It is easy to see that the set $W_{f_5} = \{\frac{19n+54}{2}, \frac{19n+82}{2}, \ldots, \frac{47n+26}{2}\}$ for even n and $W_{f_5} = \{\frac{19n+53}{2}, \frac{19n+81}{2}, \ldots, \frac{47n+25}{2}\}$ for odd n. Therefore, the graph SJ_n admits a super $(\frac{19n+54}{2}, 14) - C_4$ antimagic total labeling for $n \geq 2$ with even n. And the graph SJ_n admits a super $(\frac{19n+53}{2}, 14) - C_4$ antimagic total labeling for $n \geq 2$ with odd n. □

We continue to show the result of the super (a, d)-C_4-antimagic total labeling of disjoint union of semi Jahangir graph, SJ_n, in the following theorems.

Theorem 6. *For $m, n \geq 2$, the graph mSJ_n admits a super $(18mn + 14m + 4, 0)$-C_4 antimagic total labeling.*

Proof. For $1 \leq j \leq m$, define the vertex and edge labeling g_1 as follows

$$\begin{aligned}
g_1(p^j) &= j,\ 1 \ \leq j \leq m \\
g_1(x_i^j) &= 2mi + j - m, \text{for } 1 \ \leq i \leq n+1,\ 1 \ \leq j \leq m \\
g_1(y_i^j) &= 2mn - 2mi + 3m - j + 1, \text{for } 1 \ \leq i \leq n,\ 1 \ \leq j \leq m \\
g_1(p^j x_i^j) &= 4mn + mi + m + j, \text{for } 1 \ \leq i \leq n+1,\ 1 \ \leq j \leq m \\
g_1(x_i^j y_i^j) &= 4mn - 2mi + 4m - 2j + 2, \text{for } 1 \leq i \leq n,\ 1 \ \leq j \leq m \\
g_1(y_i^j x_{i+1}^j) &= 4mn - 2mi + 4m - 2j + 1, \text{for } 1 \leq i \leq n,\ 1 \ \leq j \leq m
\end{aligned}$$

The vertex and edge labelings g_1 are a bijective function $g_1 : V(mSJ_n) \cup E(mSJ_n) \rightarrow \{1, 2, 3, \ldots, 5mn + 3m\}$. The H-weights of mSJ_n, for $1 \leq i \leq n$ and $1 \leq j \leq m$ under the labeling g_1, constitute the following sets $w_{g_1} = g_1(p^j) + g_1(x_i^j) + g_1(x_{i+1}^j) + g_1(y_i^j) = (j) + (2mi + j - m) + (2m(i+1) + j - m) + (2mn - 2mi + 3m - j + 1) = 2mn + 2mi + 3m + 2j + 1$, and the total H-weights of mSJ_n constitute the following sets $W_{g_1} = w_{g_1} + g_1(p^j x_i^j) + g_1(p^j x_{i+1}^j) + g_1(x_i^j y_i^j) + g_1(y_i^j x_{i+1}^j) = (2mn + 2mi + 3m + 2j + 1) + (4mn + mi + m + j) + (4mn + m(i+1) + m + j) + (4mn - 2mi + 4m - 2j + 2) + (4mn - 2mi + 4m - 2j + 1) = 18mn + 14m + 4$. It is easy to see that the set $W_{g_1} = \{18mn + 14m + 4, 18mn + 14m + 4, \ldots, 18mn + 14m + 4\}$. Therefore, the graph mSJ_n admits a super $(18mn + 14m + 4, 0)$-C_4 antimagic total labeling, for $m, n \geq 2$. □

Theorem 7. *For $m, n \geq 2$, the graph mSJ_n admits a super $(17mn + 14m + 5, 2)$-C_4 antimagic total labeling.*

Proof. For $1 \leq j \leq m$, define the vertex labeling g_2 as $g_2(p^j) = g_1(p^j), g_2(x_i^j) = g_1(x_i^j), g_2(y_i^j) = g_1(y_i^j)$ and edge labeling g_2 as follows

$$\begin{aligned}
g_2(p^j x_i^j) &= 4mn + mi + 2m - j + 1, \text{for } 1 \leq i \leq n+1,\ 1 \ \leq j \leq m \\
g_2(x_i^j y_i^j) &= 3mn - mi + 2m + j, \text{for } 1 \leq i \leq n,\ 1 \ \leq j \leq m \\
g_2(y_i^j x_{i+1}^j) &= 4mn - mi + 2m + j, \text{for } 1 \leq i \leq n,\ 1 \ \leq j \leq m
\end{aligned}$$

The vertex and edge labelings g_1 are a bijective function $g_2 : V(mSJ_n) \cup E(mSJ_n) \rightarrow \{1, 2, 3, \ldots, 5mn + 3m\}$. The H-weights of mSJ_n, for $1 \leq i \leq n$ and $1 \leq j \leq m$ under the labeling g_2, constitute the following sets $w_{g_2} = w_{g_1}$, and the total H-weights of mSJ_n constitute the following sets $W_{g_2} = w_{g_2} + g_2(p^j x_i^j) + g_2(p^j x_{i+1}^j) + g_2(x_i^j y_i^j) + g_2(y_i^j x_{i+1}^j) = (2mn + 2mi + 3m + 2j + 1) + (4mn + mi + 2m - j + 1) + (4mn + m(i+1) + 2m - j + 1) + (3mn - mi + 2m + j) + (4mn - mi + 2m + j) = 17mn + 2mi + 12m + 2j + 3$. It is easy to see that the set $W_{g_2} = \{17mn + 14m + 5, 17mn + 14m + 7, \ldots, 19mn + 14m + 3\}$. Therefore, the graph mSJ_n admits a super $(17mn + 14m + 5, 2)$-C_4 antimagic total labeling, for $m, n \geq 2$. □

Theorem 8. *For $m, n \geq 2$, the graph mSJ_n admits a super $(16mn + 14m + 6, 4)$-C_4 antimagic total labeling.*

Proof. For $1 \leq j \leq m$, define the vertex labeling g_3 as $g_3(p^j) = g_1(p^j), g_3(x_i^j) = g_1(x_i^j), g_3(y_i^j) =$

$g_1(y_i^j)$ and edge labeling g_3 as follows

$$\begin{aligned}
g_3(p^jx_i^j) &= 4mn+mi+m+j;\ 1\ \le i\le n+1,\ 1\ \le j\le m,\ \text{dan } i \text{ ganjil}\\
g_3(p^jx_i^j) &= 4mn+mi+2m-j+1;\ 1\le i\le n+1, 1\le j\le m,\ \text{dan } i \text{ genap}\\
g_3(x_i^jy_i^j) &= 2mn+mi+m+j,\ \text{for } 1\le i\le n,\ 1\le j\le m\\
g_3(y_i^jx_{i+1}^j) &= 4mn-mi+2m+j,\ \text{for } 1\le i\le n,\ 1\le j\le m
\end{aligned}$$

The vertex and edge labelings g_1 are a bijective function $g_3 : V(mSJ_n)\cup E(mSJ_n) \to \{1,2,3,\dots,5mn+3m\}$. The H-weights of mSJ_n, for $1\le i\le n$ and $1\le j\le m$ under the labeling g_3, constitute the following sets $w_{g_3} = w_{g_1}$, and the total H-weights of mSJ_n constitute the following sets $W_{g_3} = w_{g_3} + g_3(p^jx_i^j) + g_3(p^jx_{i+1}^j) + g_3(x_i^jy_i^j) + g_3(y_i^jx_{i+1}^j) = (2mn+2mi+3m+2j+1)+(4mn+mi+m+j)+(4mn+m(i+1)+2m-j+1)+(2mn+mi+m+j)+(4mn-mi+2m+j) = 16mn+4mi+10m+4j+2$. It is easy to see that the set $W_{g_3} = \{16mn+14m+6, 16mn+14m+10,\dots,20mn+14m+2\}$. Therefore, the graph mSJ_n admits a super $(16mn+14m+6,4)$-C_4 antimagic total labeling, for $m,n\ge 2$. □

Theorem 9. *For $m,n\ge 2$, the graph mSJ_n admits a super $(15mn+14m+7,6)$-C_4 antimagic total labeling.*

Proof. For $1\le j\le m$, define the vertex labeling g_4 as $g_4(p^j) = g_1(p^j), g_4(x_i^j) = g_1(x_i^j), g_4(y_i^j) = g_1(y_i^j)$ and edge labeling g_4 as follows

$$\begin{aligned}
g_4(p^jx_i^j) &= 4mn+mi+m+j, \text{for } 1\le i\le n+1,\ 1\le j\le m\\
g_4(x_i^jy_i^j) &= 2mn+mi+m+j, \text{for } 1\le i\le n,\ 1\le j\le m\\
g_4(y_i^jx_{i+1}^j) &= 3mn+mi+m+j, \text{for } 1\le i\le n,\ 1\le j\le m
\end{aligned}$$

The vertex and edge labelings g_4 are a bijective function $g_4 : V(mSJ_n)\cup E(mSJ_n) \to \{1,2,3,\dots,5mn+3m\}$. The H-weights of mSJ_n, for $1\le i\le n$ and $1\le j\le m$ under the labeling g_4, constitute the following sets $w_{g_4} = w_{g_1}$, and the total H-weights of mSJ_n constitute the following sets $W_{g_4} = w_{g_4} + g_4(p^jx_i^j) + g_4(p^jx_{i+1}^j) + g_4(x_i^jy_i^j) + g_4(y_i^jx_{i+1}^j) = (2mn+2mi+3m+2j+1)+(4mn+mi+m+j)+(4mn+m(i+1)+m+j)+(2mn+mi+m+j)+(3mn+mi+m+j) = 15mn+6mi+8m+6j+1$. It is easy to see that the set $W_{g_4} = \{15mn+14m+7, 15mn+14m+13,\dots,21mn+14m+1\}$. Therefore, the graph mSJ_n admits a super $(15mn+14m+7,6)$-C_4 antimagic total labeling, for $m,n\ge 2$. □

Concluding Remarks

A least upper bound of difference d for connected and disjoint union of graphs are respectively $d\le 20$ and $d\le 25$. Apart from obtained d above, we haven't found any result yet, so we propose the following open problem:

Open Problem 1. *Apart from $d\in\{1,7,10,13,14\}$, determine a super $(a,d)-C_4$-antimagic total labeling of connected SJ_n, for $d\le 20$ and $n\ge 2$.*

Open Problem 2. *Apart from $d\in\{0,2,4,6\}$, determine a super $(a,d)-C_4$-antimagic total labeling of disjoint union of m copies of SJ_n, for $d\le 25$ and $m,n\ge 2$.*

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