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Jacobi wavelet operational matrix of fractional integration for solving fractional integro-differential equation

Loh Jian Rong[1] **and Phang Chang**[2]

Department of Science and Mathematics, Faculty of Science, Technology and Human Development, Universiti Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat, Johor, Malaysia

E-mail: [1] gw140029@siswa.uthm.edu.my

E-mail: [2] pchang@uthm.edu.my

Abstract. In this paper, we first define generalized shifted Jacobi polynomial on interval $[\frac{(n-1)L}{2^{p-1}}, \frac{nL}{2^{p-1}}]$ and then use it to define Jacobi wavelet. Then, the operational matrix of fractional integration for Jacobi wavelet is being derived to solve fractional differential equation and fractional integro-differential equation. This method can be seen as a generalization of other orthogonal wavelet operational methods, e.g. Legendre wavelets, Chebyshev wavelets of 1st kind, Chebyshev wavelets of 2nd kind, etc. which are special cases of the Jacobi wavelets. We apply our method to a special type of fractional integro-differential equation of Fredholm type.

1. Introduction

The method of solving fractional differential equations using wavelet operational matrices derived from orthogonal functions such as Legendre Polynomial[2], Legendre Wavelet[1][3], Chebyshev Wavelet[4][5][6][7][8][9][10], Jacobi Polynomial [11][12][13][14] etc. have been proposed by several researchers and have been seen as a powerful technique to find approximate solution accurately and efficiently for fractional differential equations and fractional integro-differential equations.

Recently, Bhrawy et. al. [12] have derived an operational matrix of fractional derivative based on Jacobi Polynomial and a more recent paper [13] he and other fellow researchers have derived the shifted Jacobi polynomial operational matrix of fractional integration and combined with spectral-tau method to solve linear fractional differential equations. Bhrawy et. al.[13] considered the operational matrix of fractional integration based on shifted Jacobi polynomial defined on the interval $[0, L]$ and solution of linear fractional differential equations.

In this study, we focus on extending the concept by Bhrawy et.al. [13] from the Jacobi Polynomial to Jacobi Wavelet and derive the Wavelet Operational Matrix of Fractional Integration based on a generalized shifted Jacobi Wavelets (i.e. small pulses subjected to dilations and translations and defined on the interval $\left[\frac{(n-1)L}{2^{p-1}}, \frac{nL}{2^{p-1}}\right]$) and apply it to solve the following class of linear Fredholm 2nd kind fractional integro-differential equations:

$$D^{\alpha} f(x) = g(x) + \lambda \int_0^1 K(x,t) f(t) dt \tag{1}$$

The main purpose of this method is to generalizes the wavelet operational matrix of fractional integration by Jacobi wavelets where Legendre wavelets, 1st-kind Chebyshev wavelets, 2nd-kind Chebyshev wavelets, Gegenbaur wavelets are special cases of Jacobi wavelets. Here, we will first derive the operational matrix based on Jacobi Wavelet. For this, we briefly present our preliminary result for this purpose.

To define the Jacobi Wavelet, we need to find to apply translation and dilation to the Jacobi Polynomial, i.e. $\Omega_m^{a,b}\left(\frac{2^p}{L}x-2n+1\right)$ where a,b are Jacobi polynomials parameters from its weight function $w(x)=(1-x)^a(1+x)^b$, m is degree of polynomial, n the translation, $\frac{2^p}{L}$ the dilation. So below we derive the analytic form of generalized shifted Jacobi Polynomial after applying translation and dilation.

2. Generalized Shifted Jacobi Polynomial on $\left(\frac{(n-1)L}{2^{p-1}},\frac{nL}{2^{p-1}}\right)$

The shifted Jacobi Polynomial defined on $[0,1]$ is given by:

$$\begin{aligned} P_m^{(a,b)}(x) &= \Omega_m^{a,b}\left(\frac{2}{L}x-1\right) \\ &= \sum_{k=0}^{m}(-1)^{m-k}\frac{\Gamma(m+b+1)\Gamma(m+k+a+b+1)}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!k!L^k}\,(x)^k \end{aligned} \tag{2}$$

We derive the analytic form for a generalized shifted Jacobi Polynomial $\tilde{P}_{n,m}^{(a,b,L)}(x)$ defined on $[\frac{(n-1)L}{2^{p-1}},\frac{nL}{2^{p-1}}]$ which is needed to define the Jacobi wavelet.

Theorem 2.1. *The generalized shifted Jacobi Polynomial $\tilde{P}_{n,m}^{(a,b,L)}(x)$ of degree m defined on interval $\left(\frac{(n-1)L}{2^{p-1}},\frac{nL}{2^{p-1}}\right)$ given by*

$$\tilde{P}_{n,m}^{(a,b,L)}(x)=\sum_{r=0}^{m}\sum_{k=r}^{m}S_{k,k-r}^{a,b,L,n,m,p}x^r \tag{3}$$

where

$$S_{k,k-r}^{a,b,L,n,m,p}=(-1)^{m-r}\frac{2^{(r)(p-1)}\Gamma(m+b+1)\Gamma(m+k+a+b+1)(n-1)^{k-r}}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!(r)!(k-r)!L^r} \tag{4}$$

Proof. The generalized shifted Jacobi Polynomial can be derived from the shifted Jacobi Polynomial $P_m^{(a,b)}(x)$ by using substitution $\frac{2^px}{L}-2n+1=\frac{2}{L}\left(2^{p-1}x-(n-1)L\right)-1$:

$$\tilde{P}_{n,m}^{(a,b,L)}(x)=P_m^{(a,b,L)}\left(2^{p-1}x-(n-1)L\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{m}(-1)^{m-k}\frac{\Gamma(m+b+1)\Gamma(m+k+a+b+1)}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!k!L^k}\left(2^{p-1}x-(n-1)L\right)^k \\
&= \sum_{k=0}^{m}(-1)^{m-k}\frac{\Gamma(m+b+1)\Gamma(m+k+a+b+1)}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!k!L^k}(2^{k(p-1)})\left(x-\frac{(n-1)L}{2^{p-1}}\right)^k \\
&= \sum_{k=0}^{m}(-1)^{m-k}\frac{2^{k(p-1)}\Gamma(m+b+1)\Gamma(m+k+a+b+1)}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!k!L^k}\sum_{r=0}^{k}\begin{pmatrix} k \\ r \end{pmatrix}x^{k-r}\left(-\frac{(n-1)L}{2^{p-1}}\right)^r \\
&= \sum_{k=0}^{m}\sum_{r=0}^{k}(-1)^{m-k+r}\begin{pmatrix} k \\ r \end{pmatrix}\frac{2^{k(p-1)}\Gamma(m+b+1)\Gamma(m+k+a+b+1)}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!k!L^k}x^{k-r}\frac{(n-1)^rL^r}{2^{r(p-1)}} \\
&= \sum_{k=0}^{m}\sum_{r=0}^{k}(-1)^{m-k+r}\begin{pmatrix} k \\ r \end{pmatrix}\frac{2^{(k-r)(p-1)}\Gamma(m+b+1)\Gamma(m+k+a+b+1)(n-1)^r}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!k!L^{k-r}}x^{k-r} \\
&= \sum_{k=0}^{m}\sum_{r=0}^{k}(-1)^{m-k+r}\frac{2^{(k-r)(p-1)}\Gamma(m+b+1)\Gamma(m+k+a+b+1)(n-1)^r}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!(k-r)!r!L^{k-r}}x^{k-r}
\end{aligned}
\tag{5}
$$

By interchanging the summation and substituting $k-r$ with r, we can write as:

$$
= \sum_{r=0}^{m}\sum_{k=r}^{m}(-1)^{m-r}\frac{2^{(r)(p-1)}\Gamma(m+b+1)\Gamma(m+k+a+b+1)(n-1)^{k-r}}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!(r)!(k-r)!L^r}x^r \tag{6}
$$

Set

$$
S_{k,k-r}^{a,b,L,n,m,p} = (-1)^{m-r}\frac{2^{(r)(p-1)}\Gamma(m+b+1)\Gamma(m+k+a+b+1)(n-1)^{k-r}}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!(r)!(k-r)!L^r} \tag{7}
$$

Then

$$
\tilde{P}_{n,m}^{(a,b,L)}(x) = \sum_{r=0}^{m}\sum_{k=r}^{m}S_{k,k-r}^{a,b,L,n,m,p}x^r \tag{8}
$$

□

Definition 2.2. *The orthogonality property of the generalized shifted Jacobi Polynomial* $\tilde{P}_{n,m}^{(a,b,L)}(x)$ *on interval* $\left(\frac{(n-1)L}{2^{p-1}}, \frac{nL}{2^{p-1}}\right)$ *is given by the inner product*

$$
\begin{aligned}
\left\langle \tilde{P}_{n,m_1}^{(a,b,L)}(x), \tilde{P}_{n,m_2}^{(a,b,L)}(x)\right\rangle &= \int_{\frac{(n-1)L}{2^{p-1}}}^{\frac{nL}{2^{p-1}}} \tilde{P}_{n,m_1}^{(a,b,L)}(x)\tilde{P}_{n,m_2}^{(a,b,L)}(x)\omega_n^{(a,b,L)}(x)dx \\
&= \begin{cases} h_m^{(a,b,L)} = \frac{L}{2^p}\frac{2^{a+b+1}\Gamma(m+a+1)\Gamma(m+b+1)}{(2m+a+b+1)m!\Gamma(m+a+b+1)} & \textit{for} \quad m_1 = m_2 = m \\ 0 & \textit{for} \quad m_1 \neq m_2 \end{cases}
\end{aligned}
\tag{9}
$$

where the generalized shifted weight is

$$
\omega_n^{(a,b,L)}(x) = \left(1-\left[\frac{2^px}{L}-2n+1\right]\right)^a\left(1+\left[\frac{2^px}{L}-2n+1\right]\right)^b \tag{10}
$$

3. Proposed Jacobi Wavelet Operational Matrix of Fractional Integration

3.1. Jacobi Wavelet

Bhrawy et. al. has derived the operational matrix of fractional integration for shifted Jacobi Polynomial $P_m^{a,b,L}(x)$ defined throughout the whole interval $(0, L)$. In this paper, we define

Jacobi wavelet based on the generalized shifted Jacobi Polynomial $\tilde{P}_{n,m}^{(a,b,L)}(x)$ and then derive its operational matrix of fractional integration $\psi_{n,m}^{(a,b,L}(x)$.

The Jacobi wavelet which form an orthonormal basis for space $L^2[0,1]$ can be defined as follows:

Theorem 3.1. *A Jacobi wavelet defined on the interval* $[0,1]$ *with compact support* $[\frac{(n-1)L}{2^{p-1}}, \frac{nL}{2^{p-1}}]$ *is given by:*

$$\psi_{n,m}^{(a,b,L)}(x) = \begin{cases} \frac{1}{\sqrt{h_m^{(a,b,L)}}} \sum_{r=0}^{m} \left(\sum_{k=r}^{m} S_{k,k-r}^{a,b,L,n,m,p} \right) x^r & , \quad \frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}} \\ 0 & , \quad otherwise \end{cases} \tag{11}$$

where

$$S_{k,k-r}^{a,b,L,n,m,p} = (-1)^{m-r} \frac{2^{(r)(p-1)} \Gamma(m+b+1)\Gamma(m+k+a+b+1)(n-1)^{k-r}}{\Gamma(k+b+1)\Gamma(m+a+b+1)(m-k)!(r)!(k-r)!L^r} \tag{12}$$

Proof. From the orthogonality of generalized shifted Jacobi Polynomial:

$$\begin{gathered} \int_{\frac{(n-1)L}{2^{p-1}}}^{\frac{nL}{2^{p-1}}} \tilde{P}_{n,m}^{(a,b,L)}(x) \tilde{P}_{n,m}^{(a,b,L)}(x) \omega_n^{(a,b,L)}(x) dx = h_m^{(a,b,L)} \\ \int_{\frac{(n-1)L}{2^{p-1}}}^{\frac{nL}{2^{p-1}}} \frac{1}{h_m^{(a,b,L)}} \tilde{P}_{n,m}^{(a,b,L)}(x) \tilde{P}_{n,m}^{(a,b,L)}(x) \omega_n^{(a,b,L)}(x) dx = 1 \\ \int_{\frac{(n-1)L}{2^{p-1}}}^{\frac{nL}{2^{p-1}}} \frac{1}{\sqrt{h_m^{(a,b,L)}}} \tilde{P}_{n,m}^{(a,b,L)}(x) \frac{1}{\sqrt{h_m^{(a,b,L)}}} \tilde{P}_{n,m}^{(a,b,L)}(x) \omega_n^{(a,b,L)}(x) dx = 1 \end{gathered} \tag{13}$$

Set

$$\psi_{n,m}^{(a,b,L)}(x) = \frac{1}{\sqrt{h_m^{(a,b,L)}}} \tilde{P}_{n,m}^{(a,b,L)}(x) \tag{14}$$

where $\tilde{P}_{n,m}^{(a,b,L)}(x)$ is obtained from Theorem 2.1. □

3.2. Jacobi Wavelet Operational Matrix of Fractional Integration

The operational matrix of fractional integration of degree $\alpha \geq 0$ for a Jacobi wavelet vector $\Psi_{2^{p-1}M}^{(a,b,L)}(x)$ (order $2^{p-1}M \times 1$) is defined as the matrix $P_{2^{p-1}M}^{\alpha}$ (order $2^{p-1}M \times 2^{p-1}M$) that approximates the Riemann-Liouville fractional integral operator I^{α}, i.e.

$$I^{\alpha} \Psi_{2^{p-1}M}^{(a,b,L)}(x) \simeq P_{2^{p-1}M}^{\alpha} * \Psi_{2^{p-1}M}^{(a,b,L)}(x) \tag{15}$$

where the Jacobi wavelet vector is a column matrix, i.e.

$$\Psi_{2^{p-1}M}^{(a,b,L)}(x) = \begin{bmatrix} \psi_{1,0}^{(a,b,L)}(x) \\ \psi_{1,1}^{(a,b,L)}(x) \\ \vdots \\ \psi_{2^{p-1},M-1}^{(a,b,L)}(x) \end{bmatrix} \tag{16}$$

First, we calculate the fractional integral of a single Jacobi wavelet $I^{\alpha}\psi_{n,m}^{(a,b,L)}(x)$ having compact support $\frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}}$ as follows:

$$I^{\alpha}\psi_{n,m}^{(a,b,L)}(x) = \begin{cases} 0 & , \quad t < \frac{(n-1)L}{2^{p-1}} \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{(n-1)L}{2^{p-1}}}^{x} (x-t)^{\alpha-1} \psi_{n,m}^{(a,b,L)}(t)dt & , \quad \frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}} \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{(n-1)L}{2^{p-1}}}^{\frac{nL}{2^{p-1}}} (x-t)^{\alpha-1} \psi_{n,m}^{(a,b,L)}(t)dt & , \quad x > \frac{nL}{2^{p-1}} \end{cases}$$

$$= \begin{cases} 0 & , \quad t < \frac{(n-1)L}{2^{p-1}} \\ \frac{1}{\sqrt{h_m^{(a,b,L)}}} \sum_{r=0}^{m} \left(\sum_{k=r}^{m} S_{k,k-r}^{a,b,L,n,m,p} \right) \frac{1}{\Gamma(\alpha)} \int_{\frac{(n-1)L}{2^{p-1}}}^{x} (x-t)^{\alpha-1} t^r dt & , \quad \frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}} \\ \frac{1}{\sqrt{h_m^{(a,b,L)}}} \sum_{r=0}^{m} \left(\sum_{k=r}^{m} S_{k,k-r}^{a,b,L,n,m,p} \right) \frac{1}{\Gamma(\alpha)} \int_{\frac{(n-1)L}{2^{p-1}}}^{\frac{nL}{2^{p-1}}} (x-t)^{\alpha-1} t^r dt & , \quad x > \frac{nL}{2^{p-1}} \end{cases}$$

$$= \begin{cases} 0 & , \quad t < \frac{(n-1)L}{2^{p-1}} \\ \sum_{r=0}^{m} \left(\sum_{k=r}^{m} \frac{S_{k,k-r}^{a,b,L,n,m,p}}{\sqrt{h_m^{(a,b,L)}}} \frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)} \right) \left(x^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \right) & , \quad \frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}} \\ \sum_{r=0}^{m} \left(\sum_{k=r}^{m} \frac{S_{k,k-r}^{a,b,L,n,m,p}}{\sqrt{h_m^{(a,b,L)}}} \frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)} \right) \left(\left(\frac{nL}{2^{p-1}} \right)^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \right) & , \quad x > \frac{nL}{2^{p-1}} \end{cases}$$

$$= \begin{cases} 0 & , \quad t < \frac{(n-1)L}{2^{p-1}} \\ \sum_{r=0}^{m} v_r^{m,\alpha} \left(x^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \right) & , \quad \frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}} \\ \sum_{r=0}^{m} v_r^{m,\alpha} \left(\left(\frac{nL}{2^{p-1}} \right)^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \right) & , \quad x > \frac{nL}{2^{p-1}} \end{cases}$$

$$= \begin{cases} 0 & , \quad t < \frac{(n-1)L}{2^{p-1}} \\ [\sum_{k=r}^{m} v_r^{m,\alpha}]^T \left[x^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \right] & , \quad \frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}} \\ [\sum_{k=r}^{m} v_r^{m,\alpha}]^T \left[\left(\frac{nL}{2^{p-1}} \right)^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \right] & , \quad x > \frac{nL}{2^{p-1}} \end{cases} \tag{17}$$

where

$$v_r^{m,\alpha} = \left(\sum_{k=r}^{m} \frac{S_{k,k-r}^{a,b,L,n,m,p}}{\sqrt{h_m^{(a,b,L)}}} \frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)} \right)$$

$$V = \left[\sum_{k=r}^{m} v_{k,k-r} \right]^T = \left[\begin{array}{cccc} \sum_{k=0}^{m} v_{k,k} & \sum_{k=1}^{m} v_{k,k-1} & \cdots & \sum_{k=m}^{m} v_{k,k-m} \end{array} \right].$$

$$\mathbf{X}_{\alpha} = \begin{bmatrix} x^{\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{\alpha} \\ x^{1+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{1+\alpha} \\ . \\ x^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \\ . \\ x^{m+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{m+\alpha} \end{bmatrix}, \mathbf{N}_{\alpha} = \begin{bmatrix} \left(\frac{nL}{2^{p-1}} \right)^{\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{\alpha} \\ \left(\frac{nL}{2^{p-1}} \right)^{1+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{1+\alpha} \\ . \\ \left(\frac{nL}{2^{p-1}} \right)^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{r+\alpha} \\ . \\ \left(\frac{nL}{2^{p-1}} \right)^{m+\alpha} - \left(\frac{(n-1)L}{2^{p-1}} \right)^{m+\alpha} \end{bmatrix} \tag{18}$$

(1) For $\frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}}$,

$$
\begin{aligned}
I^{\alpha}\psi_{n,m}^{(a,b,L)}(x) &= \sum_{r=0}^{m}\sum_{k=r}^{m} v_{k,k-r}\left(x^{r+\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha}\right) \\
&= \left[\ \sum_{k=0}^{m} v_{k,k} \quad \sum_{k=1}^{m} v_{k,k-1} \quad \cdot \quad \cdot \quad \sum_{k=m}^{m} v_{k,k-m}\ \right]\begin{bmatrix} x^{\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{\alpha} \\ x^{1+\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{1+\alpha} \\ \vdots \\ x^{m+\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{m+\alpha}\end{bmatrix} \\
&= \mathbf{V}\mathbf{X}_{\alpha}, r=0,1,2,...,m
\end{aligned} \tag{19}
$$

(2) For $x > \frac{nL}{2^{p-1}}$,

$$
\begin{aligned}
I^{\alpha}\psi_{n,m}^{(a,b,L)}(x) &= \sum_{r=0}^{m}\sum_{k=r}^{m} v_{k,k-r}\left(\left(\frac{nL}{2^{p-1}}\right)^{r+\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha}\right) \\
&= \left[\ \sum_{k=0}^{m} v_{k,k} \quad \sum_{k=1}^{m} v_{k,k-1} \quad \cdots \quad \sum_{k=m}^{m} v_{k,k-m}\ \right]\begin{bmatrix} \left(\frac{nL}{2^{p-1}}\right)^{\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{\alpha} \\ \left(\frac{nL}{2^{p-1}}\right)^{1+\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{1+\alpha} \\ \vdots \\ \left(\frac{nL}{2^{p-1}}\right)^{m+\alpha}-\left(\frac{(n-1)L}{2^{p-1}}\right)^{m+\alpha}\end{bmatrix} \\
&= \mathbf{V}\mathbf{N}_{\alpha}, r=0,1,2,...,m
\end{aligned} \tag{20}
$$

Therefore, the fractional integral of a single Jacobi wavelet is simplified into the following matrix notation:

$$
\begin{aligned}
I^{\alpha}\psi_{n,m}^{(a,b,L)}(x) &= \mathbf{V}\mathbf{X}_{\alpha}, \frac{(n-1)L}{2^{p-1}} \le x \le \frac{nL}{2^{p-1}} \\
&= \mathbf{V}\mathbf{N}_{\alpha}, x \ge \frac{nL}{2^{p-1}}
\end{aligned} \tag{21}
$$

Now, we have to approximate the Ξ_{α} and $\aleph_{\alpha}$ in terms of a set of Jacobi wavelets. For future use and simplicity purpose, we first calculate the following expression:

$$
\begin{aligned}
&\int_{\frac{(n-1)L}{2^{p-1}}}^{\frac{nL}{2^{p-1}}} x^{r+\alpha+s}\left(2n-\frac{2^{p}x}{L}\right)^{a}\left(\frac{2^{p}x}{L}-2n+2\right)^{b} \\
&= \left(\frac{L}{2^{p-1}}\right)^{r+\alpha+s+1} n^{r+s+\alpha}2^{a+b}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}F\left(-r-\alpha-s, a+1, a+b+2; \frac{1}{n}\right)
\end{aligned} \tag{22}
$$

where $F(a,b,c;z)$ is the hypergeometric function.

To calculate the Jacobi Wavelet Operational Matrix of Fractional Integration (Jacobi WOMFI) $P^{\alpha}_{2^{p-1}M}$ for the a Jacobi wavelet vector $\Psi(x)$, i.e. $I^{\alpha}\Psi(x) = P^{\alpha}_{2^{p-1}M}\Psi(x)$, first we determine the r-th row $\tilde{\mathbf{p}}^{\alpha}_{\mathbf{r}}$ of the operational matrix $P^{\alpha}_{2^{p-1}M}$, i.e.

$$
\begin{aligned}
P^{\alpha}_{2^{p-1}M} &= \begin{bmatrix} \tilde{\mathbf{p}}^{\alpha}_{\mathbf{1}} \\ \vdots \\ \tilde{\mathbf{p}}^{\alpha}_{\mathbf{2^{p-1}M}} \end{bmatrix}. \\
I^{\alpha}\psi^{(a,b,L)}_{n,m}(x) &= \tilde{\mathbf{p}}^{\alpha}_{\mathbf{r}}\Psi(x) \\
j &= (n-1)M+m+1
\end{aligned} \tag{23}
$$

where the wavelet vector $\Psi(x)$ the set of orthonormal Jacobi wavelets.

Then, to find $\tilde{\mathbf{p}}^{\alpha}_{\mathbf{r}}$, we calculate the following inner product:

$$
\begin{aligned}
\left\langle I^{\alpha}\psi^{(a,b,L)}_{i}(x), \psi^{(a,b,L)}_{j}(x)\right\rangle_{\omega_j(x)} &= \left\langle I^{\alpha}\psi^{(a,b,L)}_{n,m}(x), \psi^{(a,b,L)}_{u,w}(x)\right\rangle_{\omega_u(x)} \\
&= \begin{cases} \mathbf{0} & \text{if} \quad n > u \\ \mathbf{\Theta} & \text{if} \quad n = u \\ \aleph & \text{if} \quad n < u \end{cases} \\
i = (n-1)M+m+1, j &= (u-1)M+w+1
\end{aligned} \tag{24}
$$

Then the column matrix $\tilde{\mathbf{p}}^{\alpha}_{\mathbf{r}}$ is:

$$
\tilde{\mathbf{p}}^{\alpha}_{\mathbf{j}} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{\Theta} & \aleph_1 & \cdots & \aleph_{2^{p-1}-j} \end{bmatrix}. \tag{25}
$$

Thus,

$$
P^{\alpha}_{2^{p-1}M} = \begin{bmatrix} \mathbf{\Theta} & \aleph_1 & \aleph_2 & \cdots & \cdots & \aleph_{2^{p-1}-1} \\ \mathbf{0} & \mathbf{\Theta} & \aleph_1 & \cdots & \cdots & \aleph_{2^{p-1}-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Theta} & \aleph_1 & \cdots & \aleph_{2^{p-1}-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{\Theta} \end{bmatrix}. \tag{26}
$$

where

$$
\begin{aligned}
\mathbf{\Theta} &= \mathbf{V}\Upsilon \\
&= \begin{bmatrix} \sum_{k=0}^{m} v_{k,k} & \sum_{k=1}^{m} v_{k,k-1} & \cdots & \sum_{k=m}^{m} v_{k,k-m} \end{bmatrix} \cdot \begin{bmatrix} \zeta_{0,0} & \zeta_{0,1} & \cdots & \zeta_{0,2^{p-1}M} \\ \zeta_{1,0} & \zeta_{1,1} & \cdots & \zeta_{1,2^{p-1}M} \\ \vdots & \ddots & \ddots & \vdots \\ \zeta_{m,0} & \zeta_{m,1} & \cdots & \zeta_{m,2^{p-1}M} \end{bmatrix}. \\
&= \begin{bmatrix} \sum_{r=0}^{m}\sum_{k=r}^{m} v_{k,k-r}\zeta_{r,0} & \sum_{r=0}^{m}\sum_{k=1}^{m} v_{k,k-r}\zeta_{r,1} & \cdots & \sum_{r=0}^{m}\sum_{k=r}^{m} v_{k,k-r}\zeta_{r,2^{p-1}M} \end{bmatrix}. \\
\aleph &= \mathbf{V}\mathbf{\Xi} \\
&= \begin{bmatrix} \sum_{k=0}^{m} v_{k,k} & \sum_{k=1}^{m} v_{k,k-1} & \cdots & \sum_{k=m}^{m} v_{k,k-m} \end{bmatrix} \cdot \begin{bmatrix} \eta_{0,0} & \eta_{0,1} & \cdots & \eta_{0,2^{p-1}M} \\ \eta_{1,0} & \eta_{1,1} & \cdots & \eta_{1,2^{p-1}M} \\ \vdots & \ddots & \ddots & \vdots \\ \eta_{m,0} & \eta_{m,1} & \cdots & \eta_{m,2^{p-1}M} \end{bmatrix}. \\
&= \begin{bmatrix} \sum_{r=0}^{m}\sum_{k=r}^{m} v_{k,k-r}\eta_{r,0} & \sum_{r=0}^{m}\sum_{k=1}^{m} v_{k,k-r}\eta_{r,1} & \cdots & \sum_{r=0}^{m}\sum_{k=r}^{m} v_{k,k-r}\eta_{r,2^{p-1}M} \end{bmatrix}.
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\mathbf{X}_\alpha &= \Upsilon\Psi(x) \\
&= \begin{bmatrix} \zeta_{0,0} & \zeta_{0,1} & \cdots & \zeta_{0,2^{p-1}M} \\ \zeta_{1,0} & \zeta_{1,1} & \cdots & \zeta_{1,2^{p-1}M} \\ \vdots & \ddots & \ddots & \vdots \\ \zeta_{m,0} & \zeta_{m,1} & \cdots & \zeta_{m,2^{p-1}M} \end{bmatrix} \begin{bmatrix} \psi_{1,0}^{(a,b,L)}(x) \\ \psi_{1,1}^{(a,b,L)}(x) \\ \vdots \\ \psi_{2^{p-1},M-1}^{(a,b,L)}(x) \end{bmatrix}. \\
\mathbf{N}_\alpha &= \Xi\Psi(x) \\
&= \begin{bmatrix} \eta_{0,0} & \eta_{0,1} & \cdots & \eta_{0,2^{p-1}M} \\ \eta_{1,0} & \eta_{1,1} & \cdots & \eta_{1,2^{p-1}M} \\ \vdots & \ddots & \ddots & \vdots \\ \eta_{m,0} & \eta_{m,1} & \ddots & \eta_{m,2^{p-1}M} \end{bmatrix} \begin{bmatrix} \psi_{1,0}^{(a,b,L)}(x) \\ \psi_{1,1}^{(a,b,L)}(x) \\ \vdots \\ \psi_{2^{p-1},M-1}^{(a,b,L)}(x) \end{bmatrix}.
\end{aligned} \tag{28}
$$

For $\frac{(n-1)L}{2^{p-1}} \leq x \leq \frac{nL}{2^{p-1}}$,

$$
\begin{aligned}
\zeta_{r,j} &= \left\langle x^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha}, \psi_j^{(a,b,L)}(x) \right\rangle_{\omega_u}, j = (u-1)M + w + 1 \\
&= \left\langle x^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha}, \psi_{u,w}^{(a,b,L)}(x) \right\rangle_{\omega_u} \\
&= \left\langle x^{r+\alpha}, \psi_{u,w}^{(a,b,L)}(x) \right\rangle_{\omega_u} - \left\langle \left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha}, \psi_{u,w}^{(a,b,L)}(x) \right\rangle_{\omega_u} \\
&= \sum_{s=0}^{w} \sum_{v=s}^{w} S_{v,v-s}^{a,b,L,u,w,p} H_s^{r,a,b,L,p,u,\alpha} \\
H_s^{r,a,b,L,p,n,u,\alpha} &= \\
&\left(\frac{L}{2^{p-1}}\right)^{r+\alpha+s+1} \frac{u^{r+s+\alpha} 2^{a+b} \Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} F\left(-r-\alpha-s, a+1, a+b+2; \frac{1}{u}\right) \\
&- \left(\frac{L}{2^{p-1}}\right)^{r+\alpha+s+1} \frac{(n-1)^{r+\alpha} u^s 2^{a+b} \Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} F\left(-s, a+1, a+b+2; \frac{1}{u}\right)
\end{aligned} \tag{29}
$$

For $x \geq \frac{nL}{2^{p-1}}$,

$$\begin{aligned}
\eta_{r,j} &= \left\langle \left(\frac{nL}{2^{p-1}}\right)^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha}, \psi_j^{(a,b,L)}(x) \right\rangle_{\omega_u}, j = (u-1)M + w + 1 \\
&= \left\langle \left(\frac{nL}{2^{p-1}}\right)^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha}, \psi_{u,w}^{(a,b,L)}(x) \right\rangle_{\omega_u} \\
&= \sum_{s=0}^{w} \sum_{v=s}^{w} S_{v,v-s}^{a,b,L,u,w,p} E_s^{r,a,b,L,p,u,\alpha} \\
E_s^{r,a,b,L,p,n,u,\alpha} &= \left(\left(\frac{nL}{2^{p-1}}\right)^{r+\alpha} - \left(\frac{(n-1)L}{2^{p-1}}\right)^{r+\alpha} \right) \left(\frac{L}{2^{p-1}}\right)^{s+1} u^s 2^{a+b} \\
&\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} F\left(-s, a+1, a+b+2; \frac{1}{u}\right) \\
&= \left((n)^{r+\alpha} - (n-1)^{r+\alpha}\right) \left(\frac{L}{2^{p-1}}\right)^{r+\alpha+s+1} u^s 2^{a+b} \\
&\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} F\left(-s, a+1, a+b+2; \frac{1}{u}\right)
\end{aligned} \tag{30}$$

4. Numerical Solution of Linear Fredholm 2nd Kind Fractional Integro-Differential Equation based on by Jacobi WOMFI

Given the fractional integro-differential equation of the following Fredholm 2nd-kind type:

$$\begin{aligned}
D^\alpha f(x) &= g(x) + \lambda \int_0^1 K(x,t) f(t) dt, \\
f^{(i)}(0) &= d_i, i = 0, 1, \cdots, \lfloor \alpha \rfloor
\end{aligned} \tag{31}$$

where $f(x)$ is the unknown function to be found, $g(x)$ is the forced term known a priori, $\alpha \in R^+$ positive real number,and $\lfloor \alpha \rfloor = n, n-1 < \alpha \leq n$ is the smallest integer greater or equal to α.

We assume the solution $f(x)$ to be approximated by Jacobi wavelets, i.e.

$$\begin{aligned}
f(x) &= \sum_{j=0}^{2^{p-1}M} c_j \psi_j^{(a,b,L)}(x), \\
\psi_j^{(a,b,L)}(x) &= \psi_{n,m}^{(a,b,L)}(x) \\
j &= (n-1)M + m + 1
\end{aligned} \tag{32}$$

To solve the above kind of fractional integro-differential equation by Jacobi wavelet operational matrix of fractional integration P^α, we start with:

$$D^\alpha f(x) \simeq C^T \Psi(x) \quad = \begin{bmatrix} c_0 & c_1 & \dots & c_j & \dots & c_{2^{p-1}M} \end{bmatrix} \begin{bmatrix} \psi_{L,0}^{a,b}(x) \\ \psi_{L,1}^{a,b}(x) \\ \dots \\ \psi_{L,2^{p-1}M}^{a,b}(x) \end{bmatrix} \tag{33}$$

Applying Riemann-Liouville fractional integral I^α to the equation:

$$\begin{aligned}
&I^\alpha D^\alpha f(x) \simeq C^T I^\alpha \Psi(x)\\
&f(x) - \sum_{i=0}^{\lfloor\alpha\rfloor} f^{(i)}(0)\frac{x^i}{i!} \simeq C^T P^\alpha \Psi(x)\\
&f(x) \simeq C^T P^\alpha \Psi(x) + \sum_{i=0}^{\lfloor\alpha\rfloor} f^{(i)}(0)\frac{x^i}{i!}\\
&f(x) \simeq C^T P^\alpha \Psi(x) + \left[\sum_{i=0}^{\lfloor\alpha\rfloor} f^{(i)}(0)\frac{\left\langle x^i, \psi_{n,m}^{(a,b,L)}(x)\right\rangle}{i!}\right]\Psi(x)\\
&f(x) \simeq C^T P^\alpha \Psi(x) + [\epsilon_{n,m}]^T \Psi(x)\\
&f(x) \simeq C^T P^\alpha \Psi(x) + \mathbf{E}^T \Psi(x)
\end{aligned} \tag{34}$$

$$\begin{aligned}
\epsilon_{n,m} = &\sum_{i=0}^{\lfloor\alpha\rfloor}\sum_{r=0}^{m}\sum_{k=r}^{m} \frac{f^{(i)}(0)}{i!} S_{k,k-r}^{a,b,L,n,m,p}\left(\frac{L}{2^{p-1}}\right)^{i+r+1} n^{i+r}2^{a+b}\\
&\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} F\left(-i-r, a+1, a+b+2; \frac{1}{n}\right)
\end{aligned} \tag{35}$$

By approximating the following terms (i.e. fractional derivative $D^\alpha f(x)$, solution function $f(x)$, the kernel $K(x,t)$ and the forced term $g(x)$ in the fractional-integro-differential equations, using Jacobi wavelet vector $\Psi(x)$:

$$\begin{aligned}
&(1) D^\alpha f(x) \simeq C^T \Psi(x),\\
&(2) f(t) \simeq C^T P^\alpha \Psi(t) + \mathbf{E}^T \Psi(t),\\
&(3) K(x,t) \simeq \frac{1}{2^{p-1}} \Psi^T(x) \tilde{K} \Psi(t)\\
&(4) g(x) \simeq G^T \Psi(x)
\end{aligned} \tag{36}$$

The kernel matrix K is calculated by:

$$\begin{aligned}
K &= [k_{i,j}]\\
k_{i,j} &= \langle \psi_i(x), \langle K(x,t), \psi_j(t)\rangle\rangle
\end{aligned} \tag{37}$$

We reduce the original fractional integro-differential equations:

$$D^\alpha f(x) = g(x) + \lambda \int_0^1 K(x,t) f(t) dt \tag{38}$$

into

$$\begin{aligned}
&\Psi^T(x)C = \Psi^T(x)G + \lambda \int_0^1 \Psi^T(x)\tilde{K}\Psi(t)\left(\Psi^T(t)(P^\alpha)^T C + \Psi^T(t)\mathbf{E}\right)\\
&\Psi^T(x)C = \Psi^T(x)G + \lambda \Psi^T(x)\tilde{K}\left(\int_0^1 \Psi(t)\Psi^T(t)dt\right)(P^\alpha)^T C\\
&+ \lambda \Psi^T(x)\tilde{K}\left(\int_0^1 \Psi(t)\Psi^T(t)\right)\mathbf{E}\\
&\Psi^T(x)C = \Psi^T(x)G + \lambda \Psi^T(x)\tilde{K} I\left((P^\alpha)^T C + \mathbf{E}\right)\\
&\Psi^T(x)(C - G - \lambda \tilde{K} I\left((P^\alpha)^T C + \mathbf{E}\right) = 0
\end{aligned} \tag{39}$$

Thus, we obtain a system of algebraic equations

$$\Psi^T(x)(C - G - \lambda\tilde{K}I\left((P^\alpha)^T C + \mathbf{E}\right) = 0 \tag{40}$$

where $I = \int_0^1 \Psi(t)\Psi^T(t)dt$.

We apply the spectral-tau method to solve for the coefficients vector $C = [c_0, c_1, c_2, \cdots, c_{2^{p-1}M}]^T$.

Define the residue $R_N(x)$:

$$R_N(x) = C - G - \lambda\tilde{K}I\left((P^\alpha)^T C + E\right) \tag{41}$$

As in a typical tau method, we generate $2^{p-1}M - M + 1$ linear algebraic equations by applying the inner product with respect to the Jacobi wavelets, i.e.

$$\begin{aligned}\left\langle R_N, \psi_j^{(a,b,L}(x)\right\rangle_{\omega_n(x)} &= \int_0^1 R_N(x)\psi_j^{(a,b,L}(x)\omega_n(x)dx \\ &= \int_0^1 R_N(x)\psi_{n,m}^{(a,b,L}(x)\omega_n(x)dx\end{aligned} \tag{42}$$

where $j = (n-1)M + m + 1 = 0, 1, ..., M - 1$.

Thus, we will reduce the original problem of solving the fractional integro-differential equation into a reduced problem of solving this system of algebraic equations for the wavelet coefficients $C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{2^{p-1}M} \end{bmatrix}$. of the fractional derivative $D^\alpha f(x) \simeq C^T\Psi(x)$ of solution $f(x)$. Therefore, the final numerical approximate solution for $f(x)$ can be calculated as:

$$f(t) \simeq C^T P^\alpha \Psi(t) + \mathbf{E}^T\Psi(t), \tag{43}$$

Below , we solve some examples of linear Fredholm 2nd-kind fractional integro-differential equations and show the results by plotting graph of approximate solution and exact solution:

4.1. Example 1

$$D^1 f(x) = 1 - \frac{1}{3}x + \int_0^1 xtf(t)dt \tag{44}$$

where $y(0) = 0$. The exact solution is $f(x) = x$.

By employing the Jacobi WOMFI method with $k = 1, M = 6$(i.e. wavelet bases $[\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{1,4}(x), \psi_{1,5}(x)]$, $a = -\frac{1}{2}, b = -\frac{1}{2}$, we obtain the following the coefficient C:

$$C = [c_{1,0} = 1.253314138, c_{1,1} = 0, c_{1,2} = 0, c_{1,3} = 0, c_{1,4} = 0, c_{1,5} = 0] \tag{45}$$

Hence, the approximate solution for Example 1 is $\tilde{f}(x) = x + 1.128379167 * 10^{-10}x^2$.

Figure 1 shows the graph of both exact solution and approximate solution of Example 1.

4.2. Example 2

$$D^{\frac{3}{2}} f(x) = 4\sqrt{\frac{x}{\pi}} + \frac{135\sqrt{\pi}}{34}\int_0^1 xtf(t)dt \tag{46}$$

where $y(0) = 0$. The exact solution is $f(x) = x^2 + x^{\frac{5}{2}}$.

By employing the Jacobi WOMFI method with $k = 1, M = 6$(i.e. wavelet bases $[\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{1,4}(x), \psi_{1,5}(x)]$, $a = -\frac{1}{2}, b = -\frac{1}{2}$, we obtain the following the coefficient C:

$$C = [c_{1,0} = 3.888544245, c_{1,1} = 2.325202823, c_{1,2} = -0.1697652726, \\ c_{1,3} = 0.07275654540, c_{1,4} = -0.04042030300, c_{1,5} = 0.1125143234] \tag{47}$$

Hence, the approximate solution for Example 2 is $\tilde{f}(x) = -0.02905620289x + 1.673548323x^2 - 0.3164496500x^3 + 1.327035907x^4 - 0.6521067175x^5$.

Figure 2 shows the graph of both exact solution and approximate solution of Example 2.

4.3. Example 3

$$D^{\frac{3}{2}} f(x) = 4\sqrt{\frac{x}{\pi}} + \frac{160}{9\sqrt{\pi}} \int_0^1 x^{\frac{3}{2}} t f(t) dt \tag{48}$$

where $y(0) = 0$. The exact solution is $f(x) = x^3 + x^2$.

By employing the Jacobi WOMFI method with $k = 1, M = 6$(i.e. wavelet bases $[\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{1,4}(x), \psi_{1,5}(x)]$, $a = -\frac{1}{2}, b = -\frac{1}{2}$, we obtain the following the coefficient C:

$$C = [c_{1,0} = 4.204123174, c_{1,1} = 2.888255716, c_{1,2} = 0.1215817778, \\ c_{1,3} = 0.04038465087, c_{1,4} = -0.0315916043, c_{1,5} = 0.0749131814] \tag{49}$$

Hence, the approximate solution for Example 3 is $\tilde{f}(x) = -0.01649028019x + 1.255248099x^2 + 0.09134127195x^3 + 1.178779239x^4 - 0.5078540653x^5$.

Figure 3 shows the graph of both exact solution and approximate solution of Example 3.

4.4. Example 4

$$D^{\frac{3}{2}} f(x) = 4\sqrt{\frac{x}{\pi}} + \frac{105}{22} \int_0^1 \sqrt{xt} f(t) dt \tag{50}$$

where $y(0) = 1, y^{'}(0) = -1$. The exact solution is $f(x) = 2x^2 - x + 1$.

By employing the Jacobi WOMFI method with $k = 1, M = 6$(i.e. wavelet bases $[\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{1,4}(x), \psi_{1,5}(x)]$, $a = -\frac{1}{2}, b = -\frac{1}{2}$ (i.e. Chebyshev 1st kind), we obtain the following the coefficient C:

$$C = [c_{1,0} = 3.593253214, c_{1,1} = 1.693875809, c_{1,2} = -0.3387751620, \\ c_{1,3} = 0.1451893551, c_{1,4} = -0.0221655380, c_{1,5} = 0.06946570087] \tag{51}$$

Hence, the approximate solution for Example 4 is $\tilde{f}(x) = 1 - x + 1.902837228x^2 + 0.183184821x^3 - 0.055763722x^4 - 0.0391419693x^5$.

Figure 4 shows the graph of both exact solution and approximate solution of Example 4.

4.5. Conclusion

With the introduction of Jacobi Wavelet Operational Matrix of Fractional Integration (Jacobi WOMFI), we would be able to provide a generalization of wavelet operational matrix of fractional integration which includes Legendre wavelets, Chebyshev wavelets and Gegenbauer wavelets. On solving Linear Freholm 2nd kind fractional integro-differential equations, our examples showed that the proposed Jacobi WOMFI method provides a very accurate estimate of the solution. Also,with this generalization, there may exist optimal parameters (a, b) of a Jacobi Wavelets that gives the optimal numerical solution depending on the type of fractional integro-differential equations. Based on the optimal values (a, b) and application to several classes of fractional integro-differential equations, we may be able to come out with a more accurate, faster convergent numerical solution by wavelet operational matrix of fractional integration. More general properties of the wavelet operational matrix related to the class of kernel and fractional integro-differential equations are expected to be derived from this generalization.

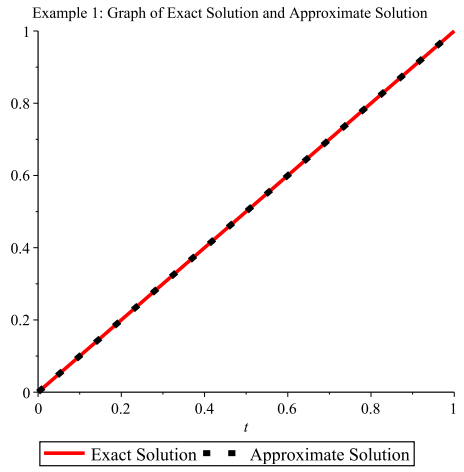

Figure 1. Example 1: Graph of exact solution and approximate solution

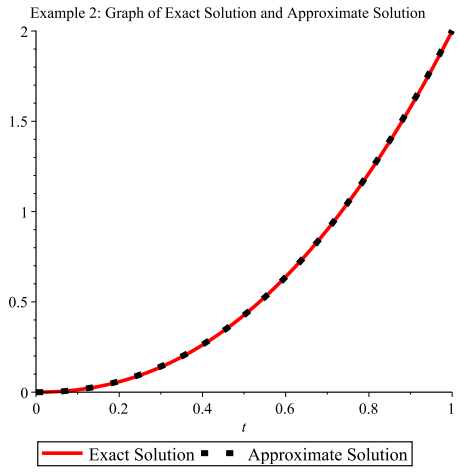

Figure 2. Example 2: Graph of exact solution and approximate solution

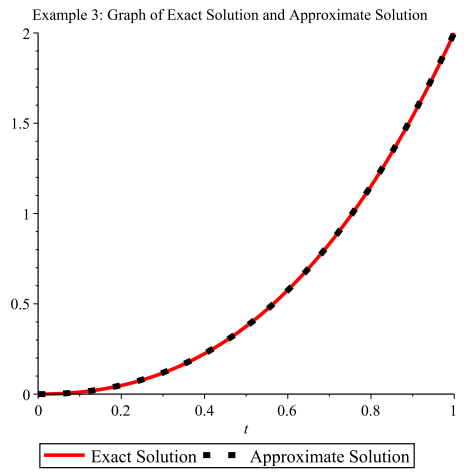

Figure 3. Example 3: Graph of exact solution and approximate solution

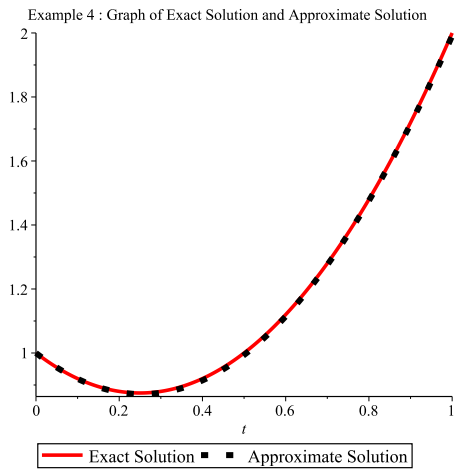

Figure 4. Example 4: Graph of exact solution and approximate solution

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