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Quantization as an emergent phenomenon due to matter–zeropoint field interaction

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Abstract. Quantization is derived as an emergent phenomenon, resulting from the permanent interaction between matter and radiation field. The starting point for the derivation is the existence of the (continuous) random zero-point electromagnetic radiation field (zpf) of mean energy $\hbar\omega/2$ per normal mode. A thermodynamic and statistical analysis leads unequivocally (and without quantum assumptions) to the Planck distribution law for the complete field in equilibrium. The problem of the quantization of matter is then approached from the same perspective: A detailed study of the dynamics of a particle embedded in the zpf shows that when the entire system eventually reaches a situation of energy balance thanks to the combined effect of diffusion and dissipation, the particle has acquired its characteristic quantum properties. To obtain the quantum-mechanical description it has been necessary to do a partial averaging and take the radiationless approximation. Consideration of the neglected radiative terms allows to establish contact with nonrelativistic quantum electrodynamics and derive the correct formulas for the first-order radiative corrections. Quantum mechanics emerges therefore as a partial, approximate and time-asymptotic description of a phenomenon that in its original (pre-quantum) description is entirely local and causal.

1. Introduction

Since the pioneering works in stochastic electrodynamics (SED) [1], [2], quantum mechanics has been conceived of as an emergent theory arising from a deeper-level dynamics. At the more fundamental level the world is assumed to be causal, deterministic and local: particles interact (locally) with (and through) the random, continuous electromagnetic zero-point radiation field (zpf), and follow well-defined (even if unknown stochastic) trajectories. At the quantum level the description becomes nonlocal, particle trajectories seem to have lost their sense (and for many even their existence), discreteness and indeterminacy enter the picture, and the classical algebra is replaced by a non-commuting one.

The program of SED has gone a long way in retrieving and explaining quantum mechanics, starting with the harmonic oscillator and including other characteristic 'quantum' phenomena such as diamagnetism and van der Waals forces, among others (see [2]). However, establishing the connection between these apparently incompatible worlds — the classical and the quantum — for the general case (and for the atomic case in particular), has not been an easy matter.

Several factors contribute to this difficulty. On the one hand the classical formalism used as a starting point, and the mathematics that come with it, do not offer a clear indication as to how to effect the transition to the quantum level from physical principles alone. On the other hand,

while the emerging (quantum) theory contains characteristic novel phenomena, some central features of the underlying dynamics disappear altogether in the transition. This is the case, in particular, with the random zpf, which is considered in SED to be precisely the ultimate cause for the emergence of the quantum phenomenon.

To try to 'complete' the description and recover causality or locality (or both) by simply adding hidden elements or re-introducing the lost ones into the quantum description, seems an exercise as futile as trying to reconstruct the raw eggs out of a baked cake. But if the cake turns out to be delicious one accepts it. The great successes of quantum mechanics are normally considered a strong enough reason to be satisfied with the theory as it is, and to renounce any attempt at looking for explanations beyond (or underneath) it.

In this paper we go back to the origins of quantum mechanics, in line with the program of SED. This means taking the existence of the random electromagnetic zpf as the starting point and carefully analysing some important consequences of its existence. We will show that quantization emerges naturally, which is in itself a remarkable outcome. But equally interesting is what one learns along the way in the transition from the deeper, 'pre-quantum' level to the higher quantum one. Because, as could be expected, quantum mechanics turns out to represent only a partly averaged, asymptotic approximation (although a very accurate one). The assumptions and approximations made in this transition help us, thus, to better understand the origin of some basic quantum features and assign a physical meaning to them.

2. Planck's law as a consequence of the zero-point field

In this first part of the paper it is shown that Planck's distribution law for the equilibrium radiation field can be derived without introducing any discontinuity in the field or in its interaction with matter. The cornerstone of the derivation is the existence of the zpf as an integral part of the radiation field. The presentation draws in part from previous work [3], [4].

2.1. The equilibrium radiation field

The existence of a nonthermal field with energy different from zero implies an extension of the classical thermodynamic description. The starting point for this extension is Wien's law for the mean energy U of the oscillators of frequency ω in equilibrium at temperature T,

$$U(\omega, T) = \omega f(\omega/T), \tag{1}$$

which is a general result of thermodynamics. At $T = 0$ this equation takes the form

$$\mathcal{E}_0 \equiv U(\omega, 0) = A\omega, \tag{2}$$

with A constant. In classical thermodynamics the constant A is taken as zero, thereby excluding a priori the existence of a nonthermal energy. In contrast, the extended treatment admits a zero-point energy (zpe) for the oscillators that is different from zero. Clearly this simple introduction of the zpe, being contrary to classical energy equipartition, opens the door to interesting physical consequences.

The thermal energy distribution of the field oscillators in thermal equilibrium inside a cavity can be written according to thermodynamics in the general form

$$W_g(\mathcal{E})d\mathcal{E} = \frac{1}{Z_g(\beta)} g(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E}, \tag{3}$$

where $Z_g(\beta) = \int g(\mathcal{E}) e^{-\beta\mathcal{E}} d\mathcal{E}$ is the partition function, $\beta = 1/(k_B T)$, and $g(\mathcal{E})$ is the structure function. The mean value of any function $f(\mathcal{E})$ is then given by

$$\langle f(\mathcal{E}) \rangle = \int_0^\infty W_g(\mathcal{E}) f(\mathcal{E}) d\mathcal{E} \tag{4}$$

which for $f(\mathcal{E}) = \mathcal{E}$ gives $\langle \mathcal{E} \rangle \equiv U$.

The (classical) Boltzmann distribution is obtained from (3) with $g(\mathcal{E})$ independent of $\mathcal{E}$, in which case

$$W_{\text{cl}}(\mathcal{E}) = W_{g=\text{cl}}(\mathcal{E}) = \frac{e^{-\beta\mathcal{E}}}{\int_0^\infty e^{-\beta\mathcal{E}} d\mathcal{E}} \tag{5}$$

and $\langle \mathcal{E} \rangle_{\text{cl}} = U_{\text{cl}}(\beta) = \beta^{-1} = k_B T$. This means that to allow for a zpe, $g(\mathcal{E})$ must depend on the energy; this will lead to a different distribution $W_g(\mathcal{E})$ and hence a different function $U(\beta)$.

The general form for the distribution $W_g(\mathcal{E})$ implies the recurrence relation

$$\langle \mathcal{E}^{r+1} \rangle = U \langle \mathcal{E}^r \rangle - \langle \mathcal{E}^r \rangle', \tag{6}$$

where the prime denotes derivative with respect to β. For $r = 1$ this gives for the thermal energy variance

$$\sigma^2 \equiv \overline{\mathcal{E}^2} - U^2 = -U', \tag{7}$$

which can be rewritten as the well-known relation $\sigma^2 = kT^2 (\partial U/\partial T)_\omega = kT^2 C_\omega$. Because the heat capacity C_ω remains finite at $T = 0$, it follows that $\sigma^2(T = 0) = 0$, meaning that the dispersion of the energy is suppressed at $T = 0$, even when $\mathcal{E}_0 \neq 0$. In other words, a purely thermodynamic treatment is limited in that it does not allow for the inclusion of a *fluctuating* zpe.

To include the zero-point fluctuations it becomes necessary to look for a statistical distribution $W_s(\mathcal{E})$ that maximizes the entropy S_s for a given mean energy U, defined (up to an arbitrary additive constant) as $S_s = -k_B \int W_s(\mathcal{E}) \ln W_s(\mathcal{E}) d\mathcal{E}$. In addition to complying with the constraints of W_s being normalized to unity and $\langle \mathcal{E} \rangle_s = U(\beta)$, the results derived from $W_s(\mathcal{E})$ should be consistent with those derived from the thermal distribution W_g, In contrast to the thermodynamic entropy S which is defined in the phase space of the particles, the *statistical entropy S_s* is interpreted as a measure of the disorder present in the system [5]. The maximum-entropy formalism [6] leads to the following result for W_s:

$$W_s(\mathcal{E}) = \frac{1}{U} e^{-\mathcal{E}/U}. \tag{8}$$

This distribution gives for the energy moments $\langle \mathcal{E}^r \rangle_s = r! U^r$, whence the total energy variance is given by

$$\sigma_s^2 = \langle \mathcal{E}^2 \rangle_s - U^2 = U^2. \tag{9}$$

At $T = 0$ this result reduces to $\sigma_s^2(T = 0) = \mathcal{E}_0^2$; these are the nonthermal energy fluctuations. The purely thermal energy variance is therefore given at any temperature by

$$\sigma^2 = \sigma_s^2 - \mathcal{E}_0^2 = U^2 - \mathcal{E}_0^2. \tag{10}$$

2.2. The Planck distribution

From Eq. (7) it follows that $U(\beta)$ can be obtained from the thermal energy variance σ^2 given in Eq. (10) as a function of U, by integrating

$$d\beta = -dU/\sigma^2(U) \tag{11}$$

and inverting $\beta = \beta(U)$. The result (subject to the limit condition $U \to \infty$ as $\beta \to 0$) is

$$U(\beta) = \mathcal{E}_0 \coth \mathcal{E}_0 \beta. \tag{12}$$

This contains the classical case as a particular solution, corresponding to $A = 0$ in Eq. (2). With $A = \hbar/2$, corresponding to an energy $\mathcal{E}_0 = \hbar\omega/2$ for the zpf modes of frequency ω, (12) gives the well-known formula

$$U(\beta, \omega) = \tfrac{1}{2}\hbar\omega \coth \tfrac{1}{2}\hbar\omega\beta, \tag{13}$$

which shows that the (complete) Planck law is an *unavoidable* consequence of the existence of the nonthermal energy $\mathcal{E}_0$. This establishes Planck's law as a physical result whose ultimate meaning — or source — is the existence of this zpe.

The corresponding partition function, obtained from Eqs. (3) and (4), is

$$Z_g = \frac{1}{2\sinh \mathcal{E}_0\beta} = \sum_{n=0}^{\infty} e^{-\beta\mathcal{E}_n} = \int_0^{\infty} \sum_{n=0}^{\infty} \delta(\mathcal{E} - \mathcal{E}_n)e^{-\beta\mathcal{E}}d\mathcal{E}, \tag{14}$$

whence $g(\mathcal{E}) = \sum_{n=0}^{\infty} \delta(\mathcal{E} - \mathcal{E}_n)$, with the energy values $\mathcal{E}_n$ given by $\mathcal{E}_n = \mathcal{E}_0(2n+1) = \hbar\omega(n + \tfrac{1}{2})$. The distribution (3) turns out to be

$$W_g(\mathcal{E}) = \frac{1}{Z_g} \sum_{n=0}^{\infty} \delta(\mathcal{E} - \mathcal{E}_n)e^{-\beta\mathcal{E}}, \tag{15}$$

giving for the mean value of any function $f(\mathcal{E})$

$$\langle f(\mathcal{E}) \rangle = \int_0^{\infty} W_g(\mathcal{E})f(\mathcal{E})d\mathcal{E} = \frac{1}{Z_g} \sum_{n=0}^{\infty} f(\mathcal{E}_n)e^{-\beta\mathcal{E}_n} = \sum_{n=0}^{\infty} w_n f(\mathcal{E}_n), \tag{16}$$

with canonical weights $w_n = e^{-\beta\mathcal{E}_n}/Z_g$. This shows that the mean value of a function of the *continuous* variable $\mathcal{E}$ (the energy) calculated with $W_g(\mathcal{E})$, can be obtained equivalently by averaging over the set of *discrete states* n, with respective weights w_n. Because we are dealing with a canonical ensemble, the structure of w_n suggests identifying $\mathcal{E}_n$ with discrete energy levels of the quantum oscillators (including the zpe). Both descriptions hold simultaneously; they are the two formal sides of the same coin.

It is important to stress that Planck's law has been obtained as a consequence of the fluctuating zero point energy without introducing any explicit quantum or discontinuity requirement.[1] The fact that Wien's law (with $A \neq 0$) is the one that opens the door to the zpe compels us to view this law as an extension of classical physics that enters into the quantum domain. Strictly speaking, as a precursor of Planck's distribution it should be considered to contain the seed of the first quantum law. Furthermore, since Wien's law is satisfied by field and material oscillators alike, Eq. (13) and the ensuing consequences are of general validity, regardless of the nature of the oscillators. The confirmation that the law which gave birth to quantum theory stems from the existence of a zpe brings thus to the fore the crucial importance of this energy for the understanding of quantum mechanics and more generally, of quantum theory.

Note that in terms of the thermal part U_T of the energy, given according to Eq. (12) by

$$U_T = U - \mathcal{E}_0 = \frac{2\mathcal{E}_0}{e^{2\mathcal{E}_0\beta} - 1}, \tag{17}$$

Eq. (10) reads

$$\sigma^2 = U_T^2 + 2\mathcal{E}_0 U_T, \tag{18}$$

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[1] See [3], [4] for previous derivations. It is interesting to note that an independent derivation of Planck's law has been obtained by Dimitrov [7], using 'averaged' statistical thermodynamics and the assumption that (classical) particles possess a random (ultrarelativistic) unobservable, unfreezeable motion.

in contrast to the classical (purely wave) expression $\sigma_{cl}^2 = U_T^2$. The identification of the term U_T^2 as a product of the interference of the modes (of frequency ω) of the thermal field suggests to interpret (the 'particle' term) $2\mathcal{E}_0 U_T$ as due to additional interferences, now between the thermal field and the zpf, whose mean energy is just $\mathcal{E}_0$.

3. Quantum mechanics as a consequence of the zero-point field

In this second part of the paper it will be shown that also the quantum laws governing the dynamics of a particle can be derived without introducing a priori any discreteness or discontinuity in the description. The cornerstone of the derivation is again the existence of the zpf in permanent interaction with an otherwise classical particle subject to an external force. The presentation made here draws in part from previous work [2],[8],[9].

3.1. The particle-zpf system in phase space

We consider a system composed of a charged particle immersed in the zpf. The motion of the mechanical subsystem (in one dimension, for simplicity) is governed in the nonrelativistic limit by

$$\dot{x} = p/m, \qquad \dot{p} = f(x) + m\tau\dddot{x} + eE(t), \tag{19}$$

where $f(x)$ is the external force, $E(t)$ is the electric component of the zpf in the long-wavelength approximation and $m\tau\dddot{x}$ is the radiation reaction force, with $\tau = 2e^2/(3mc^3)$ ($\approx 10^{-23}$ s for the electron).[2] The density of points in the particle's phase space is described by

$$\frac{\partial}{\partial t}R + \frac{\partial}{\partial x}(\dot{x}R) + \frac{\partial}{\partial p}\left(f(x) + m\tau\dddot{x}\right)R = -\frac{\partial}{\partial p}E(t)R. \tag{20}$$

We are ultimately interested in describing the particle motion, not for a specific realization of the field but for the ensemble of its realizations, or alternatively an ensemble of equivalent systems. We therefore average $R(x, p, t)$ over the realizations of E to obtain the averaged density in the phase space of the particle, $\overline{R(x, p, t)}^E \equiv Q(x, p, t)$. With the help of the projector technique (see e.g. [8], [10]), this leads after a somewhat lengthy calculation to the Fokker-Planck-type equation

$$\frac{\partial}{\partial t}Q + \hat{L}Q = e^2\frac{\partial}{\partial p}\hat{D}(t)Q, \tag{21}$$

where $\hat{L}$ is the Liouville operator

$$\hat{L} = \frac{1}{m}\frac{\partial}{\partial x}p + \frac{\partial}{\partial p}(f + m\tau\dddot{x}) \tag{22}$$

and $\hat{D}(t)$ is a complicated integro-differential diffusion operator. More precisely, the diffusion term can be written in the form [8]

$$e^2\frac{\partial}{\partial p}\hat{D}(t)Q = e\frac{\partial}{\partial p}\hat{P}E\sum_{k=0}^{\infty}\left[e\hat{G}\frac{\partial}{\partial p}(1-\hat{P})E\right]^{2k+1}Q, \tag{23}$$

where $\hat{P}$ is the smoothing operator that averages over the realizations of E ($\hat{P}A = \overline{A}^E$) and the operator $\hat{G}$ is defined by

$$\hat{G}A(x, p, t) = \left(\frac{\partial}{\partial t} + \hat{L}\right)^{-1}A(x, p.t) = \int_0^t e^{-\hat{L}(t-t')}A(x, p, t')dt'. \tag{24}$$

[2] Note that the correct (integro-differential) version of the radiation reaction term is fully causal. The noncausal behaviour that comes with the $m\tau\dddot{x}$ term is merely an artefact of the approximation made in neglecting all higher-order derivatives. For an extensive discussion of this point see [2].

Note that the diffusion term comprises at least a correlation of the form $\overline{E_i(t)E_k(t')}^E$ and hence contains a factor proportional to the spectral (energy) density of the zpf,

$$\rho_{\text{zpf}}(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3} = \frac{\omega^2}{\pi^2 c^3} \cdot \frac{\hbar\omega}{2}. \tag{25}$$

The interaction between particle and field is therefore represented in Eq. (21) by both the radiation reaction term and the diffusion term, and it is through the latter that Planck's constant enters the picture.

3.2. Particle dynamics in configuration space

Equation (21) provides in principle a (partly averaged) statistical description of the dynamics of the particle for any time t. It is clear that initially, after particle and field have started to interact, there is an irreversible process of energy and momentum exchange during which the field has a randomizing and dissipative effect on the particle dynamics, due to the terms with E and $\dddot{x}$ in Eq. (21). However, to explore the connection with quantum mechanics we are specifically interested in the solution in the time-asymptotic limit, when *and if* a regime is attained in which this exchange is no longer irreversible. In such situation a balance should be reached between the mean power radiated and that absorbed by the particle from the field; this balance condition is discussed in detail in section 3.4. When this occurs, the radiative and dissipative terms mentioned have produced their main (irreversible) effect in taking the system to such regime, and one may expect any additional effect arising from these terms to represent just a radiative correction.

In order to establish the connection with quantum mechanics, let us first go from Eq. (21) to a description in configuration space. An orderly way to make this transition is by means of the characteristic (moment-generating) function $\widetilde{Q}$ associated with the density Q,

$$\widetilde{Q}(x, z, t) = \int Q(x, p, t)e^{ipz}dp. \tag{26}$$

The marginal probability density $\rho(x, t)$ is then given by

$$\rho(x, t) = \int Q(x, p, t)dp = \widetilde{Q}(x, 0, t), \tag{27}$$

and the (partially averaged) local moments of p are

$$\langle p^n \rangle (x) = \langle p^n \rangle_x = \frac{1}{\rho} \int p^n Q dp = (-i)^n \left(\frac{1}{\widetilde{Q}} \frac{\partial^n \widetilde{Q}}{\partial z^n} \right)\bigg|_{z=0} \tag{28}$$

for any integer n. The Fourier transform of Eq. (21) is (with all surface terms assumed to vanish at infinity)

$$\frac{\partial \widetilde{Q}}{\partial t} - i\frac{1}{m}\frac{\partial^2 \widetilde{Q}}{\partial x \partial z} - iz f(x)\widetilde{Q} - \frac{\tau}{m} f' z \frac{\partial \widetilde{Q}}{\partial z} = -ie^2 z(\widetilde{\hat{D}Q}). \tag{29}$$

By expanding this equation into a power series around $z = 0$ and separating the coefficients of z^k ($k = 0, 1, 2, \ldots$) one obtains an infinite hierarchy of equations containing momenta of p of increasing order. The first two equations (coefficients of z^0 and z) read

$$\frac{\partial \rho}{\partial t} + \frac{1}{m}\frac{\partial}{\partial x}\left(\langle p \rangle_x \rho \right) = 0, \tag{30}$$

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$$\frac{\partial}{\partial t}\left(\langle p\rangle_x \rho\right) + \frac{1}{m}\frac{\partial}{\partial x}\left(\langle p^2\rangle_x \rho\right) - f\rho - \frac{\tau}{m}f'\langle p\rangle_x \rho = -e^2\left.\left(\widetilde{\hat{D}Q}\right)\right|_{z=0}. \tag{31}$$

Equation (30) is the continuity equation, with the local (partially averaged) momentum given by $\langle p\rangle_x = -i(\partial \ln \widetilde{Q}/\partial z)_{z=0}$ (see Eq. (28) with $n=1$). Equation (31) describes the transfer of momentum and contains, in addition to ρ and $\langle p\rangle_x$, the second moment $\langle p^2\rangle_x$, which reappears in the third equation along with $\langle p^3\rangle_x$, and so on. This coupling between successive equations creates a highly difficult mathematical problem. However, in the case of interest here the first two equations decouple from the rest of the hierarchy, as shown below. Therefore only the first local moments of p intervene in the description of interest, which means that one may concentrate on the behaviour of $\widetilde{Q}(x,z,t)$ for small values of z.

From Eq. (28) one obtains

$$\langle p^2\rangle_x - \langle p\rangle_x^2 = -\left(\frac{\partial^2}{\partial z^2}\ln\widetilde{Q}\right)_{z=0}. \tag{32}$$

To find an expression for the right-hand-side term as a function of x we introduce the change of variables

$$z_+ = x + \eta z, \qquad z_- = x - \eta z, \tag{33}$$

where η is an as yet undetermined parameter having dimensions of action. We now write $\widetilde{Q}$ in the general form

$$\widetilde{Q}(x,z,t) = q_+(z_+,t)q_-(z_-,t)\chi(x,z,t), \tag{34}$$

where, according to Eq. (26),

$$q_+(z_-,t) = q_-^*(z_-,t), \quad q_-(z_+,t) = q_+^*(z_+,t), \quad \chi^*(x,z,t) = \chi(x,-z,t). \tag{35}$$

Equations (27) and (32) become then

$$\rho(x,t) = \widetilde{Q}(x,0,t) = q_+^*(x,t)q_+(x,t)\chi_0,$$

$$\langle p^2\rangle_x - \langle p\rangle_x^2 = -\eta^2\frac{\partial^2}{\partial x^2}\ln\rho + \Sigma, \tag{36}$$

with $\Sigma = 4\eta^2(\partial_{z_+}\partial_{z_-}\ln\chi)_{z=0}$, $\chi_0 = \chi(x,0,t)$. Inserting these results into Eq. (31) one obtains

$$m\frac{\partial}{\partial t}(\langle p\rangle_x \rho) + m\frac{\partial}{\partial x}\left(\langle p\rangle_x^2 \rho\right) - \frac{\eta^2}{m}\frac{\partial}{\partial x}\left(\rho\frac{\partial^2}{\partial x^2}\ln\rho\right) - f\rho =$$

$$= \frac{\tau\langle p\rangle}{m}\frac{\partial f}{\partial x}\rho - e^2\left.\left(\widetilde{\hat{D}Q}\right)\right|_{z=0} - \frac{1}{m}\frac{\partial}{\partial x}\Sigma\rho. \tag{37}$$

3.3. Quantum mechanics in the radiationless approximation

Following the discussion at the beginning of the previous section, we consider that in the time-asymptotic limit the effect of the terms in Eq. (37) due to radiation reaction (proportional to $\tau \sim e^2$) and diffusion (also proportional to e^2) has become a mere radiative correction of order $\alpha \sim e^2$ (the fine structure constant). In addition, we introduce the assumption that the contribution of the nonfactorizable function $\chi(z_+,z_-,t)$, and hence of Σ in Eq. (37), has also become small, of order e^2. We shall come back to this assumption below (section 4.3). Under these conditions one may write $\chi \simeq \chi_0 = 1$ (by absorbing the dependence on x and t in q_- and q_+) and neglect the three terms on the right-hand side of Eq. (37); this is the meaning of the radiationless approximation.

In the radiationless approximation Eq. (36) becomes

$$\left\langle p^2 \right\rangle_x = \left\langle p \right\rangle_x^2 - \eta^2 \frac{\partial^2}{\partial x^2} \ln \rho, \tag{38}$$

so that $\left\langle p^2 \right\rangle_x$ decouples from the rest of the hierarchy. Equation (29) reduces then to

$$\frac{1}{\widetilde{Q}} \partial_t \widetilde{Q} - \frac{i}{m\widetilde{Q}} \partial_x \partial_z \widetilde{Q} = izf(x). \tag{39}$$

Writing explicitly $\widetilde{Q}(x,z,t) = q_+(z_+,t)q_-(z_-,t)$, according to (34), this equation gives

$$\frac{1}{q_+ q_-} \partial_t (q_+ q_-) - \frac{i\eta}{m} \left[\frac{1}{q_+} \partial_+^2 (q_+) - \frac{1}{q_-} \partial_-^2 (q_-) \right] = \frac{i}{2\eta}(z_+ - z_-)f[\tfrac{1}{2}(z_+ + z_-)]. \tag{40}$$

Recall that we are interested in small values of z only. Assuming $f(z)$ to be well behaved in the interval $[z_-, z_+]$, the mean-value theorem applied to the term on the right-hand side gives to lowest order in z

$$(z_+ - z_-)f[\tfrac{1}{2}(z_+ + z_-)] = \int_{z_-}^{z_+} f(u)du = -\left[V(z_+) - V(z_-) \right]. \tag{41}$$

This result holds for any (well-behaved) potential $V(x)$. One thus gets from Eq. (40)

$$-\frac{i\eta}{m}\frac{1}{q_+}\partial_+^2 q_+ + \frac{i}{2\eta}V(z_+) + \frac{1}{q_+}\partial_t q_+ = -\frac{i\eta}{m}\frac{1}{q_-}\partial_-^2(q_-) + \frac{i}{2\eta}V(z_-) - \frac{1}{q_-}\partial_t q_-. \tag{42}$$

The two sides of this equation depend on different variables and hence separate; let K be the (real) separation constant. In the limit $z \to 0$ both z_+ and z_- reduce to x, whence putting $q_+(x,t) \equiv e^{iKt}\psi(x,t)$, $q_-(x,t) \equiv e^{-iKt}\psi^*(x,t)$ (see Eq.(35)) one gets in this limit

$$-2\frac{\eta^2}{m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = 2i\eta \frac{\partial \psi}{\partial t} \tag{43}$$

and its complex conjugate, with $\rho(x,t) = \psi^*(x)\psi(x)$. Except for the (constant) factor η to be determined (see the discussion in the next section), we recognize in (43) the Schrödinger equation for $\psi(x,t)$. This means that, indeed, a regime of unitary (time-reversible) evolution has been attained.

3.4. Detailed energy balance
The full equivalence between Eq. (43) and the Schrödinger equation entails the relation $\eta = \hbar/2$. To prove that this value is correctly predicted by the theory we resort to the condition of energy balance, which has been assumed to hold in the time-asymptotic limit (see section 3.2).

From the Fokker-Planck-type equation (21) multiplied by p^2 and integrated over the entire phase space it follows (assuming again all surface terms to vanish at infinity) that

$$\frac{1}{2m}\frac{d}{dt}\left\langle p^2 \right\rangle = \frac{1}{2m}\frac{d}{dt}\int p^2 Q dx dp = \frac{1}{m}\left\langle fp + m\tau p\dddot{x} - \frac{e^2}{2}p\hat{D} \right\rangle, \tag{44}$$

where it is understood that $\hat{D}$ acts on $Q(x,p,t)$. Since $d\left\langle V \right\rangle /dt = -\left\langle fp \right\rangle /m$, the total energy gained or lost in the mean by the particle through radiation exchange is

$$\frac{d}{dt}\left\langle H \right\rangle = \frac{d}{dt}\left\langle \frac{1}{2m}p^2 + V \right\rangle = m\tau \left\langle \dot{x}\dddot{x} \right\rangle - \frac{e^2}{2m}\left\langle p\hat{D} \right\rangle, \tag{45}$$

with H representing the mechanical Hamiltonian. The terms on the right-hand side represent the average power dissipated by the particle due to Larmor radiation and the average power absorbed by the particle from the field, respectively. When these terms compensate each other,

$$m\tau \langle \dot{x}\,\dddot{x} \rangle = \frac{e^2}{2m} \left\langle p\hat{D} \right\rangle, \tag{46}$$

$\langle H \rangle$ becomes constant and energy balance is reached.

In line with the radiationless approximation used above to arrive at Eq. (43), it suffices to calculate both sides of (46) to lowest order in $\tau \sim e^2$. The left-hand side is readily calculated using the solutions of Eq. (43) (in terms of η) and gives for the ground state

$$m\tau \langle \dot{x}\,\dddot{x} \rangle = -m\tau_k \omega_{0k}^4 \left| x_{0k} \right|^2, \tag{47}$$

where $\omega_{0k} = (\mathcal{E}_0 - \mathcal{E}_k)/2\eta$, $x_{0k} = \int \psi_0^* x \psi_k dx$, $\mathcal{E}_k$ are the energy eigenvalues and ψ_k the corresponding eigenfunctions. To calculate the right-hand side of (46) one must resort to Eq. (23) to lowest order in e^2, introduce Eq. (25) for the spectral density of the zpf, and take into account that energy balance is established in the time-asymptotic limit, so that the integral over time extends to infinity. Using again the solutions of Eq. (43), one arrives finally at

$$\frac{e^2}{2m} \left\langle p\hat{D} \right\rangle = -\frac{\hbar m\tau}{2\eta} \sum_k \omega_{k0}^4 \left| x_{0k} \right|^2. \tag{48}$$

Equating this result with Eq. (47) one obtains for the parameter η the value $\eta = \hbar/2$, thus recovering the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \tag{49}$$

for any potential $V(x)$. Notice that the Planck constant in Schrödinger's equation comes directly from the zpf through Eq. (48). It is notable that the dipole approximation for the zpf has been sufficient to arrive at this result. Equations (47) and (48) have exactly the same structure, which means that balance is achieved term by term; one speaks therefore of *detailed* energy balance. In fact a spectrum proportional to ω^3, being responsible for the ω_{k0}^4 factor in Eq. (48), is the *only* one that guarantees detailed energy balance. This establishes a fundamental difference with classical systems, for which equilibrium is reached only for the Rayleigh-Jeans law, proportional to ω^2. Since a spectrum $\sim \omega^3$ is the single Lorentz-invariant one [1],[11], it is verified that Lorentz invariance and detailed equilibrium with matter are intimately related properties of the zpf. In particular, atomic stability is reached for those specific (stationary and discrete) states for which the Larmor radiation along the orbit is compensated in the mean by the energy absorbed from the field and turned into fluctuating motions.

3.5. Quantum nonlocality

With the vanishing of the radiative terms on the right-hand side, Eq. (37) reduces (now with $\eta = \hbar/2$) to

$$m\frac{\partial}{\partial t}(\langle p \rangle_x \rho) + m\frac{\partial}{\partial x}\left(\langle p \rangle_x^2 \rho\right) - \frac{\hbar^2}{4m}\frac{\partial}{\partial x}\left(\rho\frac{\partial^2}{\partial x^2}\ln\rho\right) - f\rho = 0. \tag{50}$$

The behaviour of the quantum system is therefore fully described (in the radiationless approximation) by this equation along with the continuity equation (30) (plus the energy-balance condition), or alternatively by the Schrödinger equation and its complex conjugate, with $\rho(x,t) = \psi^*\psi$.

Note that the only term that distinguishes Eq. (50) from the corresponding classical equation is the one containing $\ln \rho$. From Eq. (38) it is clear that this term originates in the fluctuations of the momentum, transcribed to configuration space. The source of these fluctuations is the zpf, which also determines their size through Planck's constant. Hence the (momentum) quantum fluctuations are conventional fluctuations having a *causal* origin. Their mean value is given by

$$\left\langle \sigma_p^2(x) \right\rangle = \left\langle -\frac{\hbar^2}{4m}\frac{\partial^2}{\partial x^2}\ln\rho \right\rangle = \frac{\hbar^2}{4}\int \rho \left(\frac{1}{\rho}\frac{\partial\rho}{\partial x}\right)^2 dx. \tag{51}$$

The connection of this term with the quantum potential is readily established by applying the Madelung transformation $\psi(x,t) = \sqrt{\rho}\exp(iS/\hbar)$ to the Schrödinger equation, which leads to the Hamilton-Jacobi-type equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2} + V = 0. \tag{52}$$

The term $Q_p = \left(-\hbar^2/2m\right)(1/\sqrt{\rho})(\partial^2\sqrt{\rho}/\partial x^2)$, the so-called *quantum potential*, ascribes to this (otherwise classical) equation a statistical and nonlocal meaning. This is precisely the term responsible for the nonlocality of Bohm's theory, where it is also known as 'Bohm potential' [12].[3] As is clear from the present derivation, however, it is of *kinematic* origin.[4] It is important to observe that this nonlocality is generated by the reduction of the statistical description to the configuration space, and appears even in the case of a single particle — with the exception of the free particle described by a plane wave, $\rho = $ const. For a system composed of two or more particles, additional nonlocalities arise. For instance, in the two-particle case the phase-space density described by the respective Fokker-Planck-type equation — which is an extension of Eq. (21) — depends on the variables of both particles. This will be reflected in general in the configuration-space description by means of a nonfactorizable joint probability density $\rho(x_1, x_2) = \rho_1(x_1)\rho_2(x_2)\rho_{12}(x_1, x_2)$. The corresponding 'quantum potential' for particle 1, for instance, will be of the form

$$Q_{p1}(1,2) = -\frac{\hbar^2}{2m}\left[\frac{\partial_1^2\sqrt{\rho_1}}{\sqrt{\rho_1}} + \frac{\partial_1^2\sqrt{\rho_{12}}}{\sqrt{\rho_{12}}} + \frac{1}{2}\frac{\partial_1\rho_1}{\rho_1}\frac{\partial_1\rho_{12}}{\rho_{12}}\right], \tag{53}$$

containing two additional terms that depend on the presence of particle 2. These terms represent a nonlocal action of particle 2 on particle 1. Indeed, a more detailed analysis of the two-particle problem, following a procedure slightly different from the one exposed here, has led to a clarification of the mechanism leading to the entanglement of states which is normally associated with Bell's inequalities [14].

4. Radiative corrections

This third part of the paper is devoted to the analysis of further (first-order) effects deriving from the particle-field interaction in the quantum regime. Firstly the condition of energy balance is analysed for a particle in an excited state. More generally, absence of detailed balance is shown to give rise to radiative transitions, with the respective transition probabilities exactly as given by nonrelativistic quantum electrodynamics (QED). Finally, other radiative corrections to first order include the Lamb shift and a (nonrelativistic) mass correction.

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[3] As mentioned in section 3.2, $\langle p \rangle_x/m$ is the local mean velocity of the subensemble of particles moving in the neighbourhood of x at time t; hence the instantaneous velocity of an individual particle differs from it by a stochastic component $\delta\dot{x}$. This is important with regard to Bohm's theory, which gives information only about the mean trajectories by neglecting $\delta\dot{x}$. Due to this stochastic component, quantum trajectories are entirely free to intersect.

[4] This result is in line with previous conclusions arrived at independently by other authors; see Ref. [13].

4.1. Energy balance for a particle in an excited state

In section 3.4, detailed energy balance was shown to hold with both particle and field in their ground state. Assume now that the particle has by some external mechanism been transferred to an excited state n, the background field still being in its ground state. Then both terms on the right-hand side of (45) must be recalculated. For the first term one obtains instead of (47)

$$m\tau \left\langle \dot{x}\,\dddot{x} \right\rangle_n = -m\tau \sum_k \omega_{nk}^4 \left| x_{nk} \right|^2 . \tag{54}$$

For the second term, on the other hand, one obtains instead of (48) (with $\eta = \hbar/2$)

$$\frac{e^2}{2m} \left\langle p\hat{D} \right\rangle_n = -m\tau \sum_k \omega_{kn}^4 \left| x_{nk} \right|^2 \mathrm{sign}\,\omega_{kn}. \tag{55}$$

Both expressions have again the same structure, except that the latter contains now a mixture of positive and negative terms, whilst in Eq. (54) all terms have the same sign. The upshot is that there cannot be detailed balance between the zpf and a particle in an excited state. This confirms that only the ground state of the particle is stable in the sole presence of the zpf.

Let us now investigate whether there is *any* background field of spectral density $\rho(\omega) = \rho_{\mathrm{zpf}}(\omega)\gamma(\omega)$ (with $\gamma(\omega)$ to be determined), with which a mechanical system in a particular excited state n can be in equilibrium. The radiation-reaction term is again given by Eq. (54). For the 'diffusion' term, which depends on the spectral density, we obtain instead of Eq. (55)

$$\frac{e^2}{2m} \left\langle p\hat{D} \right\rangle_n = -m\tau \sum_k \omega_{kn}^4 \left| x_{nk} \right|^2 \gamma(|\omega_{kn}|)\mathrm{sign}\,\omega_{kn}. \tag{56}$$

Both Eqs. (54) and (56) contain in general mixtures of terms of different frequencies (ω_{nk} for different values of k), but with different signs, so that detailed balance cannot be satisfied in general. Only for the particular case in which all values of $|\omega_{nk}|$ coincide, the possibility of detailed balance exists in principle. This possibility is realized for the harmonic oscillator, the single system for which all $|\omega_{nk}|$ are equal (and coincide with the oscillator frequency ω_0). Using again the solutions provided by the Schrödinger equation, Eqs. (54) and (56) give

$$m\tau \left\langle \dot{x}\,\dddot{x} \right\rangle_n = -\frac{1}{2}\hbar\tau\omega_0^3(2n+1), \quad \frac{e^2}{2m} \left\langle p\hat{D} \right\rangle_n = -\frac{1}{2}\hbar\tau\omega_0^3\gamma(\omega_0). \tag{57}$$

On equating these two results we conclude that indeed, detailed balance exists between a harmonic oscillator in its excited state n and a background field with spectral energy density

$$\rho_n(\omega) = \rho_{\mathrm{zpf}}(\omega)(2n+1) = \frac{\hbar\omega^3}{2\pi^2 c^3}(2n+1) \tag{58}$$

This result should not come as a surprise, since it corresponds to a field with precisely an energy per normal mode $\frac{1}{2}\hbar\omega(2n+1)$ as derived in section 2, equal to the energy of the mechanical oscillators with which it is in equilibrium. This is the secret of the Planck distribution and field quantization: equilibrium with matter is reached only for those oscillators that have energy $\frac{1}{2}\hbar\omega(2n+1)$.

4.2. Radiative transitions: Einstein A and B coefficients

Now we investigate some important implications of the absence of detailed balance. This can be done by introducing Eqs. (54) and (56) into (45), to determine the average rate of energy loss or gain for a particle in state n; the result is

$$\frac{dH_n}{dt} = -m\tau \sum_k \omega_{nk}^4 \left| x_{nk} \right|^2 \left[1 - \gamma(|\omega_{nk}|)\mathrm{sign}\,\omega_{kn} \right]. \tag{59}$$

With $\gamma(\omega) = 1 + \gamma_a(\omega)$, where $\gamma_a(\omega)$ represents the contribution from the excited (or additional) background field to the spectral energy density, this equation takes the form

$$\frac{dH_n}{dt} = m\tau \sum_k \omega_{nk}^4 \left|x_{nk}\right|^2 \left[(\gamma_a)_{\omega_{kn>0}} - (2 + \gamma_a)_{\omega_{kn<0}} \right]. \tag{60}$$

The first term within brackets represents the absorptions and the second one, the emissions. It is clear from this result that there can be absorptions only when the background field is excited (or there is an external component), whilst the emissions can be either 'spontaneous' (in presence of just the zpf, the term with coefficient 2) or stimulated by the additional field. The coefficients appearing in the various terms determine the respective rates of energy gain and energy loss; therefore, they should be directly related with Einstein's A and B coefficients.

Indeed, for the coefficient A, which is defined as the time rate for 'spontaneous' emissions via $dH_n = -\hbar \sum_k \omega_{nk} A_{nk} dt$, Eq. (60) with $\gamma_a = 0$ gives

$$A_{nk} = \frac{4e^2 \left|\omega_{nk}\right|^3}{3\hbar c^3} \left|x_{nk}\right|^2 = \frac{2\tau m}{\hbar} \left|\omega_{nk}\right|^3 \left|x_{nk}\right|^2, \tag{61}$$

which coincides with the result given by nonrelativistic QED [15]. For the coefficients B, which give the rate of energy gain or loss due to transitions induced by the external field and are defined through $dH_n = \pm\hbar \sum_k \omega_{nk} B_{nk} \rho_a(\omega_{nk}) dt$, Eq. (60) gives

$$B_{nk} = B_{kn} = \frac{m\tau \omega_{nk}^4 \left|x_{nk}\right|^2 \gamma_a(\omega_{nk})}{\hbar \left|\omega_{nk}\right| \rho_a(\omega_{nk})} = \frac{4\pi^2 e^2}{3\hbar^2} \left|x_{nk}\right|^2, \tag{62}$$

which again coincides with the respective formula of QED. This confirms the key role played by *both* radiative terms in determining the rates of transition. According to traditional wisdom, the A coefficient is due to the commutator of the creation and annihilation field operators. As a mathematical rule this is of course a nice description. From a present point of view its meaning becomes transparent: the factor 2 comes directly from the combined effect of Larmor radiation and the zpf (in equal parts) in 'inducing' emissions from an excited state.

The expressions for the Einstein coefficients involve each a single frequency $\left|\omega_{nk}\right|$, which confirms that the system as a whole reaches a state of *detailed balance*, i.e., energy balance of matter with the field at every separate frequency. The theory has thus led us from global balance to detailed balance in the quantum regime. We recall that this demand was one of Einstein's major hypotheses in his pioneering work where he introduced the absorption and emission coefficients [16].

4.3. Lamb shift and mass correction

In the context of the present work, the radiative corrections can be traced to the terms that were discarded from the right-hand side of Eq. (37) when taking the radiationless approximation that led to Eq. (50). The first and the second terms are the ones that intervene in the energy balance equation (45), being responsible for the finite lifetimes of excited states and the transitions between states. The third term, the one containing Σ in Eq. (37), represents a contribution to the momentum fluctuations, whence to first order in perturbation theory it adds to the mean kinetic energy of a stationary state an amount given by $\langle \Sigma \rangle / 2m$. This term accounts for another important radiative correction, namely the Lamb shift of the atomic levels (and an accompanying mass correction).

Because of the complexity of the calculations (which are the subject matter of a future paper) we report here the results for a simple system, namely the harmonic oscillator. An elementary procedure can be followed in this case to calculate the value of the energy shift, by

going back to the Langevin equation (19) and carrying out a perturbative treatment [8] (see also Refs. [1], [2]). The correction to the mean kinetic energy can be shown to be given by the term $\delta E = -\frac{1}{2} \langle \mathbf{d} \cdot \mathbf{E} \rangle$, which originates in the coupling of the instantaneous fluctuations of the electric dipole $\mathbf{d}$ to the electric component of the zpf. Qualitatively this corresponds to one of the most popular interpretations of the Lamb effect in QED, and in particular, to the noted model of Welton [17]. However, by noting that for the harmonic oscillator under energy balance $\langle \mathbf{d} \cdot \mathbf{E} \rangle = (\tau m/\omega) \langle \ddot{\mathbf{x}}^2 \rangle$, one can write this shift in the alternative form $\delta E = -(\tau m/2\omega) \langle \ddot{\mathbf{x}}^2 \rangle$ and interpret it as due to Larmor radiation. This is the kind of ambiguity that appears in QED due to the freedom to reorder the operators.

The calculations of the above term are readily performed to lowest order of approximation and give two corrections, namely a mass renormalization of value $\delta m = 3m/(8\pi\alpha)$ and a level (Lamb) shift of value

$$\delta E = \frac{\alpha \hbar^2 \omega^2}{\pi m c^2} \ln \frac{\omega_c}{\omega}, \tag{63}$$

where $\alpha = e^2/\hbar c$ and ω_c is some cutoff frequency. The reason for the large value of both results is the long tail of the spectral density $\rho(\omega) \sim \omega^3$ at high frequencies. Fixing the cutoff at $\omega_c = mc^2/\hbar$, reasonable for a nonrelativistic calculation, one obtains $\delta m/m \sim \alpha$ and for δE the same value as predicted by nonrelativistic QED [15],

$$\frac{\delta E}{\frac{1}{2}\hbar\omega} = \frac{2\alpha}{\pi} \frac{\hbar\omega}{mc^2} \ln \frac{mc^2}{\hbar\omega}. \tag{64}$$

This represents a very small correction,[5] but one that played an important role during the foundation of QED and is frequently considered one of the best successes of QED. With this, the approximation $\chi = 1$ made above becomes fully justified.

5. Concluding remarks

Several important conclusions and observations can be drawn from the results here presented. Of course the main one is that the quantum behaviour can be understood as emerging from the interaction matter-zpf. The ubiquitous presence of the zpf implies thus the omnipresence of the quantum phenomenon. The quantum properties are not intrinsic to matter or field, but are something more interesting and rich: they belong to a world of emerging properties.

In the present account the quantum fluctuations of matter lose their cryptic meaning to become a causal phenomenon, produced by the action of the random zpf. Analogously, quantum indeterminism arises as an anticipated property of the statistical description of the stochastic dynamics. The Heisenberg inequalities find thus a natural explanation: the minimal variances of the dynamical variables determined by the inequalities are those impressed by the zpf, which itself fluctuates with an intensity proportional to $\hbar$.

Nonlocality, that other quantum mystery, is explained in the present context as an outcome of the description at the quantum level. At the deeper level the theory is entirely local, yet an effective nonlocality arises as a result of the reduction of the statistical description from phase space to configuration space. This nonlocality is encapsulated in the quantum potential, which represents a kinetic contribution to the energy due to the momentum fluctuations. The correlations among dynamical variables that pertain to two or more entangled particles give rise to additional nonlocalities, e.g. of the EPR or Bell type.

The quantum behaviour of matter emerges in the time-asymptotic limit, when the combined diffusive and dissipative action of radiation has driven the system to a regime of unitary time

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[5] Note that this result is independent of the state, hence it does not lead to any observable shift for the harmonic oscillator.

invariance. Herein lies the secret of the stability of atomic states, which quantum mechanics is unable to reveal due to its unawareness of the zpf: stationarity is reached for certain orbital motions only. Quantum mechanics is derived once the radiative terms can be neglected. Taking due account of these neglected terms to first order provides correct results for the rates of transition between stationary states and the corresponding radiative corrections of (nonrelativistic) QED.

In between the initial time (when particle and zpf start to interact) and the quantum regime there is a new physics, which is neither the purely classical one — since it contains Planck's constant and the zpf as important ingredients — nor the quantum one — since for short times even the Heisenberg inequalities can be violated. In particular the function $Q(x, p, t)$ should furnish a better phase-space description, an important matter that requires further consideration. Moreover, there are other properties of quantum systems that from the present perspective should emerge from a deeper level, such as the spin of particles and all that comes with it. In summary, there is a wide interesting field open to study.

One should of course bear in mind that ours is not the only possible deeper-level explanation of the quantum phenomenon. In recent times the notion that quantization is an emergent phenomenon, possibly rooted on a classical dynamics, has gained much ground and several interesting — alternative or complementary — theories have been put forward [18]. Doubtlessly a finer and richer picture of the quantum world, and hopefully a physically more satisfactory one, will emerge as a result of these developments.

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