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Freak waves at the surface of deep water

A.I. Dyachenko^{1,2}, D.I. Kachulin¹ and V.E. Zakharov^{1,2,3,4}

¹Novosibirsk State University, Novosibirsk, Russia

²Landau Institute for Theoretical Physics, Moscow, Russia

³Physical Institute of RAS, Moscow, Russia

⁴Department of Mathematics, University of Arizona, Tucson, USA

E-mail: d.kachulin@gmail.com

Abstract. In the paper [1] authors applied canonical transformation to water wave equation not only to remove cubic nonlinear terms but to simplify drastically fourth order terms in Hamiltonian. After the transformation well-known but cumbersome Zakharov equation is drastically simplified and can be written in X -space in compact way. This new equation is very suitable for analytic study as well as for numerical simulation. At the same time one of the important issues concerning this system is the question of its integrability. The first part of the work is devoted to numerical and analytical study of the integrability of the equation obtained in [1]. In the second part we present generalization of the improved Zakharov equation for the "almost" 2-D water waves at the surface of deep water. When considering waves slightly inhomogeneous in transverse direction, one can think in the spirit of Kadomtsev-Petviashvili equation for Korteweg-de-Vries equation taking into account weak transverse diffraction. Equation can be written instead of classical variables $\eta(x, y, t)$ and $\psi(x, y, t)$ in terms of canonical normal variable $b(x, y, t)$. This equation is very suitable for robust numerical simulation. Due to specific structure of nonlinearity in the Hamiltonian the equation can be effectively solved on the computer. It was applied for simulation of sea surface waving including freak waves appearing.

1. Compact Zakharov equation

It is well known that the Hamiltonian describing the propagation of gravitational waves in deep water can be represented by an infinite series expansion. Hamiltonian variables here are the elevation of the free surface and the velocity potential at the surface. Analytical and numerical analysis are often limited to only a few terms of the expansion up to the fourth order as usual:

$$H = \frac{1}{2} \int g\eta^2 + \psi \hat{k} \psi dx - \frac{1}{2} \int \{(\hat{k}\psi)^2 - (\psi_x)^2\} \eta dx + \frac{1}{2} \int \{\psi_{xx} \eta^2 \hat{k} \psi + \psi \hat{k} (\eta \hat{k} (\eta \hat{k} \psi))\} dx \quad (1)$$

In the paper [1] authors applied canonical transformation to water wave equations not only to remove cubic nonlinear terms but to simplify drastically fourth order terms in Hamiltonian. This transformation explicitly uses the fact of vanishing exact four waves interaction for water gravity waves for 2D potential fluid. After the transformation well-known but cumbersome Zakharov equation is drastically simplified and can be written in X -space in compact way.

$$i \frac{\partial b}{\partial t} = \hat{\omega}_k b + \frac{i}{4} \hat{P}^+ \left[b^* \frac{\partial}{\partial x} (b'^2) - \frac{\partial}{\partial x} (b^{*'} \frac{\partial}{\partial x} b^2) \right] - \frac{1}{2} \hat{P}^+ \left[b \cdot \hat{k} (|b'|^2) - \frac{\partial}{\partial x} (b' \hat{k} (|b|^2)) \right], \quad (2)$$



Here $\hat{P}^{+2} = \hat{P}^+ = \frac{1}{2}(1 - i\hat{H})$ - projection operator to the upper half-plane. Hamiltonian in X-space is the following:

$$\mathcal{H} = \int b^* \hat{\omega}_k b dx + \frac{1}{2} \int \left| \frac{\partial b}{\partial x} \right|^2 \left[\frac{i}{2} \left(b \frac{\partial b^*}{\partial x} - b^* \frac{\partial b}{\partial x} \right) - \hat{k} |b|^2 \right] dx. \quad (3)$$

This equation is very suitable for analytic study as well as for numerical simulation. At the same time one of the important issues concerning this system is the question of its integrability. The next subsection is devoted to numerical and analytical study of the integrability of the equation obtained in [1].

1.1. Analysis of integrability of the compact Zakharov equation

At first we developed the spectral algorithm to simulate the equation. Numerical integration of the equation were carried out on the base of Runge-Kutta method 4th order accurate in time. Then we try to find breather solutions of the equation (2). Breather is the localized solution of the form:

$$b(x, t) = B(x - Vt)e^{i(k_0x - \omega_0t)}, \quad (4)$$

Here k_0 is the wavenumber of the carrier wave, V is the group velocity. In the Fourier space breather can be written as follow:

$$b_k(t) = e^{-i(\Omega + Vk)t} \phi_k, \quad (5)$$

where Ω is close to $\frac{\omega_{k_0}}{2}$. For ϕ_k the following equation is valid:

$$(\Omega + Vk - \omega_k) \phi_k = \int \tilde{T}_{kk_1}^{k_2k_3} \phi_{k_1}^* \phi_{k_2} \phi_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3. \quad (6)$$

Where $\tilde{T}_{kk_1}^{k_2k_3}$ four-wave interactions coefficient. Localized in space breather-type solutions with different group velocities and amplitudes were found by iterative Petviashvili method (n - is the number of iteration):

$$(\Omega + Vk - \omega_k) \phi_k^{n+1} = M^n \int \tilde{T}_{kk_1}^{k_2k_3} \phi_{k_1}^{*n} \phi_{k_2}^n \phi_{k_3}^n \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3, \quad (7)$$

Petviashvili coefficient M^n is the following:

$$M^n = \left[\frac{\langle \phi_k^n (\Omega + Vk - \omega_k) \phi_k^n \rangle}{\langle \phi_k^n \int \tilde{T}_{kk_1}^{k_2k_3} \phi_{k_1}^{*n} \phi_{k_2}^n \phi_{k_3}^n \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 \rangle} \right]^{\frac{3}{2}}. \quad (8)$$

In the limit of NLSE breather solution (4) is nothing but well-known NLSE soliton. Numerical simulations of collisions of such breathers were studied. If the Eq.(2) is integrable collisions of breathers would be pure elastic. However, numerical simulation shows the collisions are not elastic (Figs. 1 and 2). Figure 2 shows the zoomed Fig. 1. To confirm that, we study analytically amplitudes of six-wave interactions for this equation (see for details [2]). It was found that six-wave amplitude is not canceled for this equation. So, the kernel of six-wave element of scattering matrix is nonzero $\tilde{T}_{q_1q_2q_3}^{p_1p_2p_3} \neq 0$ on the resonant manifold:

$$p_1 + p_2 + p_3 = q_1 + q_2 + q_3$$

$$\omega_{p_1} + \omega_{p_2} + \omega_{p_3} = \omega_{q_1} + \omega_{q_2} + \omega_{q_3} \quad (9)$$

Thus, 1-D Zakharov equation is not integrable. However the question about integrability of fully nonlinear system is still unclear. Exact equation for deep gravitational waves has his own six wave term which would change total six wave coefficient.

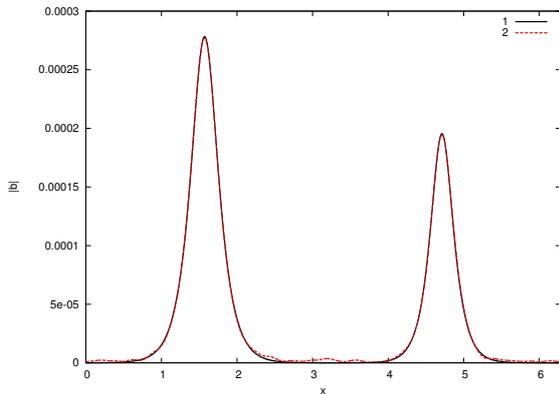


Figure 1. Modulus of $b(x,t)$ for two point of time. Curve 1 corresponds the initial statement ($t = 0$), curve 2 corresponds to the statement after 100 breather collisions ($t = 88000$).

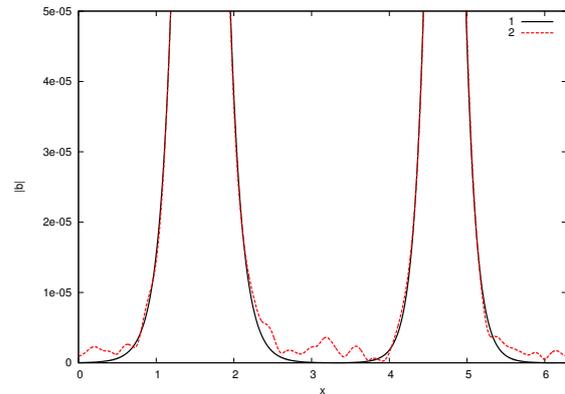


Figure 2. Modulus of $b(x,t)$ for two point of time. Curve 1 corresponds the initial statement ($t = 0$), curve 2 corresponds to the statement after 100 breather collisions ($t = 88000$).

2. Zakharov equation for the "almost" 2-D water waves at the surface of deep water

The equation (2) can be generalized for almost 2D waves. Waves that slightly inhomogeneous in the transverse direction can be considered in the spirit of the Kadomtsev-Petviashvili equation for the Korteweg-de-Vries equation: the frequency ω_k can be treated as 2D ω_{k_x, k_y} , while the four-wave interactions coefficient $\tilde{T}_{kk_1}^{k_2 k_3}$ not dependent on y and $b(x, y, t)$ is a function of x and y :

$$\mathcal{H} = \int b^* \hat{\omega}_{k_x, k_y} b dx dy + \frac{1}{4} \int |b'_x|^2 \left[\frac{i}{2} (bb'^* - b^* b'_x) - \hat{K}_x |b|^2 \right] dx dy. \quad (10)$$

Corresponding equation of motion is the following:

$$i \frac{\partial b}{\partial t} = \hat{\omega}_{k_x, k_y} b + \frac{i}{4} \hat{P}^+ \left[b^* \frac{\partial}{\partial x} (b'^2) - \frac{\partial}{\partial x} (b_x'^* \frac{\partial}{\partial x} b^2) \right] - \frac{1}{2} \hat{P}^+ \left[b \cdot \hat{k} (|b'|^2) - \frac{\partial}{\partial x} (b'_x \hat{k} (|b|^2)) \right]. \quad (11)$$

For Fourier harmonics Hamiltonian can be written as following:

$$H = \int \omega_{k_x, k_y} |b_{k_x, k_y}|^2 dk_x dk_y + \frac{1}{2} \int T_{k_x_1 k_x_2}^{k_x_3 k_x_4} b_{k_x_1, y_1}^* b_{k_x_2, y_2}^* b_{k_x_3, y_3} b_{k_x_4, y_4} \delta_{k_x_1 + k_x_2 - k_x_3 - k_x_4} \times \\ \times \delta_{k_y_1 + k_y_2 - k_y_3 - k_y_4} dk_{x_1} dk_{x_2} dk_{x_3} dk_{x_4} dk_{y_1} dk_{y_2} dk_{y_3} dk_{y_4} \quad (12)$$

Here

$$T_{k_x_2 k_x_3}^{k_x k_x_1} = \frac{\theta(k_x) \theta(k_{x_1}) \theta(k_{x_2}) \theta(k_{x_3})}{16\pi^2} [(k_x k_{x_1} (k_x + k_{x_1}) + k_{x_2} k_{x_3} (k_{x_2} + k_{x_3})) - (k_x k_{x_2} |k_x - k_{x_2}| + \\ + k_x k_{x_3} |k_x - k_{x_3}| + k_{x_1} k_{x_2} |k_{x_1} - k_{x_2}| + k_{x_1} k_{x_3} |k_{x_1} - k_{x_3}|)]. \quad (13)$$

θ - functions in (13) is the manifestation of waves moving in the same direction. Corresponding evolution equation is the following:

$$i \frac{\partial b_{k_x, k_y}}{\partial t} = \omega_{k_x, k_y} b_{k_x, k_y} + \int T_{k_x_1 k_x_2}^{k_x_3 k_x_4} b_{k_x_1, y_1}^* b_{k_x_2, y_2}^* b_{k_x_3, y_3} b_{k_x_4, y_4} \delta_{k_x_1 + k_x_2 - k_x_3 - k_x_4} \times$$

$$\times \delta_{k_{y_1}+k_{y_2}-k_{y_3}-k_{y_4}} dk_{x_1} dk_{x_2} dk_{x_3} dk_{x_4} dk_{y_1} dk_{y_2} dk_{y_3} dk_{y_4}. \tag{14}$$

Let try to find solutions:

$$b(x, y, t) = f(y, t)e^{i(k_{x_0}x - \omega_0 t)} \tag{15}$$

where $\omega_0 = \omega_{k_{x_0},0}$. Expand ω_{k_x, k_y} around $\omega_{k_{x_0},0}$:

$$\omega_{k_{x_0}, k_y} = \omega_0 + \frac{\omega_0}{4k_{x_0}^2} k_y^2 + O(k_y^3) \tag{16}$$

Substituting (15) and using (16) into Eq.(11) yields defocusing NLS equation:

$$i \frac{\partial f(y, t)}{\partial t} = -\frac{\omega_0}{4k_{x_0}^2} \frac{\partial^2 f(y, t)}{\partial y^2} + k_{x_0}^3 |f(y, t)|^2 f(y, t) \tag{17}$$

The solutions of equation (17) are well-known (kink, grey soliton). Also, the equation (11) is very suitable for robust numerical simulation. Due to specific structure of nonlinearity in the Hamiltonian (10) viz. four-wave interactions coefficient $\tilde{T}_{kk_1}^{k_2 k_3}$ not dependent on y , the equation (11) can be effectively solved on the computer by using 1D Fast Fourier Transform in the periodic domain $x \in [0, 2\pi], y \in [0, 2\pi]$.

2.1. Modulational instability of monochromatic wave

Monochromatic wave

$$b(x, y, t) = B_0 e^{i(k_{x_0}x + k_{y_0}y - \omega_0 t)} \tag{18}$$

is the simplest solution of (11). Indeed, plugging (18) in to the equation (11) one can get the following relation

$$\omega_0 = \omega_{k_{x_0}, k_{y_0}} + k_{x_0}^3 |B_0|^2. \tag{19}$$

We consider perturbation to the solution

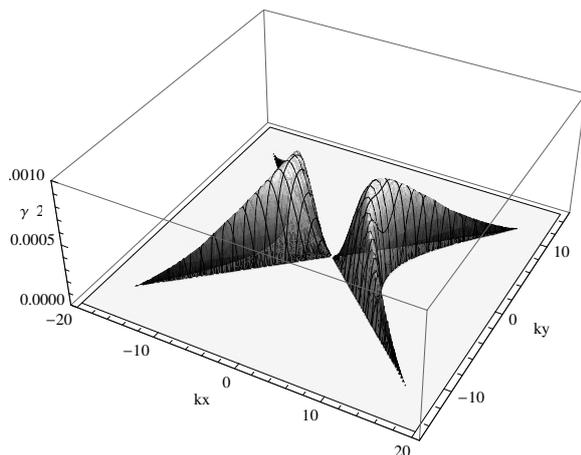


Figure 3. The growth rate γ_{k_x, k_y}^2 for case $k_{x_0} = 25, k_{y_0} = 0, |B_0| = 0.0015 (\mu \sim 0.12)$.

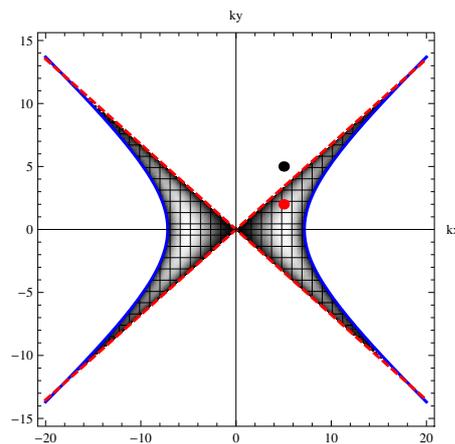


Figure 4. The growth rate γ_{k_x, k_y}^2 for case $k_{x_0} = 25, k_{y_0} = 0, |B_0| = 0.0015 (\mu \sim 0.12)$.

$$b(x, y, t) = B_0 e^{i(k_{x_0}x + k_{y_0}y - \omega_0 t)}, \tag{20}$$

or, in K -space:

$$b(k_x, k_y, t) = 2\pi B_0 \delta_{k_x - k_{x_0}} \delta_{k_y - k_{y_0}} e^{-i\omega_0 t}, \tag{21}$$

Perturbed solution has the following form:

$$b \Rightarrow (b_{k_{x_0}, k_{y_0}} + \delta b_{k_{x_0}+k_x, k_{y_0}+k_y} e^{-i\Omega_+ t + \gamma_{k_x, k_y} t} + \delta b_{k_{x_0}-k_x, k_{y_0}-k_y} e^{-i\Omega_- t + \gamma_{k_x, k_y} t}) e^{-i\omega_0 t}. \quad (22)$$

with the following condition:

$$\Omega_+ = -\Omega_- \quad (23)$$

If we introduce steepness of the carrier wave $\mu^2 = 2|B_0|^2 \frac{k_{x_0}^3}{\omega_{k_0}}$ then for growth rate is

$$\gamma_{k_x, k_y}^2 = \frac{1}{8} \frac{\omega_{k_0}^2}{k_0^4} k_x^2 \left[\left(1 - 2 \left(\frac{k_y}{k_x} \right)^2 \right) - 6\mu^2 \right] \left[\mu^2 \left(k_0 - \frac{|k_x|}{2} \right)^2 - \frac{k_x^2}{8} \left(1 - 2 \left(\frac{k_y}{k_x} \right)^2 \right) \right]. \quad (24)$$

There is the growth rate function (24) on figures 3, 4 for the case $k_{x_0} = 25, k_{y_0} = 0, |B_0| = 0.0015 (\mu \sim 0.12)$. The black dot ($k_x = 5, k_y = 5$) on the figure 4 corresponds to "stable" region and red ($k_x = 5, k_y = 2$) dot corresponds to "modulational unstable" region.

2.2. Numerical simulation of the freak wave appearing

We have performed the numerical simulation of the freak wave formation in the framework of equation (11). Initial condition was chosen as slightly perturbed monochromatic wave (18) with the steepness of the carrier wave $\mu \sim 0.12$ and $k_{x_0} = 25$ as shown in figure 5. After several

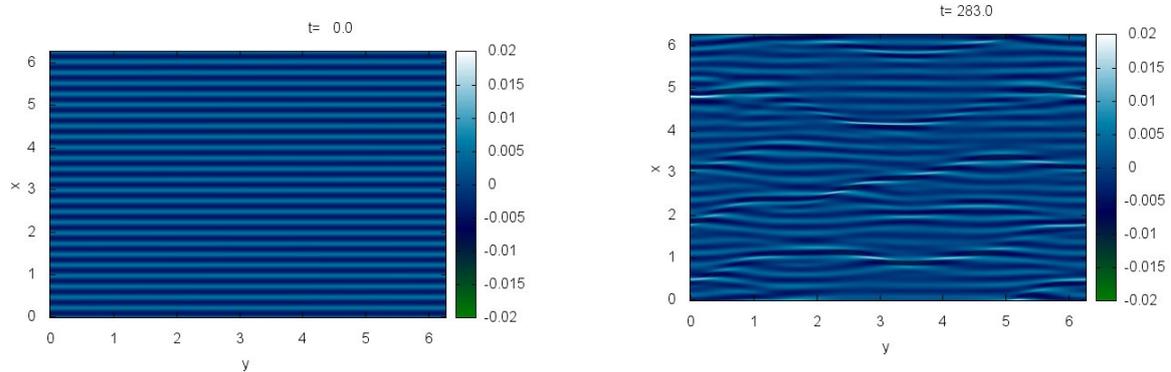


Figure 5. Initial free surface $\eta(x, y, t = 0)$

Figure 6. Extreme wave formation at the surface $\eta(x, y, t = 565)$

hundreds of carrier wave periods we observed the freak wave formation with the steepness of $\mu \sim 0.5$ (shown in figure 6).

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