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To cite this article: Dieter Schuch 2012 *J. Phys.: Conf. Ser.* **361** 012020

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Nonlinear quantum mechanics, complex classical mechanics and conservation laws for closed and open systems

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Abstract. Phase and amplitude of the complex wave function are not independent of each other, but coupled, which becomes obvious looking at Madelung's hydrodynamic formulation of quantum mechanics. In the time-independent case, this leads to a kind of conservation law that allows for the reformulation of the linear Schrödinger equation in terms of a nonlinear Ermakov equation which is equivalent to a complex Riccati equation where the quadratic term in this equation explains the origin of the phase-amplitude coupling. A similar conservation law and corresponding nonlinear equations can also be found in the time-dependent case. The gain from the nonlinear formulations emerges when open systems with dissipation and irreversibility are considered. Describing this kind of systems by an effective nonlinear Schrödinger equation leads to a modification of the above-mentioned equations with new qualitative effects like Hopf bifurcations.

1. Introduction

This is my contribution to the proceedings of a conference held in Vienna in November 2011 in honour of the 100th birthday of Heinz von Foerster. Almost 25 years earlier, in March 1987, I attended a conference in London celebrating the centenary of the birth of another Austrian polymath, Erwin Schrödinger. One of the talks there was given by the Nobel Laureate Chen Ning Yang entitled "Square root of minus one, complex phases and Erwin Schrödinger" [1] in which he stated that with quantum mechanics, for the first time the imaginary unit enters physics in a fundamental way. So, he claims: "Complex numbers became a conceptual element of the very foundation of physics: the fundamental equations of matrix mechanics and wave mechanics,

$$p q - q p = -i\hbar \quad (1)$$

$$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi \quad (2)$$

both explicitly contain the imaginary unit $i = \sqrt{-1}$. It is to be emphasized that the very meaning of these equations would be totally destroyed if one tries to get rid of i by writing (1) and (2) in terms of real and imaginary parts."

I totally agree with this statement, but want to go even a step further in this paper. The reason why it is not sufficient to simply write these equations in terms of two real equations shall

be explained in the case of the Schrödinger equation (SE), Eq.(2). In Section 2, Madelung's hydrodynamic formulation [2] of Schrödinger's theory will be used, where the complex wave function is written in polar form to show that phase and amplitude (or real and imaginary parts), and hence the corresponding equations of motion are not independent of each other but uniquely coupled. First, for the time-independent (TI)SE, it will be shown that this coupling can be explained in terms of a conservation law that leads to a nonlinear (NL) formulation of quantum mechanics that can be rewritten in terms of a complex NL Riccati equation, where the origin of the coupling becomes obvious. Formally, a similar situation is also found in the case of the time-dependent (TD)SE, at least for quadratic Hamiltonians with Gaussian wave packet solutions. In this case, spatial derivatives are replaced by temporal ones, but the system of interrelated differential equations looks very much like in the TI case.

All realistic physical systems are in contact with some kind of environment, thus introducing phenomena like irreversibility and dissipation. There are different approaches in the literature for including these aspects in a quantum mechanical context. The ansatz that will be discussed in Section 3 is based on a modified equation for the probability density and leads to an effective NLSE with complex logarithmic nonlinearity. In the cases discussed in Section 2 where Gaussian wave packet solutions exist, there are also the same type of solutions for our NLSE. In Section 4, the modifications of the results found in Section 2 for the TD case will be discussed if dissipation and irreversibility are included and a particular example leading to Hopf bifurcation will be considered in detail. From the form of the modifications in the TD case one can draw conclusions for the TI case and find hints for the derivation of the corresponding equations. Section 5 will summarize the results and show possible further developments.

2. Complex Riccati equations and NL Ermakov equations in time-independent and time-dependent quantum mechanics

Already in the same year (1926) when Schrödinger's communications on quantum mechanics appeared, Madelung [2] published a paper where he used the polar form

$$\Psi(\mathbf{r}, t) = \varrho^{1/2}(\mathbf{r}, t) \exp\left(\frac{i}{\hbar} S(\mathbf{r}, t)\right) \quad (3)$$

of the complex wave function that fulfils the TDSE

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left(-\frac{\hbar^2}{2m} \Delta + V \right) \Psi(\mathbf{r}, t) = H_L \Psi(\mathbf{r}, t) . \quad (4)$$

Inserting (3) into (4), one obtains two real equations,

$$\frac{\partial}{\partial t} S + \frac{1}{2m} (\nabla S)^2 + V - \frac{\hbar^2}{2m} \frac{\Delta \varrho^{1/2}}{\varrho^{1/2}} = 0 \quad (5)$$

and

$$\frac{\partial}{\partial t} \varrho + \frac{1}{m} \nabla(\varrho \nabla S) = 0 . \quad (6)$$

The first one essentially determines the evolution of the phase $S(\mathbf{r}, t)$ and has almost the form of the Hamilton–Jacobi equation, only that there is an additional term, the last one on the rhs, that depends on the amplitude $\varrho^{1/2}$ of the wave function and is often deceptively called "quantum potential" (V_{qu}) although it is essentially related to the kinetic energy of the quantum system (see below). This equation reflects the classical mechanical (deterministic) trajectory aspect of the system however, via ϱ , also a probabilistic element enters. The quantity $\varrho = \Psi^* \Psi$ represents the probability density and fulfils the continuity equation (6). This equation that essentially

determines the evolution of the amplitude of the wave function, contains the (gradient of) the phase in the probability current $\mathbf{j} = \varrho(\frac{1}{m}\nabla S)$, where the gradient of the action S is usually associated with the momentum of the system. So, the Hamilton–Jacobi equation (5) and the continuity equation (6) (or, phase and amplitude of the wave function Ψ) are coupled. This coupling, however, is not arbitrary but related to a conservation law which will be shown in the following for the TI case.

2.1. Nonlinear formulations and invariants in time-independent quantum mechanics

In 1994, G. Reinisch [3] presented a NL formulation of TI quantum mechanics. Since in this case $\frac{\partial}{\partial t}\varrho = 0$ and $\frac{\partial}{\partial t}S = -E$ are valid, the continuity equation (6) (we use now the notation $\varrho^{1/2} = |\Psi| = a$) turns into

$$\nabla(a^2\nabla S) = 0 \quad (7)$$

and the modified Hamilton–Jacobi equation into

$$-\frac{\hbar^2}{2m}\Delta a + (V - E)a = -\frac{1}{2m}(\nabla S)^2 a. \quad (8)$$

Equation (7) is certainly fulfilled for $\nabla S = 0$, turning (8) into the usual TISE for the real wave function $a = |\Psi|$ with position-independent phase $S(\mathbf{r})$. (NB: the kinetic energy term $-\frac{\hbar^2}{2m}\Delta a$ is just identical to $V_{qu} a$!)

However, Eq.(7) can also be fulfilled for $\nabla S \neq 0$ if only the conservation law

$$\nabla S = \frac{C}{a^2} \quad (9)$$

with constant (or, at least, position-independent) C is fulfilled. This relation now shows explicitly the coupling between phase and amplitude of the wave function. Inserting (9) into the rhs of Eq.(8) changes this into the real NL Ermakov equation

$$\Delta a + \frac{2m}{\hbar}(E - V)a = \left(\frac{1}{\hbar}\nabla S\right)^2 a = \left(\frac{C}{\hbar}\right)^2 \frac{1}{a^3} \quad (10)$$

with inverse cubic nonlinearity. (This equation had been studied by V. Ermakov [4] long before quantum mechanics was developed and we will come back to it in the TD case.) Knowing the solution of (10), via (9) the phase (up to an integration constant) can be determined and thus the complex wave function can be obtained.

For a better understanding of the origin of the coupling expressed by Eq.(9) it should be mentioned that the real NL Ermakov equation (10) is equivalent to a complex NL Riccati equation. In our case, this quadratic NL equation would read

$$\nabla\left(\frac{\nabla\Psi}{\Psi}\right) + \left(\frac{\nabla\Psi}{\Psi}\right)^2 + \frac{2m}{\hbar}(E - V) = 0 \quad (11)$$

with the complex variable $\left(\frac{\nabla\Psi}{\Psi}\right) = \frac{\nabla a}{a} + i\frac{1}{\hbar}\nabla S$.

First, it is straightforward to show that Eq.(11) can be linearised to yield the usual TISE,

$$-\frac{\hbar^2}{2m}\Delta a + V a = E a, \quad (12)$$

but inserting the definition of the complex variable into Eq.(11) and separating real and imaginary parts shows that the imaginary part leads to the conservation law (9), whereas the

real part leads to the Ermakov equation (10). The origin of the coupling between phase and amplitude is obviously the quadratic nonlinearity since it introduces a term proportional to $(\nabla S)^2$ into the real part (Eq.(10)) and is responsible for the product term proportional to $(\nabla a)(\nabla S)$ in the imaginary part, leading to Eq.(9).

The same type of Riccati and Ermakov equations also occurs in the TD case, as will be shown in the following.

2.2. Nonlinear formulations and invariants in time-dependent quantum mechanics

The TDSE (4) always possesses for at most quadratic Hamiltonians exact analytic solutions in the form of Gaussians wave packets that can be written (in the following in one dimension) as

$$\Psi(x, t) = N(t) \exp \left\{ i \left[y(t) \tilde{x}^2 + \frac{1}{\hbar} \langle p \rangle \tilde{x} + K(t) \right] \right\}, \quad (13)$$

where the maximum is at $\eta(t)$ which is identical with the mean value of the position, $\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^* x \Psi dx = \eta$, $\tilde{x} = x - \eta$ and the width of the WP is related with the imaginary part of the complex quantity $y(t)$ via $y_I = \frac{1}{4\langle \tilde{x}^2 \rangle}$ with $\langle \tilde{x}^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$. (N is a, possibly TD, normalization factor, $\langle p \rangle$ the classical momentum and K a purely TD function that will not be relevant for the following.) Inserting wave packet (13) into the SE (4) leads to the equations of motion for $\eta(t)$ and $y(t)$. Particularly for the harmonic oscillator (i.e., $V = \frac{m}{2}\omega^2 x^2$ with possibly TD $\omega(t)$), one obtains the usual Newtonian equation of motion

$$\ddot{\eta} + \omega^2(t)\eta = 0 \quad (14)$$

for the maximum and a complex Riccati equation, but now for the TD variable $\left(\frac{2\hbar}{m}y\right)$,

$$\frac{2\hbar}{m}\dot{y} + \left(\frac{2\hbar}{m}y\right)^2 + \omega^2(t) = 0, \quad (15)$$

where overdots denote time-derivatives.

Introducing a new variable $\alpha_L(t)$ via $\frac{2\hbar}{m}y_I = \frac{\hbar}{2m\langle \tilde{x}^2 \rangle_L} = \frac{1}{\alpha_L^2}$ allows one to determine the real part $\frac{2\hbar}{m}y_R$ from the imaginary part of Eq.(15) as $\frac{2\hbar}{m}y_R = \frac{\dot{\alpha}_L}{\alpha_L}$. Inserting this into the real part of (15) yields again an Ermakov equation,

$$\ddot{\alpha}_L + \omega^2(t)\alpha_L = \frac{1}{\alpha_L^3}. \quad (16)$$

Eliminating $\omega^2(t)$ from Eqs.(14) and (16) leads to the dynamical invariant

$$I_L = \frac{1}{2} \left[(\dot{\eta} \alpha_L - \eta \dot{\alpha}_L)^2 + \left(\frac{\eta}{\alpha_L} \right)^2 \right] = \text{constant} \quad (17)$$

that also exists for TD $\omega(t)$, whereas the corresponding Hamiltonian is no longer a dynamical invariant (for further details see, e.g., [4, 5]).

Also the NL Riccati equation (15) can be linearised via the ansatz $\left(\frac{2\hbar}{m}y\right) = \frac{\dot{\lambda}}{\lambda}$ with complex $\lambda(t)$ to yield the **complex** Newtonian equation

$$\ddot{\lambda} + \omega^2(t)\lambda = 0. \quad (18)$$

With the polar form $\lambda = \alpha e^{i\varphi}$ the complex variable $\frac{2\hbar}{m}y$ can be written as $\frac{\dot{\lambda}}{\lambda} = \frac{\dot{\alpha}}{\alpha} + i \dot{\varphi}$. Inserting this into Eq.(15) shows that the imaginary part leads to the **conservation law**

$$\dot{\varphi} = \frac{1}{\alpha^2} \quad (19)$$

which represents a kind of conservation of angular momentum for the motion of $\lambda(t)$ in the complex plane [6]. From the real part of Eq.(15) one finally obtains the Ermakov equation (16) with $\alpha \equiv \alpha_L$.

3. Irreversibility, dissipation and effective NLSEs

Since realistic physical systems are always in contact with some kind of environment and this coupling usually introduces the phenomena irreversibility and dissipation (but not necessarily both simultaneously!), the question arises of how this can be taken into account particularly in a quantum mechanical context. On the classical level, one must go beyond the established formalism of canonical transformations to be able to include these aspects into the Hamiltonian formalism if one wants to avoid a rather unwieldy and computationally time-consuming inclusion of a large number of environmental degrees of freedom. There are, however, numerous approaches (see, e.g., [7, 8, 9] and references cited therein) that only take into account the effect of the environment on the system of interest by modifying the respective equation of motion without including the environmental degrees of freedom explicitly. In classical Newtonian mechanics, this can be done, e.g., by introducing a dissipative friction force. If it is assumed that this force depends linearly on the velocity (or momentum), this leads to

$$m\ddot{x} = m\dot{v} = -m\gamma v - \frac{\partial}{\partial x}V \quad (20)$$

which is the Langevin equation without stochastic contribution (γ is the friction coefficient).

This equation was used to describe Brownian motion from a trajectory point of view. An equivalent description of the phenomenon can also be given in terms of (classical) probability distributions ϱ_{cl} via the Fokker-Planck equations that contain irreversible diffusion terms. Particularly in position-space this can be written in form of the Smoluchowski equation [10] for $\varrho_{cl}(x, t)$,

$$\frac{\partial}{\partial t}\varrho_{cl} + \frac{\partial}{\partial x}\left(\frac{F(x)}{m\gamma}\varrho_{cl}\right) - \frac{kT}{m\gamma}\frac{\partial^2}{\partial x^2}\varrho_{cl} = 0, \quad (21)$$

with the conservative force $F(x)$, Boltzmann's constant k and temperature T . Comparison with Einstein's theory of Brownian motion [11] shows that the coefficient of the diffusion term fulfils the Einstein relation $D = \frac{kT}{m\gamma}$.

Equation (21) provided the basis for our modification of the SE in order to include irreversibility and dissipation. We started from the continuity equation (6) for the probability density $\varrho(\mathbf{r}, t)$, for which Madelung [12] and Mrowka [13] had shown that with a bilinear ansatz for the probability density, $\varrho = \Psi^*\Psi$, and for the probability current density, $\mathbf{j} = \varrho\left(\frac{1}{m}\nabla S\right) = \varrho\mathbf{v}_- = \left(\frac{\hbar}{2mi}\right)(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)$, this can be separated into the TDSE and its complex conjugate, where the separation "constant" is proportional to the potential V . Keeping Eq.(21) in mind, we tried to break the time-symmetry by adding an irreversible diffusion term leading (in the following again in one dimension) to

$$\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x}(\varrho v_-) - D\frac{\partial^2}{\partial x^2}\varrho = 0. \quad (22)$$

However, due to the diffusion term, Ψ - and Ψ^* -dependent terms can generally no longer be separated. But if one introduces an additional separation condition

$$-D \frac{\partial^2 \varrho}{\partial x^2} = \gamma (\ln \varrho - \langle \ln \varrho \rangle) , \quad (23)$$

where $\ln \varrho = \ln \Psi + \ln \Psi^*$, separation is possible and leads to an additional complex logarithmic term in the SE [8] for $\Psi_{NL}(x, t)$,

$$i\hbar \frac{\partial}{\partial t} \Psi_{NL} = \{H_L + \gamma \frac{\hbar}{i} (\ln \Psi_{NL} - \langle \ln \Psi_{NL} \rangle)\} \Psi_{NL} = \{H_L + W\} \Psi_{NL} , \quad (24)$$

where H_L is the usual linear Hamiltonian. Now the question arises, has this additional term W , that can be written as real and imaginary contributions in the form

$$W = W_R + W_I = \frac{\gamma \hbar}{2i} \left(\ln \frac{\Psi_{NL}}{\Psi_{NL}^*} - \langle \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} \rangle \right) + \frac{\gamma \hbar}{2i} (\ln \varrho_{NL} - \langle \ln \varrho_{NL} \rangle) , \quad (25)$$

any physical meaning?

The real part was already known in the literature (see: Kostin [14]) and leads, according to Ehrenfest, to a linear velocity dependent term in the equation of motion for the mean value of position, thus turning Eq.(14) into a Langevin equation like (20).

Since the mean value of W_R (as well as of W_I) vanishes, the mean value of the energy is still given by $\langle H_L \rangle$, but now calculated with Ψ_{NL} , the solution of the NLSE (24). Therefore, the energy is no longer constant but decreases in a way that would be expected from the classical counterpart. However, W_R alone does not completely describe the physics of this system, as has been criticized in comments concerning Kostin's work [7, 8]. In particular, W_R would still allow the solutions of the undamped harmonic oscillator to be also solutions of the damped one and, since W_R is real, the equation of motion for $\varrho_{NL} = \Psi_{NL}^* \Psi_{NL}$ would still be the reversible continuity equation (6). All these problems are eliminated when the imaginary part W_I is also taken into account.

A term formally like the imaginary part W_I (but in the context of density operators) was independently discussed around the same time by Beretta [15] to describe non-equilibrium systems (without dissipation). An interesting interpretation of W_I can be found if one identifies, according to Grössing et al [16], the Einstein diffusion coefficient with the quantum mechanical one (if the SE is considered a diffusion equation with imaginary diffusion coefficient), i.e., $D = \frac{kT}{m\gamma} = \frac{\hbar}{2m}$. Then, W_I turns into

$$W_I = -iTk (\ln \varrho_{NL} - \langle \ln \varrho_{NL} \rangle) , \quad (26)$$

where $-k \langle \ln \varrho \rangle = -k \int_{-\infty}^{+\infty} \varrho \ln \varrho dx$ has a form like the definition of entropy, $\mathcal{S}$. So, the mean value of the linear Hamiltonian that still represents the energy of the system, $\langle H_L \rangle = E$, together with the second term of (26) would look like $E - iTS$, i.e., it has similarity with an expression for the free energy only that here, again, the imaginary unit i turns up in the quantum mechanical context. This point still needs further investigation.

It should be emphasized that the imaginary part by itself does not introduce any dissipation of energy but a kind of non-unitarity in the time-evolution of the wave function, however, still guaranteeing normalizability (due to the subtraction of $\langle \ln \varrho \rangle$).

The fact that both real and imaginary parts of W are necessary to describe correctly dissipation as well as irreversibility of the system, where W_R depends only on the phase of the wave function, while W_I only depends on its amplitude, shows again that phase and amplitude are not independent of each other, but coupled, as already shown in the conservative reversible case.

How including the environment modifies the NL relations, and thus the coupling between phase and amplitude that were discussed in Sections 2.1 and 2.2, will be shown in the following.

4. Modification of the complex Riccati and nonlinear Ermakov equations

4.1. Time-dependent case

In the cases where the TDSE (4) possesses exact Gaussian-type wave packet solutions of the form (13), the same also holds for the NLSE (24). Due to the logarithmic nonlinearity, the equations of motion for the wave packet maximum and the wave packet width are modified. The maximum now follows the classical trajectory obeying the Newton-type equation

$$\ddot{\eta} + \gamma \dot{\eta} + \omega^2 \eta = 0 \quad (27)$$

with an additional friction force and the complex NL Riccati equation gains an additional linear term according to

$$\frac{2\hbar}{m} \dot{y} + \gamma \left(\frac{2\hbar}{m} y \right) + \left(\frac{2\hbar}{m} y \right)^2 + \omega^2(t) = 0 \quad (28)$$

(here again for the harmonic oscillator potential).

With a definition analogous to the one in the conservative case, a new variable α_{NL} is introduced via $\frac{2\hbar}{m} y_I = \frac{\hbar}{2m \langle \tilde{x}^2 \rangle_{NL}} = \frac{1}{\alpha_{NL}^2}$. Inserting this into Eq.(28) yields from its imaginary part $\frac{2\hbar}{m} y_R = \frac{\dot{\alpha}_{NL}}{\alpha_{NL}} - \frac{\gamma}{2}$ which, inserted into the real part of (28), turns it into an Ermakov equation with shifted frequency

$$\ddot{\alpha}_{NL} + \left(\omega^2(t) - \frac{\gamma^2}{4} \right) \alpha_{NL} = \frac{1}{\alpha_{NL}^3} . \quad (29)$$

Also in this dissipative case, a dynamical Ermakov invariant can be obtained by eliminating ω^2 between Eqs.(27) and (29), leading to

$$I_{NL} = \frac{1}{2} e^{\gamma t} \left[\left(\dot{\eta} \alpha_{NL} - \left(\dot{\alpha}_{NL} - \frac{\gamma}{2} \alpha_{NL} \right) \eta \right)^2 + \left(\frac{\eta}{\alpha_{NL}} \right)^2 \right] = \text{constant} . \quad (30)$$

An interesting qualitative change occurs when considering the modified Riccati equation (28), which shall be shown in the following for the damped free motion. Knowing a particular solution $\left(\frac{2\hbar}{m} \hat{y} \right)$ of the inhomogeneous differential equation (28) allows it to be turned into the homogeneous Bernoulli equation

$$\frac{2\hbar}{m} \dot{v} + \left(\gamma + \frac{4\hbar}{m} \hat{y} \right) \left(\frac{2\hbar}{m} v \right) + \left(\frac{2\hbar}{m} v \right)^2 = 0 \quad (31)$$

with $\gamma + \frac{4\hbar}{m} \hat{y} = A$, where the general solution of the Riccati equation is obtained from $\frac{2\hbar}{m} y = \frac{2\hbar}{m} \hat{y} + \frac{2\hbar}{m} v(t)$. For $\frac{2\hbar}{m} \hat{y} = \text{constant}$ one finds

$$A = \pm 2 \left(\frac{\gamma^2}{4} - \omega^2 \right)^{1/2} \quad (32)$$

or, in our case for $\omega = 0$, $A = \pm \gamma$. In the conservative case, for $\gamma = 0$, only one solution exists with $A = 0$, but now there are two different values possible for A (like in a Hopf bifurcation) and, therefore, two different solutions which both correspond to different wave packets with different widths and different contributions to the energy. For comparison, the position uncertainty $\langle \tilde{x}^2 \rangle(t)$ and the minimum energy that is proportional to the momentum uncertainty via $\tilde{E} = \frac{1}{2m} \langle \tilde{p}^2 \rangle = \frac{1}{2m} (\langle p^2 \rangle - \langle p \rangle^2)$ shall be given for the conservative ($\gamma = 0$) and dissipative ($\gamma \neq 0$) case, respectively.

For $A = 0$, one obtains

$$\langle \tilde{x}^2 \rangle_L = \langle \tilde{x}^2 \rangle_0 \{1 + (\beta_0 t)^2\} \quad (33)$$

$$\tilde{E}_L = \frac{\langle \tilde{p}^2 \rangle_0}{2m} = \frac{\hbar}{4} \beta_0 \quad (34)$$

with the constant frequency-type quantity $\beta_0 = \frac{\hbar}{2m\langle \tilde{x}^2 \rangle_0}$, depending on the initial position uncertainty $\langle \tilde{x}^2 \rangle_0$.

For $A = +\gamma$, this changes into

$$\langle \tilde{x}^2 \rangle_+ = \langle \tilde{x}^2 \rangle_0 \left\{ e^{\gamma t} + \left(\frac{\beta_0}{\gamma/2} \right)^2 \sinh^2 \frac{\gamma}{2} t \right\} \quad (35)$$

$$\tilde{E}_+ = \frac{\hbar}{4} \beta_0 e^{-\gamma t}, \quad (36)$$

for $A = -\gamma$ into

$$\langle \tilde{x}^2 \rangle_- = \langle \tilde{x}^2 \rangle_0 \left\{ e^{-\gamma t} + \left(\frac{\beta_0}{\gamma/2} \right)^2 \sinh^2 \frac{\gamma}{2} t \right\} \quad (37)$$

$$\tilde{E}_- = \frac{\hbar}{4} \beta_0 \left(1 + \left(\frac{\gamma}{\beta_0} \right)^2 \right) e^{-\gamma t}. \quad (38)$$

It becomes obvious that the coupling to the environment in both cases causes a faster spreading of the wave packet width (due to a faster dephasing; for details see [17]) and an exponential decay of $\tilde{E}_\pm$ (nevertheless, the uncertainty product always fulfils the relation $U_\pm = \langle \tilde{x}^2 \rangle_\pm \langle \tilde{p}^2 \rangle_\pm \geq \frac{\hbar^2}{4}$).

To further compare these results with the conservative case, we consider the time $t = 0$ with $\tilde{E}_+(0) = \frac{\hbar}{4} \beta_0 = \tilde{E}_L$ and $\tilde{E}_-(0) = \frac{\hbar}{4} \beta_0 \left(1 + \left(\frac{\gamma}{\beta_0} \right)^2 \right) > \tilde{E}_L$. The "splitting" of energy level is given by

$$\Delta \tilde{E}_0 = \tilde{E}_-(0) - \tilde{E}_+(0) = \frac{\hbar}{4} \frac{\gamma^2}{\beta_0} = \frac{m}{2} \gamma^2 \langle \tilde{x}^2 \rangle_0 \quad (39)$$

and is thus independent of $\hbar$, but only depending on γ , the parameter that characterizes the environment! From the Smoluchowski equation (22) and the NLSE (24) it follows that $D = \frac{\gamma}{2} \langle \tilde{x}^2 \rangle$ (in particular for time-dependent $\langle \tilde{x}^2 \rangle(t)$, the diffusion coefficient also becomes time-dependent). Assuming for $t = 0$ an equilibrium condition where the Einstein relation applies, we then obtain

$$\Delta \tilde{E}_0 = kT. \quad (40)$$

So, the coupling to the environment (thermal bath) breaks the time-symmetry and removes a kind of "degeneracy" of the ground state energy, introducing a new state where this second state is initially kT higher in energy than the original quantum mechanical ground state and both energies, $\tilde{E}_+$ and $\tilde{E}_-$, decay exponentially.

Finally, it shall be mentioned that also the Riccati equation (28) can be linearised via $\left(\frac{2\hbar}{m} y \right) = \frac{\dot{\tilde{\lambda}}}{\tilde{\lambda}}$ with $\tilde{\lambda} = \lambda e^{-\gamma t/2} = \alpha_{NL} e^{-\gamma t/2 + i\varphi}$ to yield the complex Newtonian equation

$$\ddot{\tilde{\lambda}} + \gamma \dot{\tilde{\lambda}} + \omega^2 \tilde{\lambda} = 0. \quad (41)$$

Also in this case, phase and amplitude of $\tilde{\lambda}$ are coupled via $\dot{\varphi} = \frac{1}{\alpha_{NL}^2}$ and α_{NL} can be determined by solving the Ermakov equation (29).

4.2. Time-independent case

Due to the formal similarities between the systems of equations that we discussed in Section 2 for the TDSE and TISE, the modifications we obtained in the TD case when a dissipative environment is included may provide a hint in which way the corresponding equations in the TI situation must be modified. From the Riccati equation (28) one can conclude that a term proportional to $\left(\frac{\nabla\Psi}{\Psi}\right)$ must be added to Eq.(11), or, from Eqs.(27) and (41), that a term proportional to $\nabla\Psi$ must be added to the TISE (12). Since Ψ is complex, from the imaginary part one would obtain an additional contribution to the continuity equation (6). In the following, only a short outline of the reasoning shall be given as to how the additional terms can be obtained; further details will be presented elsewhere [18]. Since the SE can be obtained from the continuity equation via separation, as well as the NLSE from the Smoluchowski equation (plus separation condition), the additional term in the continuity equation should take a separable form proportional to $\nabla\varrho = (\Psi^*\nabla\Psi + \Psi\nabla\Psi^*)$, leading to a $\nabla\Psi$ -term with imaginary coefficient in the corresponding modified TISE. Furthermore, at least for the cases with Gaussian wave packet solutions, this additional term should have the same effect as the diffusion term or the $\ln\varrho$ -term, i.e.,

$$-D\Delta_x\varrho = \gamma(\ln\varrho - \langle \ln\varrho \rangle)\varrho = \frac{\gamma}{2}\left(1 - \frac{\tilde{x}^2}{\langle \tilde{x}^2 \rangle}\right)\varrho \quad (42)$$

should be reproduced. Comparison shows that $\nabla_x(\frac{\gamma}{2}\tilde{x}\varrho)$ yields the desired result.

To compare the modified equations with the situation discussed in Section 2.1., we consider again the TI case with $\frac{\partial}{\partial t}\varrho = 0$ (and use again the notation $\varrho^{1/2} = |\Psi| = a$), where Eq.(7) is now replaced by

$$\nabla\left(a^2\left(\nabla S + m\frac{\gamma}{2}\tilde{x}\right)\right) = \nabla\left(a^2\nabla S'\right) = 0 \quad (43)$$

with $S' = S + m\frac{\gamma}{4}\tilde{x}^2 + f(t)$. So now the conservation law (9) must be replaced by $\nabla S' = \frac{C}{a^2}$ for the modified action S' , i.e. a change of the phase, corresponding to a gauge transformation, must be performed.

Consequently, also the modified Hamilton–Jacobi equation (8) must be changed which (with $\frac{\partial}{\partial t}S = -E$) finally replaces the Ermakov equation (10) by

$$\Delta a + \frac{2m}{\hbar}\left(E - \left(V - \frac{m}{2}\frac{\gamma^2}{4}\tilde{x}^2\right)\right)a = \left(\frac{1}{\hbar}\nabla S'\right)^2 a = \left(\frac{C}{\hbar}\right)^2 \frac{1}{a^3}. \quad (44)$$

5. Conclusions

We start out with the statement that a fundamental difference between classical physics and quantum mechanics exists because the latter is based essentially on complex quantities where the imaginary unit cannot simply be eliminated by writing down two equations for the real and imaginary parts of the complex quantities, or for the phase and amplitude, respectively. The reason for this already becomes evident in Madelung’s hydrodynamic formulation of Schrödinger’s wave equation using a polar form for the complex wave function. Obviously, there is a coupling between the two components that can be related to some kind of conservation law. The origin of this coupling can however be understood immediately when the TISE is rewritten as a complex NL Riccati equation. In particular, the quadratic nonlinearity shows right away how the mixing of phase and amplitude occurs.

In a further step it has been shown that similar formal relationships also occur in TD quantum mechanics, at least (but not only) in cases where Gaussian wave packet solutions exist. Here, the complex Riccati equation essentially describes the time-evolution of the quantum uncertainties and can again be rewritten in terms of a (real) NL Ermakov equation.

In this case, this equation, together with the corresponding Newtonian equation that describes the motion of the wave packet maximum, allows for the construction of a dynamical invariant that still exists in cases where the corresponding Hamiltonian does not have this property.

A different way of treating the Riccati equation is to linearise it to a Newtonian equation for a complex quantity ($\lambda(t)$) where, formally, the same relations between its phase and amplitude exist as in the TI case between phase and amplitude of the wave function. The corresponding conservation law can be interpreted as a kind of conservation of angular momentum for the motion of λ in the complex plane.

Next, the systems considered were coupled to a dissipative environment. In the TD case, this was studied using an effective NLSE with complex logarithmic nonlinearity. The changes for the afore-mentioned Riccati, Ermakov and Newtonian equations due to the coupling to the environment were discussed. Now the quadratic nonlinearity in the Riccati equation leads to new qualitative effects like Hopf bifurcations since two physically distinguishable solutions occur. A possible interpretation of this phenomenon has been given.

From the results obtained in the TD case, conclusions for corresponding modifications in the TI case have been drawn leading to the results given in Eqs.(43) and (44). One interesting aspect of this comparison is that the same type of Ermakov equation can be obtained either by changing the phase of the complex quantity (in the TI case, $S \rightarrow S' = S + m\frac{\gamma}{4}\tilde{x}^2$) or by changing the amplitude (in the TD case, $\alpha \rightarrow \alpha e^{-\gamma t/2}$). Consequences for the relations between gauge transformations and non-unitary transformations are still under investigation.

Acknowledgments

The author wishes to thank the organizers of EmerQuM 2011 for the invitation to participate in this really stimulating conference.

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